Two-dimensional gyrokinetic turbulence

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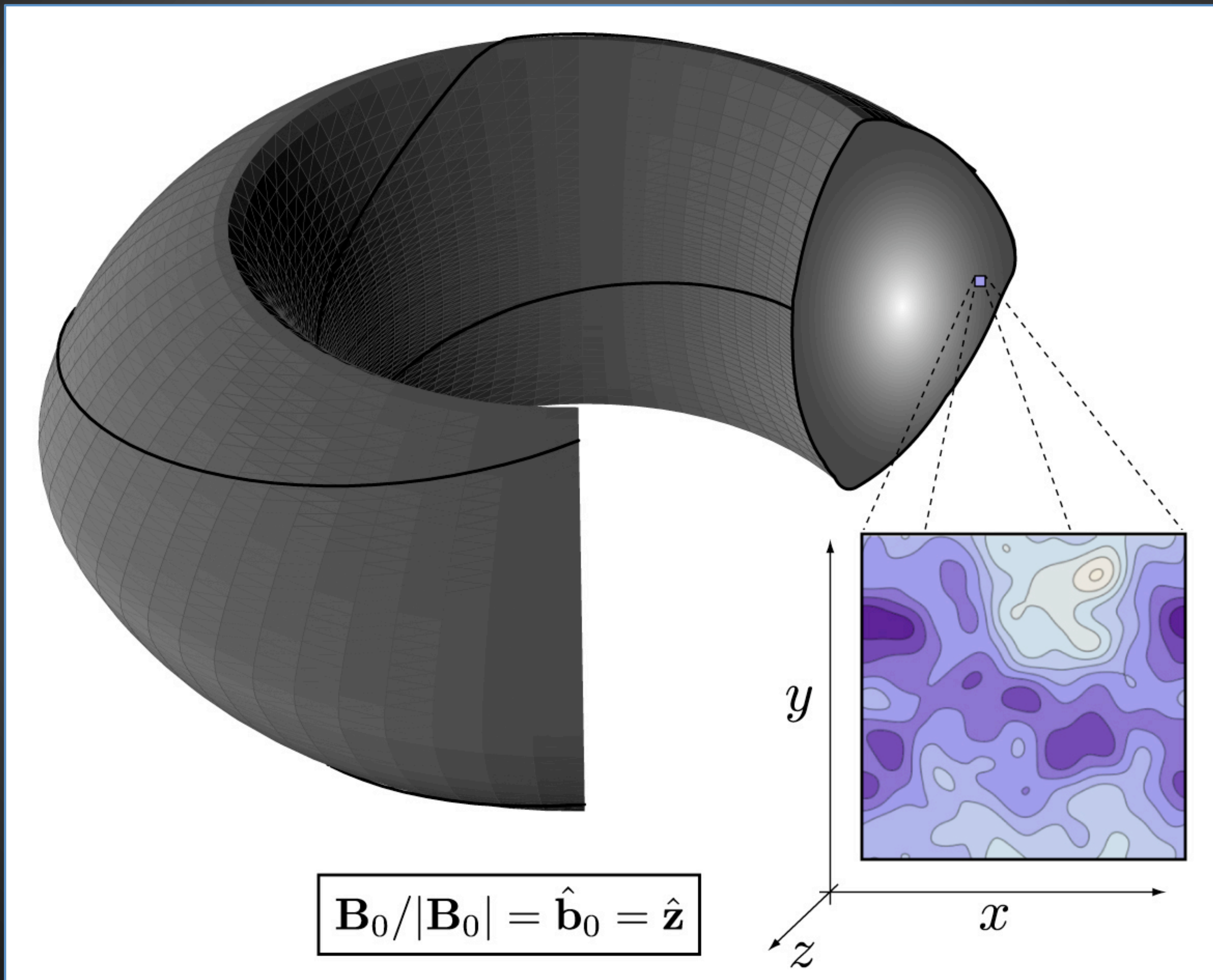
1 [Plunk, Cowley, Schekochihin, Tatsuno, *almost submitted* JFM (2009)]
2 [T. Tatsuno, W. Dorland, A. A. Schekochihin, G. G. Plunk, M. Barnes, S. C. Cowley, and G. G. Howes, *submitted* POP (2009); *submitted* PRL. (2008) arXiv:0811.2538]

Outline

- Two-dimensional gyrokinetics: Dual cascade by nonlinear phase-mixing
  - Phenomenology
  - Relationship to Charney-Hasegawa-Mima
  - Formalism for phase-space spectrum
  - Predictions
  - Numerical simulations

Local 2D Geometry



The Equations

The two-dimensional gyrokinetic system:

$$\frac{\partial g}{\partial t} + \mathbf{v}_E \cdot \boldsymbol{\nabla} g = \langle C[h] \rangle_{\mathbf{R}} + \mathcal{F} \qquad (1)$$

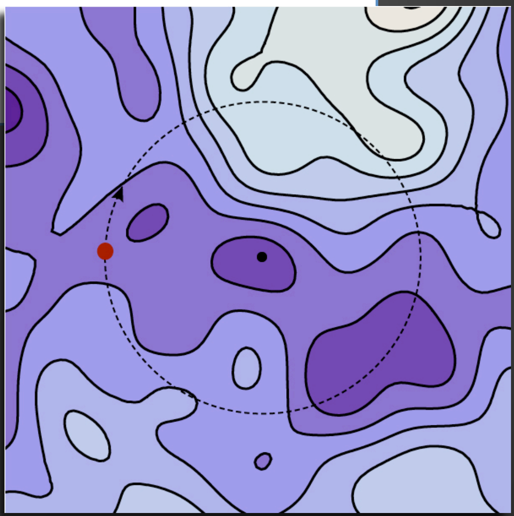
$$2\pi \int v dv \, \langle g \rangle_{\mathbf{r}} = \alpha\varphi - \Gamma_0 \varphi \qquad (2)$$

$$\mathbf{v}_E = \hat{\mathbf{z}} \times \boldsymbol{\nabla} \langle \varphi \rangle_{\mathbf{R}}$$

"Gyro-averaged" ExB velocity

Random homogeneous forcing

Quasi-neutrality



$g(v_\perp, v_\|, \mathbf{R}, t)$ is the kinetic distribution of charged rings.

Collisionless Invariants of 2D Gyrokinetics

"Generalized free energy"

$$W_g = 2\pi \int v dv \int \frac{d^2\mathbf{r}}{V} \frac{\left\langle g^2 \right\rangle_{\mathbf{r}}}{2F_0}$$

