The Influence of Magnetic Islands on Ion Temperature Gradient Mode Stability

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Outline

- Introduction:
  - Motivation
  - magnetic islands in slab geometry

- Theory:
  - Magnetic island equilibrium
  - Modified eigenmode equation for ITG mode in presence of island
  - numerical results
  - WKB theory of mode structure

- Summary

Motivation: spans several areas

- Theories predict that small scale (~ρ_i) magnetic islands may exist in tokamaks
  - if present, there are a number of consequences for transport barrier formation, NTM formation and core confinement

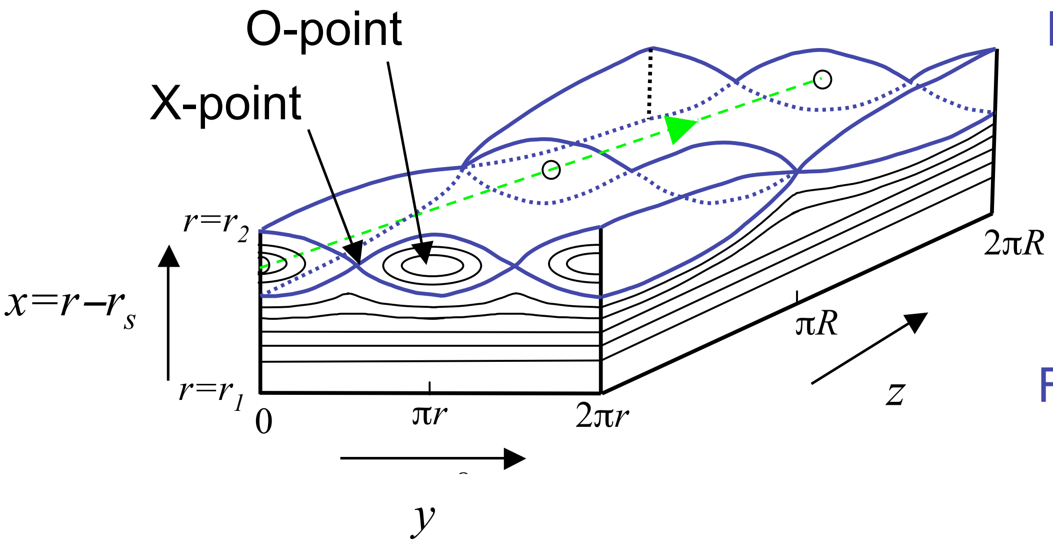
- NTM threshold
  - We know that cross-field (turbulent) transport provides a threshold to NTMs (healing small scale "seed islands")
  - But how does the island itself affect the turbulence?

- Transport barrier formation/core confinement
  - what if magnetic flux surfaces are not "good" (ie not nested)?
  - small islands have an associated flow shear: what is influence of this on electrostatic (eg ITG) turbulence?

- RMP ELM suppression
  - RMPs likely create small islands
  - How do these influence the pedestal confinement?

Geometry: magnetic islands in a sheared slab

- We adopt a "sheared slab" geometry with a magnetic island



Magnetic field

$$\boldsymbol{B} = B_0 \nabla z - \nabla \psi \times \nabla z$$

$$\psi = -\frac{B_0 x^2}{2L_s} + \tilde{\psi} \cos K_y y$$

Flux quantity

$$\chi = -\frac{\psi}{\tilde{\psi}} = \frac{2x^2}{w^2} - \cos K_y y$$

- Consider long, thin islands: weak variation of equilibrium in y-direction

[redacted]

- Work in island rest frame (ie, a "slab-equilibrium" radial electric field exists)

Electrostatic potential has 3 pieces:

$$\frac{e\Phi}{T_e} = \frac{e\Phi_0}{T_e} + \overline{\Phi} + \varphi\, e^{-i\omega t}$$

Slab-equilibrium

Island-equilibrium

fluctuation (linearised)

$$\Phi_0 = -E_r x$$

$$\lim_{x\to\infty} \overline{\Phi} = 0$$

$$\lim_{x\to\infty} \tilde{\varphi} = 0$$

Methodology (1)

- Our starting point is an equilibrium of the standard sheared slab, where distribution functions are Maxwellian with density and temperature functions of x alone, and a constant "radial" electric field, E_r

- We treat both the island and the fluctuations as perturbations to this state

- Thus the distribution functions are expressed as:

$$f_j = (1 + \overline{\Phi} + \tilde{\varphi} e^{-i\omega t}) F_{0j} + \overline{G}_j + g_j e^{-i\omega t} \qquad F_{0j} = \frac{n_{oj}(x)}{\pi^{3/2} v_{th,j}^3(x)} \exp\left[-\frac{v^2}{v_{th,j}^2} \right]$$