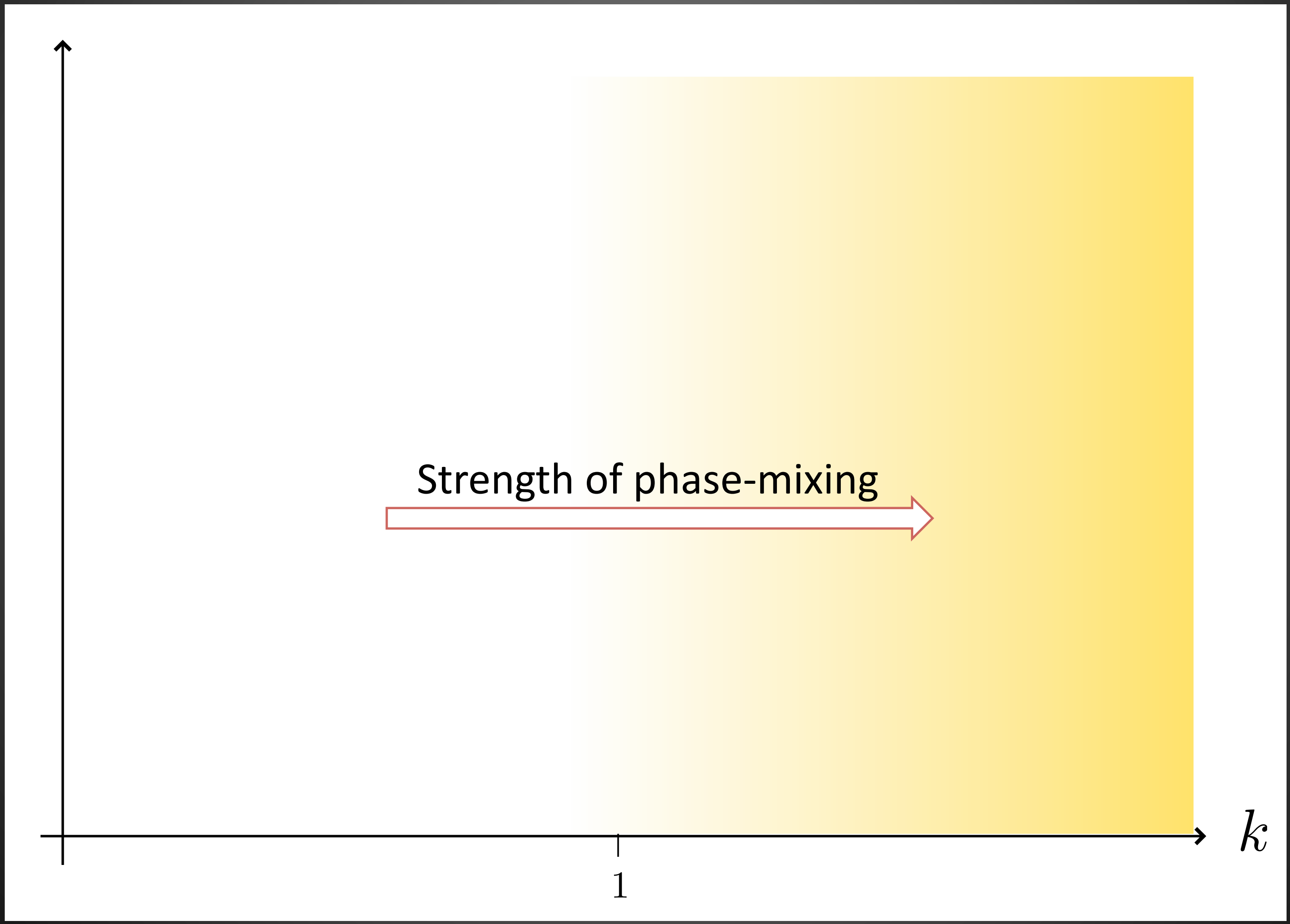
Electrostatic invariant (only in 2D!)

$$E = \frac{1}{2} \int \frac{d^2\mathbf{r}}{V} \left[\alpha\varphi^2 - \varphi\Gamma_0\varphi\right]$$

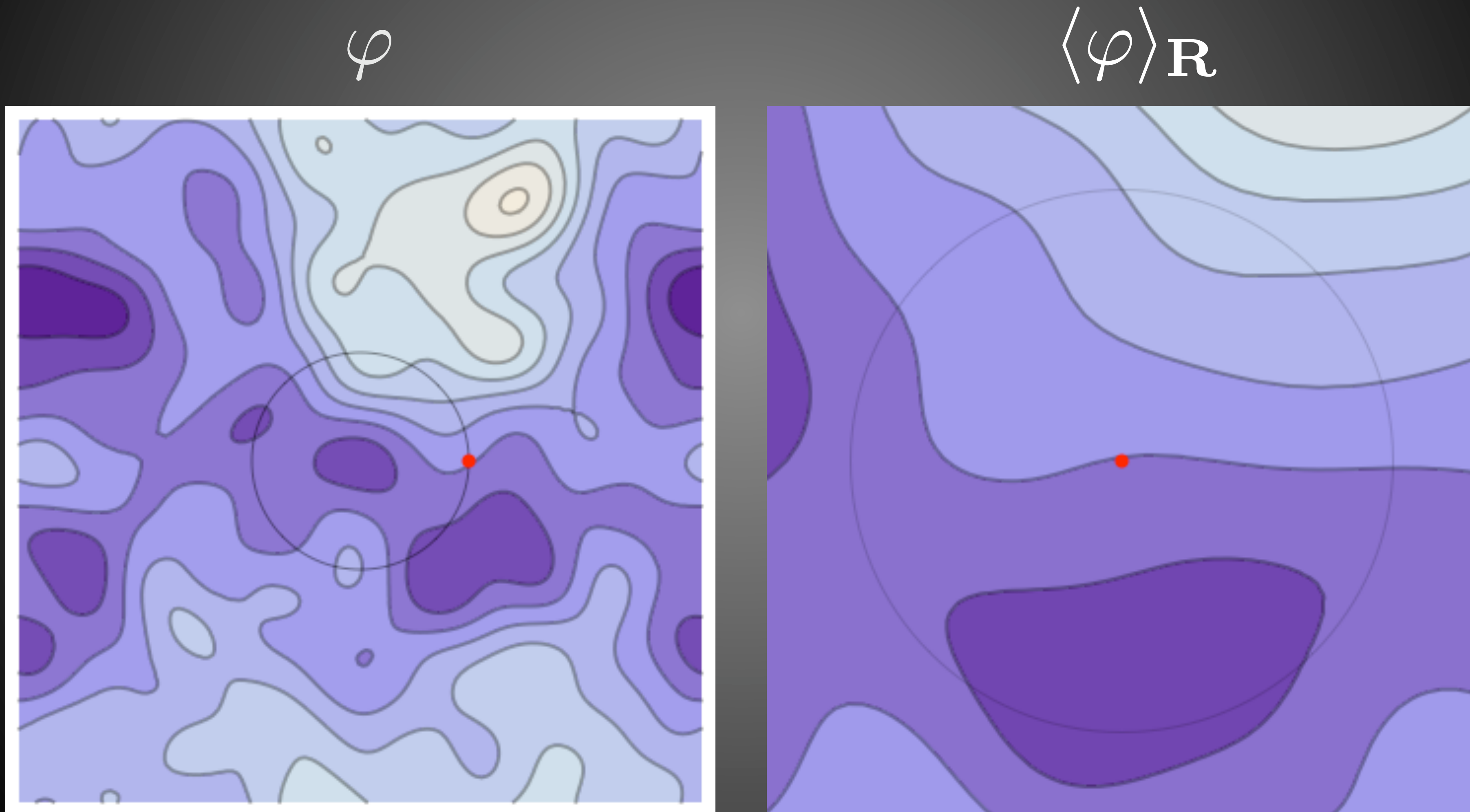
Actual free energy:

$$W = T_0 \delta S = -\int \frac{d^2\mathbf{r}}{V} \frac{\delta f^2}{F_0} = -\int \frac{d^2\mathbf{r}}{V} \left[2\pi \int v dv \frac{\left\langle h^2 \right\rangle_{\mathbf{r}}}{F_0} - \alpha\varphi^2\right]$$

Nonlinear phase-mixing range

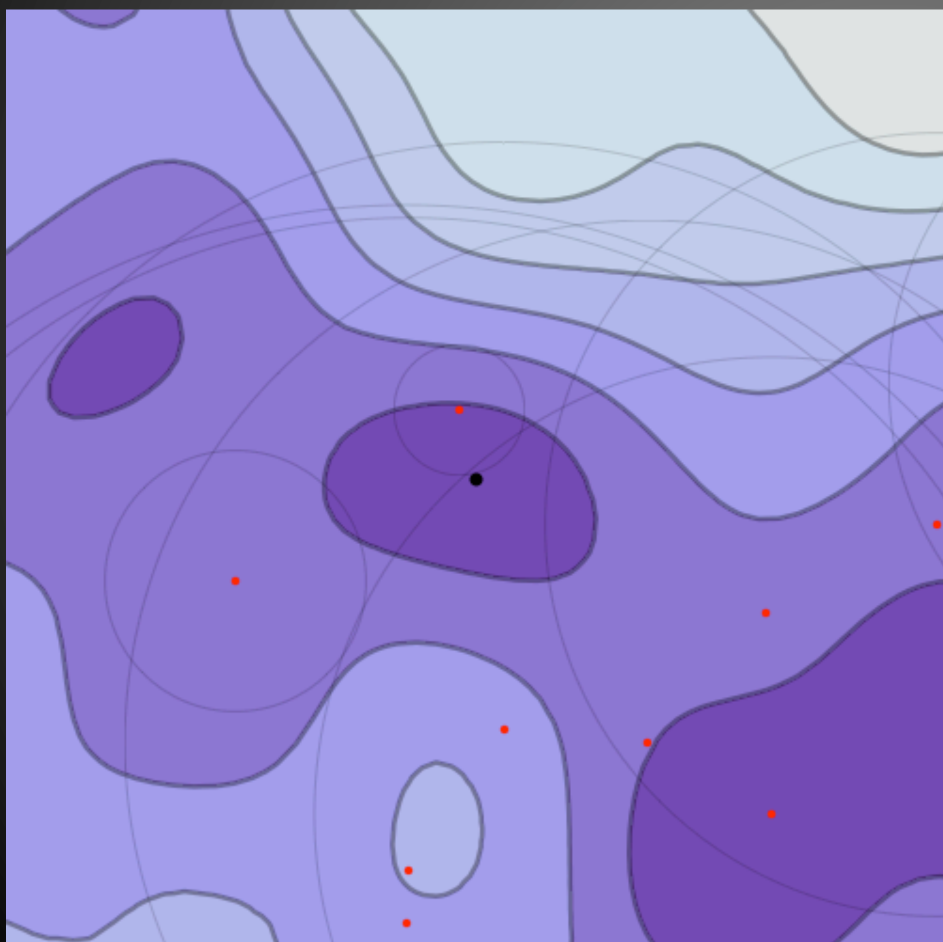


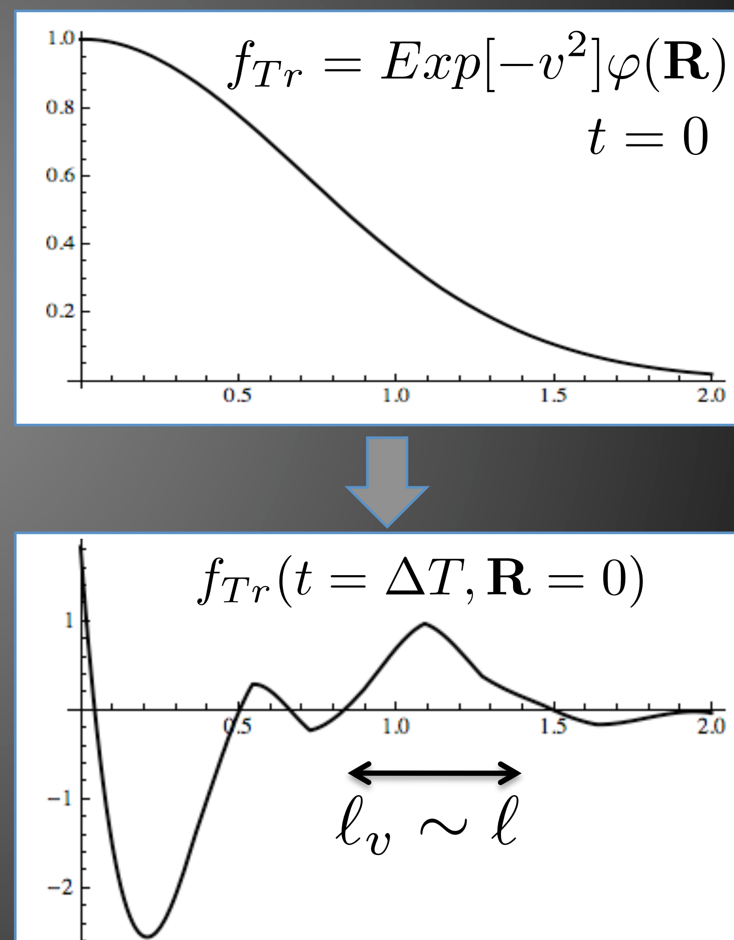
Gyrokinetic particle motion



Cartoon of Phase Mixing

(Passive advection of *tracer* particle distribution f_{Tr})

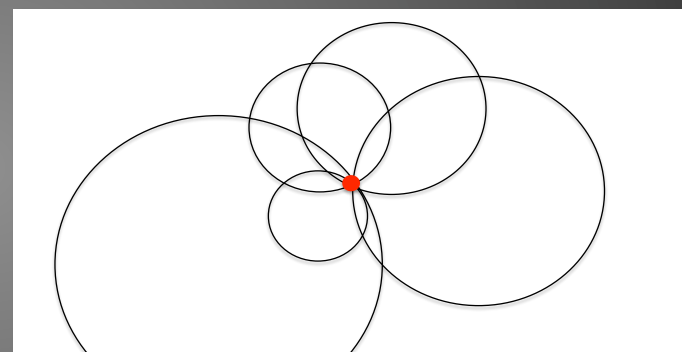




The importance of phase-mixing at fine scales, k >> 1

Quasi-neutrality: the potential arises from the collection of particles whose gyro-orbits intersect at the coordinate **r**.

$$\varphi \sim \int v dv \, \langle g \rangle_{\mathbf{r}}$$



Significant accumulation of this integral depends on a *resonance* between the velocity *and* spatial distribution of gyro-centers. E.g., a Maxwellian g produces no contribution:

If $\hat{\varphi}(k) \sim \int v dv J_0(kv) \hat{g}(k, v)$ and $\hat{g} \sim \mathrm{e}^{-v^2}$ then $\hat{\varphi} \sim \mathrm{e}^{-k^2}$

Phenomenology (ideology) in the "nonlinear phase-mixing range," k >> 1

Scale-by-scale relationship implied by quasi-neutrality:

$$\varphi_\ell \sim \int v dv J_0(v/\ell) g_\ell(v)$$
$$\sim \int \ell^{1/2} v^{1/2} dv \cos\left(v/\ell - \pi/4\right) g_\ell(v)$$

Integral accumulates as random walk*, so:

$$\varphi_\ell \sim \ell^{1/2} \ell_v^{1/2} g_{0\ell} \sim \ell g_{0\ell}$$

Dual cascade ***already*** implied:

$$W_g(k) \sim k^2 E(k)$$

* see Eqn (267) of [Schekochihin, et al ApJ (2009)]

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[Fjørtoft 1953]

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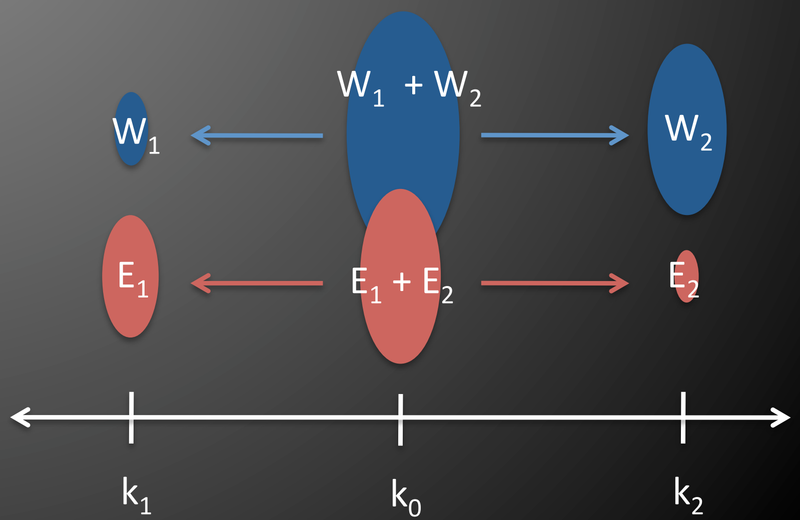

Phenomenology (Continued…)

Forward cascade, $k > k_i$ (k_i = injection scale)

$$\frac{g_\ell}{\tau_{NL}} \sim \ell^{-2}\varphi_\ell J_0(v/\ell) g_\ell \sim v^{-1/2}\ell^{-3/2}\varphi_\ell \cos\left(v/\ell - \pi/4\right) g_\ell$$

Constancy of free energy flux

$$\varepsilon_W = \int v dv \overline{\frac{g_\ell^2}{\tau_{NL}}} = \text{Const.}$$

Scaling relationship

$$\varphi_\ell \sim \ell g_{0\ell}$$

$$g_{\ell 0} \sim \ell^{1/6} \longrightarrow W_g(k) \sim k^{-4/3}$$

$$\varphi_\ell \sim \ell^{7/6} \longrightarrow E(k) \sim k^{-10/3}$$

"Collisional scale" ℓ_c : Balance collisions against NL term

$$\ell_c^{-3/2}\varphi_{\ell_c} \sim \nu \ell_c^{-2}$$

$$\varphi_\ell \sim \varphi_1 \ell^{7/6}$$

$$\ell_c \sim \left(\frac{\nu}{\phi_1}\right)^{3/5}$$

Phenomenology (Continued…)

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$$g_{\ell 0} \sim \ell^{1/6}$$

$$\varphi_\ell \sim \ell^{7/6}$$

Scaling relationship

Estimate: $\varphi_1 \sim \gamma \sim \sqrt{L_T/R}$

$$\frac{\ell_c}{\rho_i} \sim \left(\frac{\nu_{ii}\sqrt{L_T R}}{v_{th}}\right)^{3/5}$$

ITER: $5\text{x}10^{-3}$

JET: $8\text{x}10^{-4}$

DIIID: $2\text{x}10^{-2}$

MAST: $3\text{x}10^{-2}$

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Kraichnan-Batchelor theory for inverse cascade (forced stationary state):

$$\frac{\varphi_\ell}{\tau_{NL}} \sim \ell^{-2}\varphi_\ell \int v dv J_0^2(v/\ell) g_\ell(v)$$
$$\sim \ell^{-1/2}\varphi_\ell g_{\ell 0} \sim \ell^{-3/2}\varphi_\ell^2$$

$$\varepsilon_E = \frac{\varphi_\ell^2}{\tau_{NL}} = \text{Const.}$$

$\Rightarrow$

$$\varphi_\ell \sim \ell^{1/2}$$
$$g_{\ell 0} \sim \ell^{-1/2}$$

$\Rightarrow$

$$E_\varphi(k) \sim k^{-2}$$
$$W_g(k) \sim k^0 = \text{Const.}$$

2D Gyrokinetics related to Charney-Hasegawa-Mima (CHM) turbulence

Gyrokinetic system in CHM limit: $k^2 \sim \tau << 1$

$$\frac{\partial g}{\partial t} + \mathbf{v}_{E0} \cdot \boldsymbol{\nabla} g = C[g] + \mathcal{F} \qquad \mathbf{v}_{E0} = \hat{\mathbf{z}} \times \boldsymbol{\nabla}\varphi$$

$$2\pi \int v dv g = \lambda^2 \varphi - \nabla^2 \varphi$$

Inviscid CHM equation:

$$\partial_t (\lambda^2 - \nabla^2)\varphi + \mathbf{v}_{E0} \cdot \boldsymbol{\nabla}(-\nabla^2 \varphi) = \mathcal{F}_{\mathrm{CHM}}$$

Invariants:

$$E_{\mathrm{CHM}} = \frac{1}{2} \int \frac{d^2\mathbf{r}}{V} (\lambda^2 \varphi^2 + |\boldsymbol{\nabla}\varphi|^2)$$

$$Z_{\mathrm{CHM}} = \frac{1}{2} \int \frac{d^2\mathbf{r}}{V} (\lambda^2 |\boldsymbol{\nabla}\varphi|^2 + (\nabla^2 \varphi)^2)$$

Also:

$$\tilde{g} = g - F_0 (\lambda^2 - \nabla^2)\varphi$$

$$W_{\tilde{g}} = \frac{1}{2} \int \frac{d^2\mathbf{r}}{V} \int v dv \frac{\langle \tilde{g}^2 \rangle_{\mathbf{r}}}{F_0}$$

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Extra invariant!

$$W_{\tilde{g}} = \frac{1}{2} \int \frac{d^2\mathbf{r}}{V} \int v dv \frac{\langle \tilde{g}^2 \rangle_{\mathbf{r}}}{F_0}$$

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$$2\pi \int v dv g = \lambda^2 \varphi - \nabla^2 \varphi$$

Relationship Between CHM and 2D Gyrokinetics

$$W_g = W_{\tilde{g}} + \lambda^2 E_{\rm CHM} + Z_{\rm CHM}$$

$$E \approx E_{\rm CHM}$$

$$= \mathcal{F}_{\rm CHM}$$

Invariants:

$$E_{\rm CHM} = \frac{1}{2} \int \frac{d^2\mathbf{r}}{V} (\lambda^2 \varphi^2 + |\boldsymbol{\nabla}\varphi|^2$$

$$Z_{\rm CHM} = \frac{1}{2} \int \frac{d^2\mathbf{r}}{V} (\lambda^2 |\boldsymbol{\nabla}\varphi|^2 + (\nabla^2\varphi)^2)$$

Also:

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Extra invariant!

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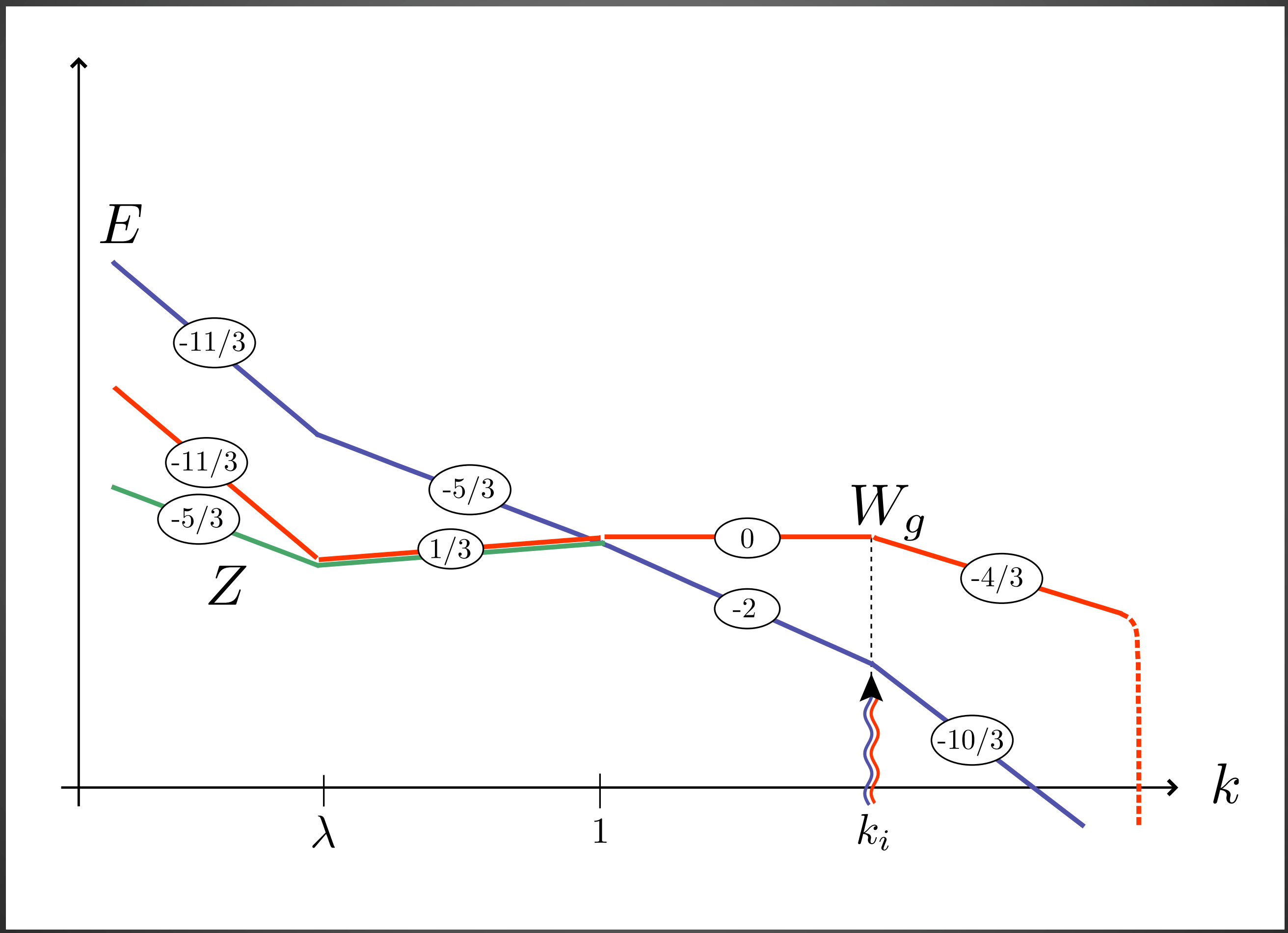
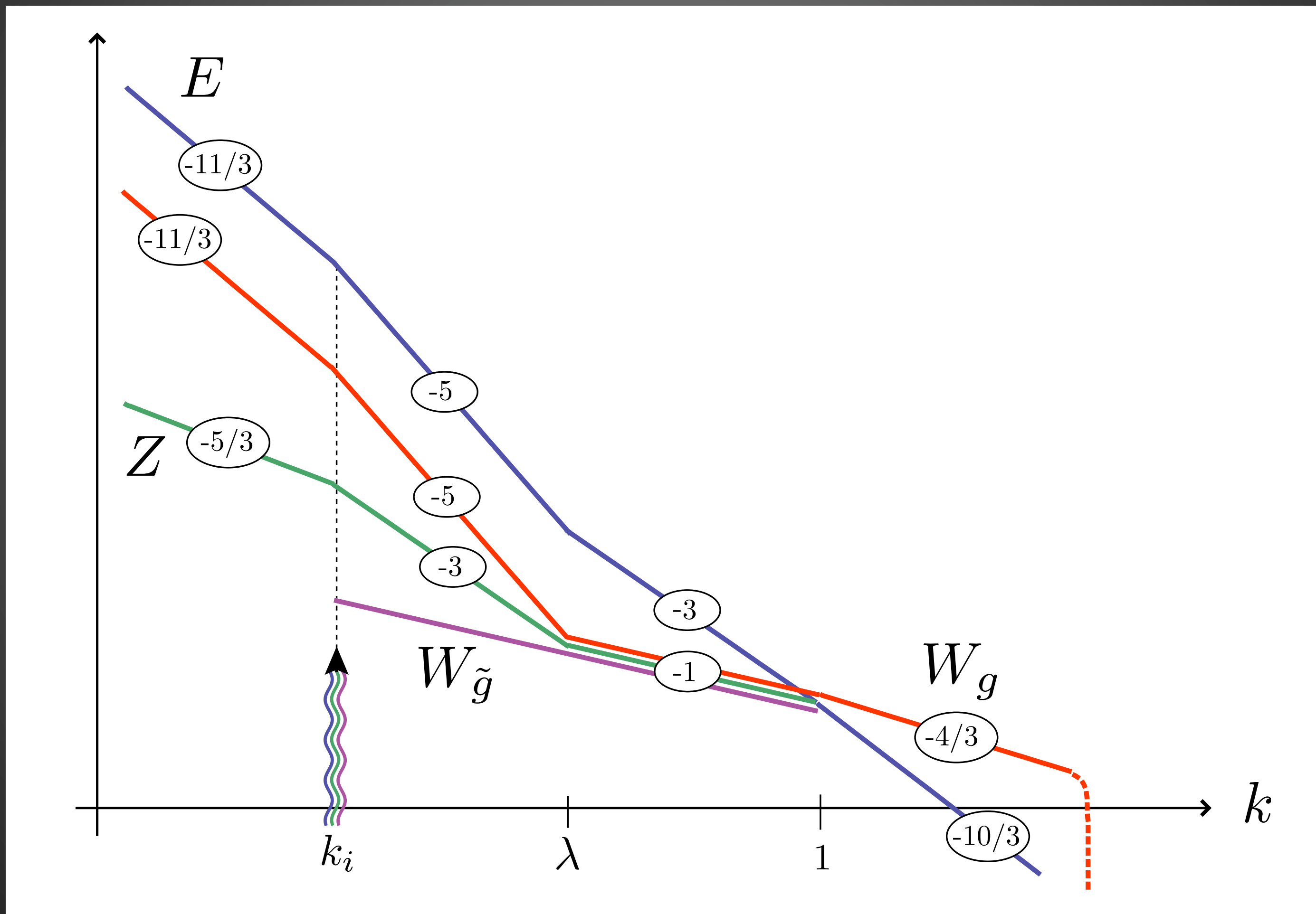
Cascade from small scales, k >> 1



Cascade from large scales, k << λ



Defining the phase-space spectrum

Problem: Define wavenumber spectrum for both position and velocity space

$$W_g = \int dk dp\, W_g(k,p)$$

Solution: Hankel-Fourier transform

$$\hat{g}(\mathbf{k},p) \equiv \frac{1}{2\pi}\int_{\mathbb{R}} d^2\mathbf{R} \int_0^\infty v dv\, J_0(pv) \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{R}} g(\mathbf{R},v)$$

Thus the spectra are defined:

$$W_g(k,p) = \pi\ pk\ \overline{|\hat{g}(k,p)|^2}$$

$$E(k) = \beta(k)^2 k^{-1} W_g(k,k)$$

Core Theoretical Arguments

homogeneity, isotropy and stationarity

third order result

$$\overline{\delta \mathbf{v}_E \delta g^2} = \varepsilon \ell + \mathrm{C}$$

self-similarity + scaling symmetry

dual self-similarity hypothesis

$$\delta g(\lambda \ell, \lambda \ell_v) \doteq \lambda^{h_g} \delta g(\ell, \ell_v)$$

Phase-space third order expression

$$\overline{\delta \mathbf{v}_E \delta g^2} = \varepsilon \ell + \varepsilon' \ell_v$$

NL phase-mixing range

$$\ell, \ell_v \ll 1$$

Spectral Power Law Predictions

(Nonlinear phase-mixing range, k >> 1)

$$W_g(\lambda k, \lambda p) = \lambda^{-7/3} W_g(k,p)$$

$$W_g(k,k) \propto k^{-7/3} \qquad W_g(k) \propto k^{-4/3}$$

$$E(k) \propto k^{-10/3} \qquad W_g(p) \propto p^{-4/3}$$

$$W_g(k,p) \propto \begin{cases} k^{-2} p^{-1/3}, & \text{for} \quad k \gg p \\ p^{-2} k^{-1/3}, & \text{for} \quad k \ll p \end{cases}$$

Final note: Freely decaying (unforced) inverse cascade

- $W(k) \sim k^2 E(k)$ determined by phase-mixing process – unlike fluid turbulence
- Steady-state relationship of invariants possibly set by initial condition of phase-space spectrum
- Preservation of initial spectrum [P.D. Ditlevsen*, et al,* Physica A 342 (2004)]

AstroGK

Developed and maintained by M. Barnes, W. Dorland, G. Howes, R. Numata and T. Tatsuno based on GS2.

- Fourier spectral in x-y (perp. to field line)
- 2nd order centered FD in z (along field line)
- Legendre + Laguerre spectral integral in velocity space
- 2nd order implicit trapezoidal scheme (linear convection)
- 3rd order Adams-Bashforth scheme (nonlinear term)
- implicit Euler scheme
  (p-a scatt. + energy diff. w/ mom. cons. collision)

Open source code: http://www.physics.uiowa.edu/~ghowes/astrogk/

Collision operator

Requirements:

- smoothes small structures
- conserves particle number, momentum, and energy
- satisfies Boltzmann's H-theorem

$$C = L + E + U_L + U_E + V_E$$

$$L = \frac{\nu_D}{2}\frac{\partial}{\partial\xi}\left[(1-\xi^2)\frac{\partial}{\partial\xi}\right]$$

$$E = \frac{1}{2v^2}\frac{\partial}{\partial v}\left(\nu_{\|} v^4 F_0 \frac{\partial}{\partial v}\frac{1}{F_0}\right)$$

U_L ,U_E : Momentum restoring terms,V_E : Energy restoring term

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Abel *et al.*, Phys. Plasmas **15**, 122509 (2008); Barnes *et al.*, arXiv:0809.3945.

Dissipation scale

Collisional dissipation

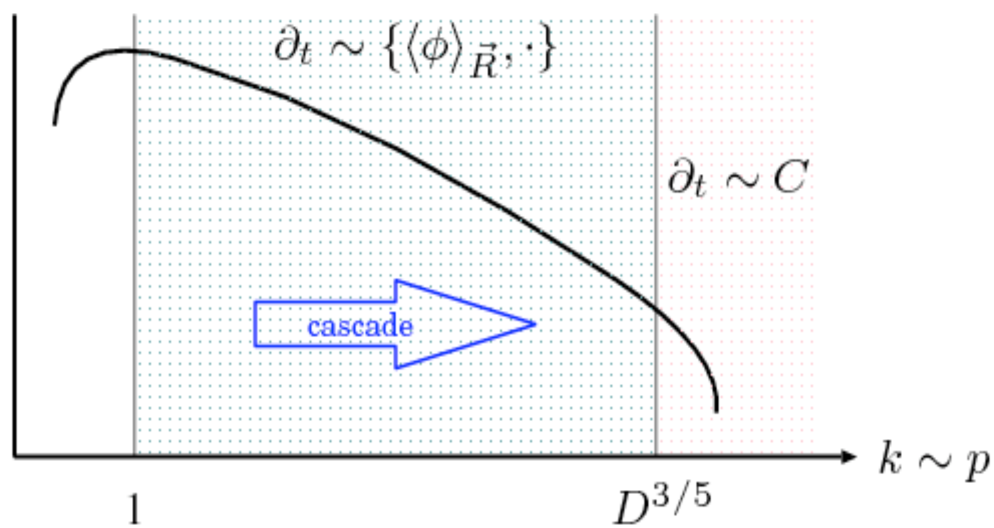
Correlation btw. position & velocity

$$\frac{\nu}{\ell_v^2} \sim \frac{1}{\tau_\ell}$$

$$\ell_v \sim \ell$$

Collisional cutoff

$$\ell_{v,\mathbf{c}} \sim \ell_{\mathbf{c}} \sim D^{-3/5}$$



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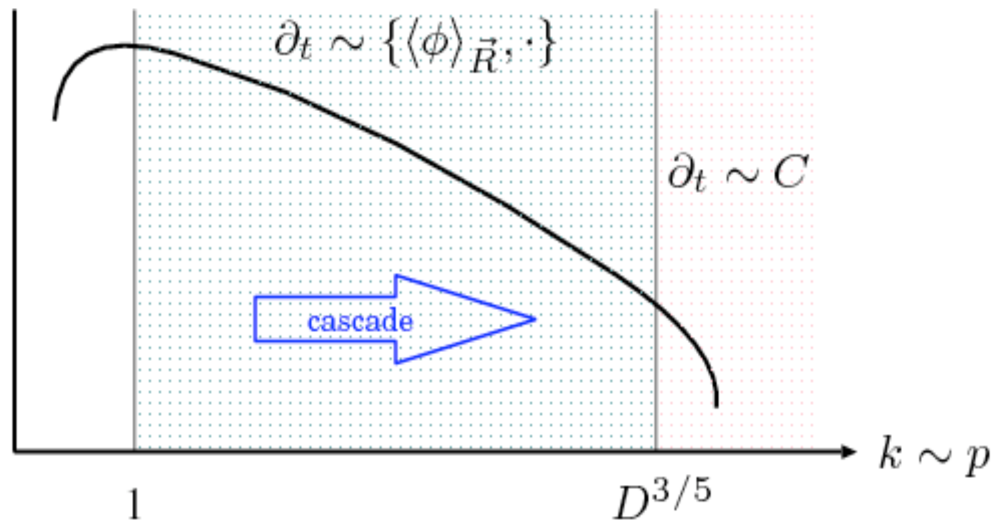
Dorland number

$$D := \frac{1}{\nu \tau_\rho}$$



Dorland & Hammett, Phys. Fluids B **5**, 812 (1993).

Direct cascade

Setup

- Straight homogeneous slab: $L_x = L_y = 2\pi$
- Initial condition (decaying turbulence)

$$g_{\text{init}} = g_0 \left[\cos 2x + \cos 2y + \text{noise}\right] F_0$$

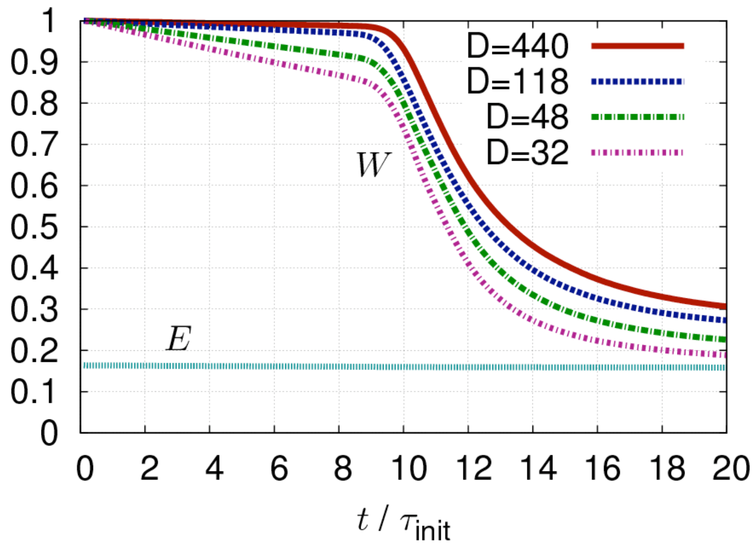
Run table

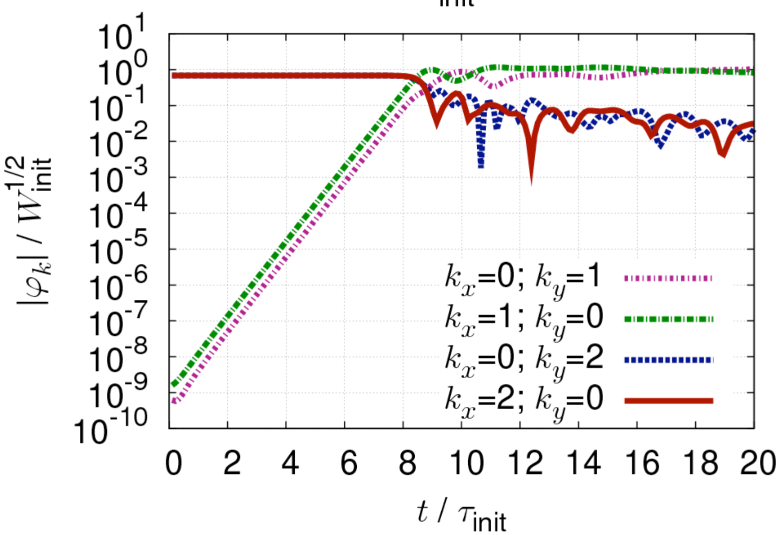
case	$\nu_{ii}\,\tau_{\text{init}}$	D	k_c	N_x X N_y	N_E X N_ξ
(i)	5.2×10^{-3}	32	16	64^2	32^2
(ii)	3.0×10^{-3}	48	21	64^2	32^2
(iii)	1.0×10^{-3}	118	42	128^2	64^2
(iv)	4.2×10^{-4}	440	72	256^2	128^2

Direct cascade

Time evolution

Results [E and ϕ_k from case (iv): D = 440]





Collisionless invariants

- decrease of W
  ⇒ creation of entropy
- initial decrease in W ∝ ν
- turbulent decrease $t/\tau_{init} \geq 10$
- constancy of E
  ⇒ inverse cascade

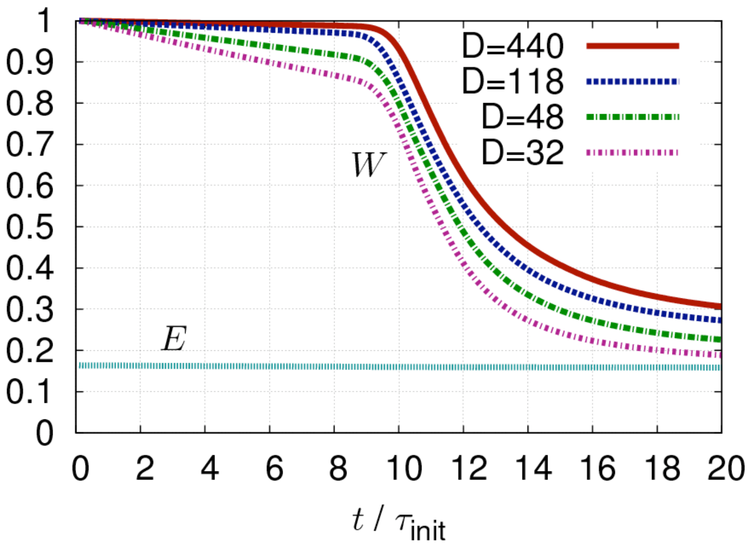
Evolution of ϕ_k

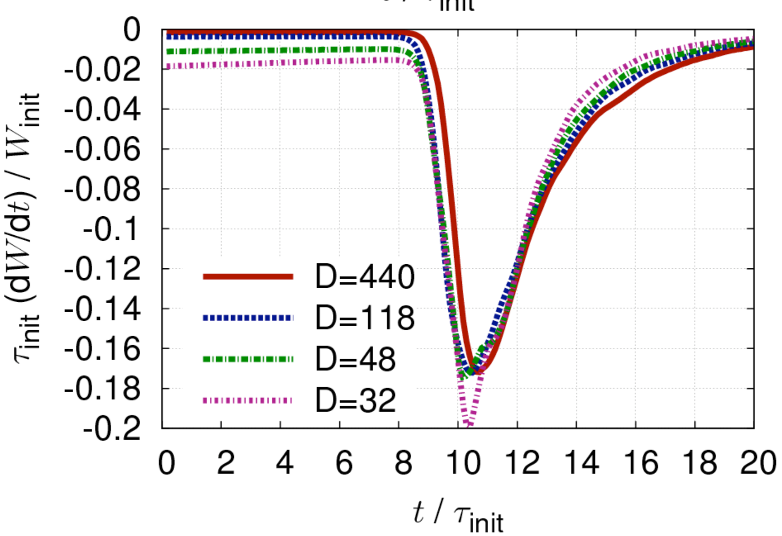
- instability & saturation

Direct cascade

Time evolution

Results [E from case (iv): D = 440]





Collisionless invariants

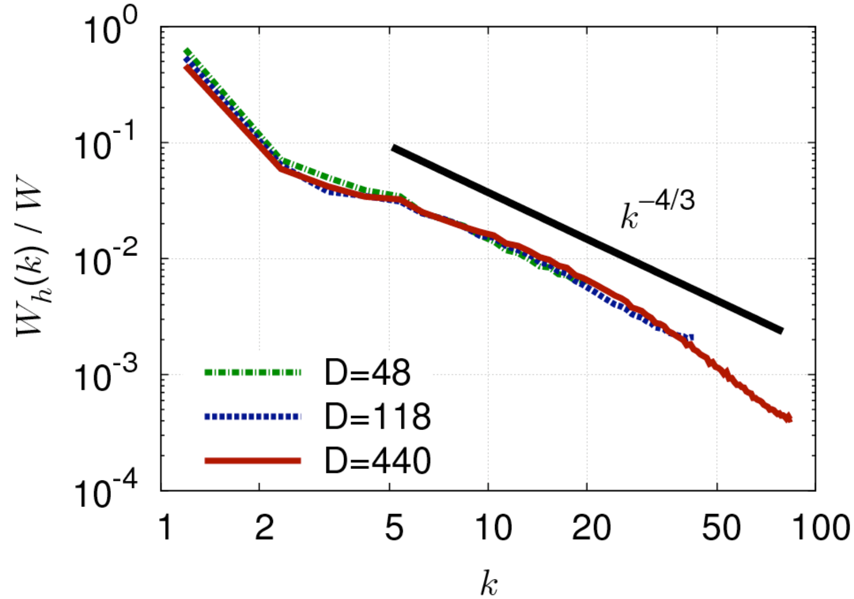
- decrease of W
  ⇒ creation of entropy
- initial decrease in W ∝ ν
- turbulent decrease $t/\tau_{init} \geq 10$
- constancy of E
  ⇒ inverse cascade
- $dW/dt \not\to 0$ as $D \to \infty$
  ⇒ finite dissipation indep. ν

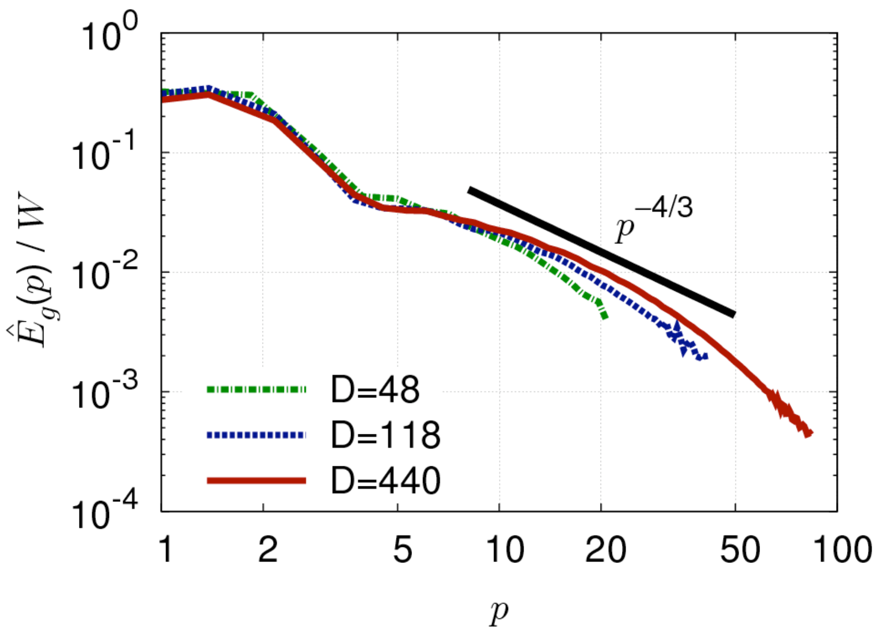
Direct cascade

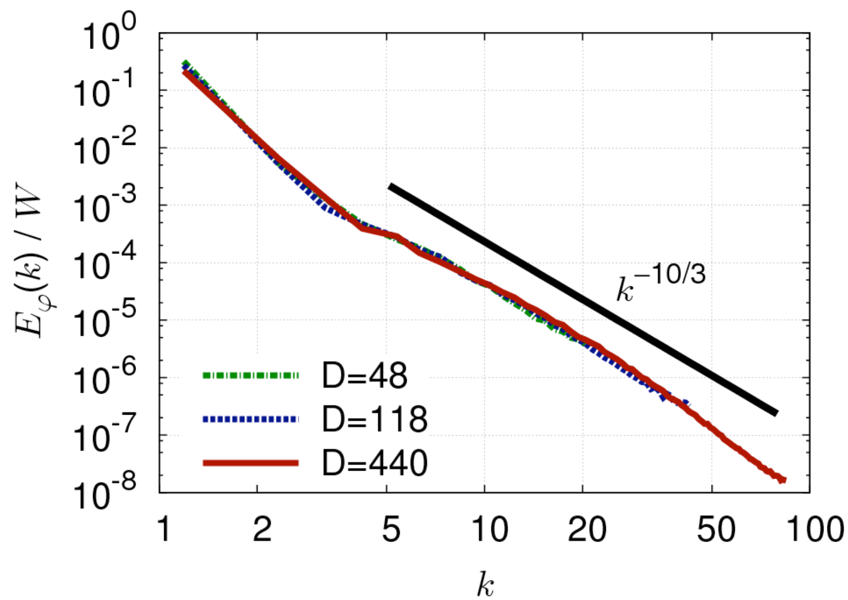
Averaged spectra

Largest resolution run:

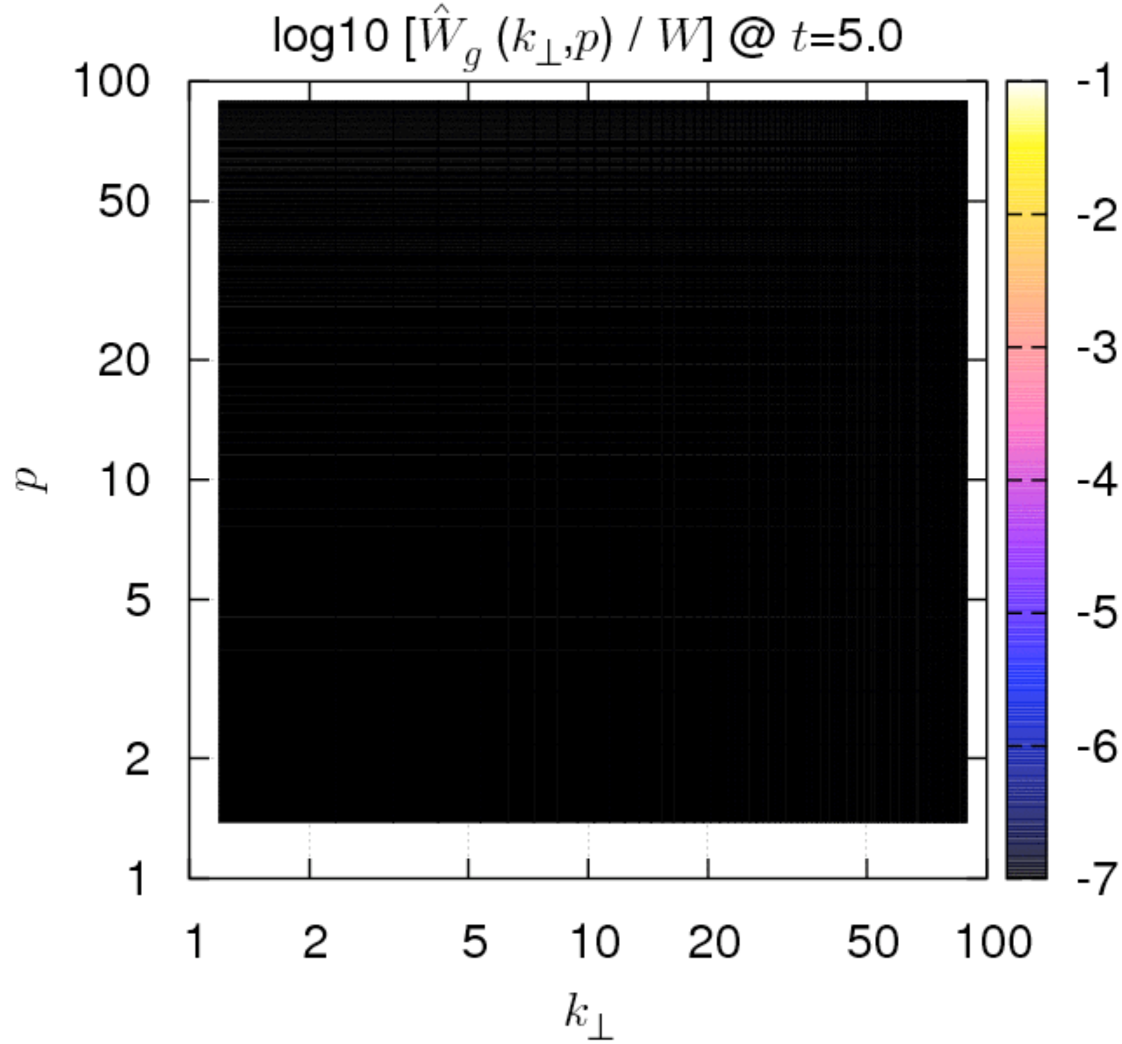
- D = 440
- N_x X N_y = 256^2, N_E X N_ξ= 128^2
- 36 hrs on 8,192 processor cores





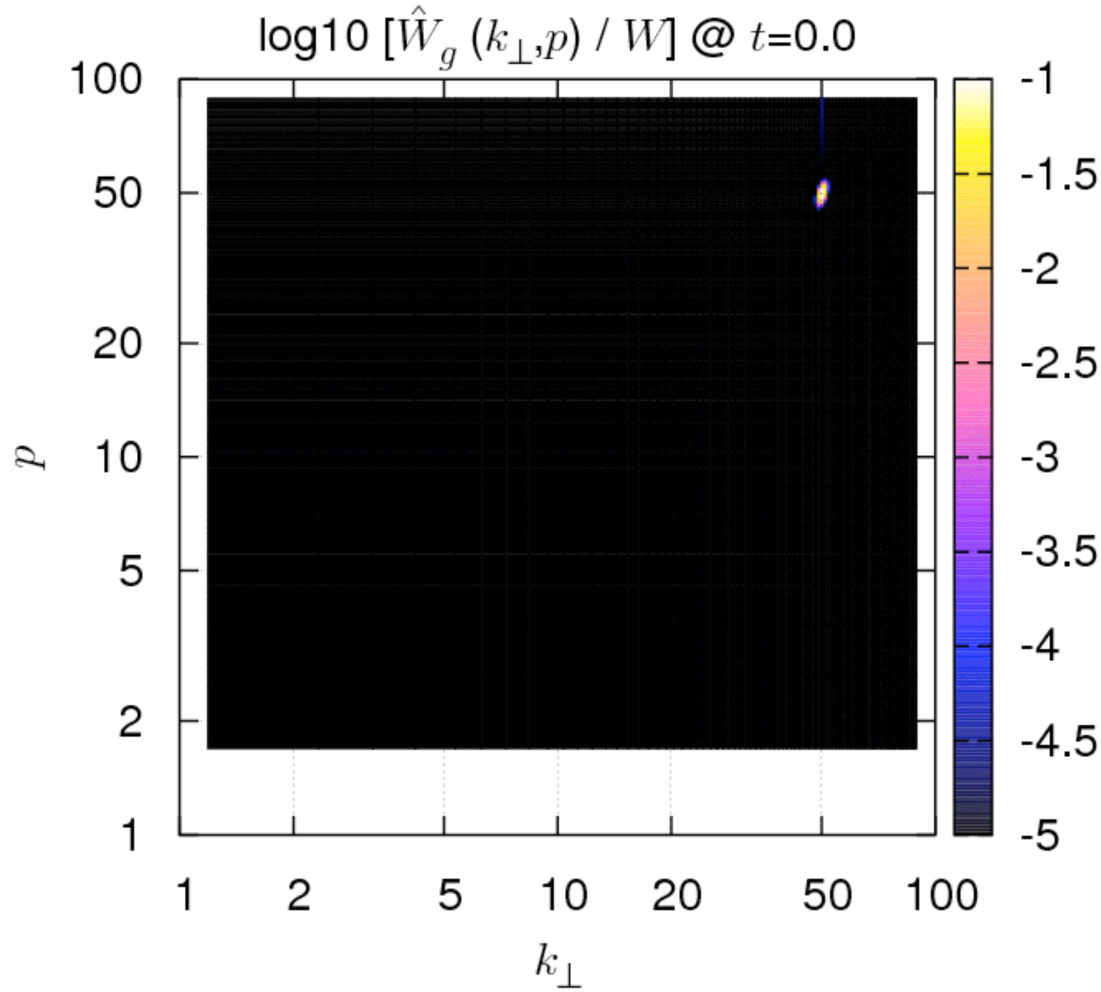


Free Energy Phase-Space Spectrum Evolution



Preview: Inverse Cascade Demonstrated

(Freely decaying)



Inverse cascade

Magical normalization:

$$\hat{E}_h(\hat{k}_\perp) = \frac{E_h(k_\perp, t)}{Wl}, \quad \hat{E}_\varphi(\hat{k}_\perp) = \frac{E_\varphi(k_\perp, t)}{El}, \quad \hat{E}_h(\hat{p}) = \frac{E_h(p, t)}{Wl_v},$$

where $l = l_v = E/W$.