- The non-adiabatic part of the distribution function is split into two pieces:
  - a time independent response to the island perturbation: [illegible]
    - retain (almost) full non-linear dynamics associated with this piece
    - provides the new, self-consistent island equilibrium with self-consistent flows
  - a fluctuating response associated with ITG mode: [illegible]
    - linearise with respect to these so time-dependence ~$e^{-i\omega t}$

- Use gyro-kinetic theory to derive particle responses

Methodology (2)

- For fluctuating pieces, we Fourier transform perturbations ~

- We can, and have, developed the theory for arbitrary $k\rho_i$, but here we consider a simpler model for illustration

  – order $k_x\rho_i \sim k_y\rho_i \ll 1$; $K_y w \ll 1$, $\rho_i/w \ll 1$

- We can define diamagnetic frequencies both for fluctuations and the island:

$$\omega_{*e} = -\frac{k_y \rho_s c_s}{L_n} \qquad \overline{\omega}_{*e} = -\frac{K_y \rho_s c_s}{L_n} \qquad \omega_{*i} = -\frac{1}{\tau}\omega_{*e} \qquad \tau = \frac{T_e}{T_i}$$

- The equilibrium radial electric field can be used to define a drift frequency (normalised to the island diamagnetic drift frequency)

$$\omega_E = -\frac{eL_n}{T_e}E_r$$

A measure of the island rotation frequency in ExB rest frame (ie an O(1) parameter)

We expect $\omega_E \sim 1$: an input parameter here

Electron response

- For electrons, assume parallel transport dominates: $k_{||}v_{the}$>>ω, ω_E
- We then find (neglecting FLR) $$\frac{\delta n_e}{n_e} = \overline{\Phi} + \varphi\, e^{-i\omega t} + \frac{w}{L_n}(\omega_E - 1)\left(\frac{x}{w} - h(\chi)\right)$$
- Second part describes flattening of electron distribution function across island
  - $h(\chi)$ depends on transport processes, not included
  - we adopt a simple model that satisfies boundary conditions:

$$h(\chi) = \frac{\sqrt{\chi} - 1}{\sqrt{2}}\Theta(\chi - 1)$$

Recall: $$\chi = \frac{2x^2}{w^2} - \cos K_y y$$

- We now proceed to consider the ions
- Quasi-neutrality will then determine *both*
  - $\overline{\Phi}$ and hence the self-consistent ExB flow around the island
  - the fluctuating potential φ, and the complex mode frequency ω as the solution of an eigenmode equation

Ion response

- For ions, we take gyro-kinetic equation to derive δn_i
- We illustrate main physics assuming

$$\omega, \omega_E >> k_{\|} v_{th,i} \qquad k_x \rho_i \sim k_y \rho_i << 1$$

- We then find the following equation for the non-adiabatic ion response:

$$-i\omega \tilde{G}_i e^{-i\omega t} - \frac{K_y E_r}{B} \left.\frac{\partial \tilde{G}_i}{\partial \xi}\right|_x e^{-i\omega t} - \frac{K_y E_r}{B} \left.\frac{\partial \overline{G}_i}{\partial \xi}\right|_x + \rho_s c_s \left(\boldsymbol{b} \times \nabla \langle \overline{\Phi} \rangle_\alpha \right) \nabla \overline{G}_i$$

$$+ \rho_s c_s \left(\boldsymbol{b} \times \nabla \langle \overline{\Phi} \rangle_\alpha \right) \nabla \tilde{G}_i e^{-i\omega t} + \rho_s c_s \left(\boldsymbol{b} \times \nabla \langle \tilde{\Phi} \rangle_\alpha \right) \nabla \overline{G}_i e^{-i\omega t}$$

$$= -i\omega \langle \tilde{\Phi} \rangle_\alpha F_{0i} e^{-i\omega t} - \left(\omega_{*i}^T + \tau \omega_E \right) F_{0i} \left[\left.\frac{\partial \langle \overline{\Phi} \rangle_\alpha}{\partial \xi}\right|_x + \left.\frac{\partial \langle \tilde{\Phi} \rangle_\alpha}{\partial \xi}\right|_x e^{-i\omega t} \right]$$

$$\xi = K_y y$$

Fluctuating terms (linearised)

Time independent (non-linear)

Ion response: ExB flow around the island

- Equating the time-independent terms

$$\overline{G}_i = \left(\tau + \frac{\overline{\omega}_{*i}^T}{\overline{\omega}_{*i}\omega_E} \right) \langle \overline{\Phi} \rangle_\alpha F_{0i} + K\left(\frac{e\Phi_0}{T_e} + \langle \overline{\Phi} \rangle_\alpha \right)$$

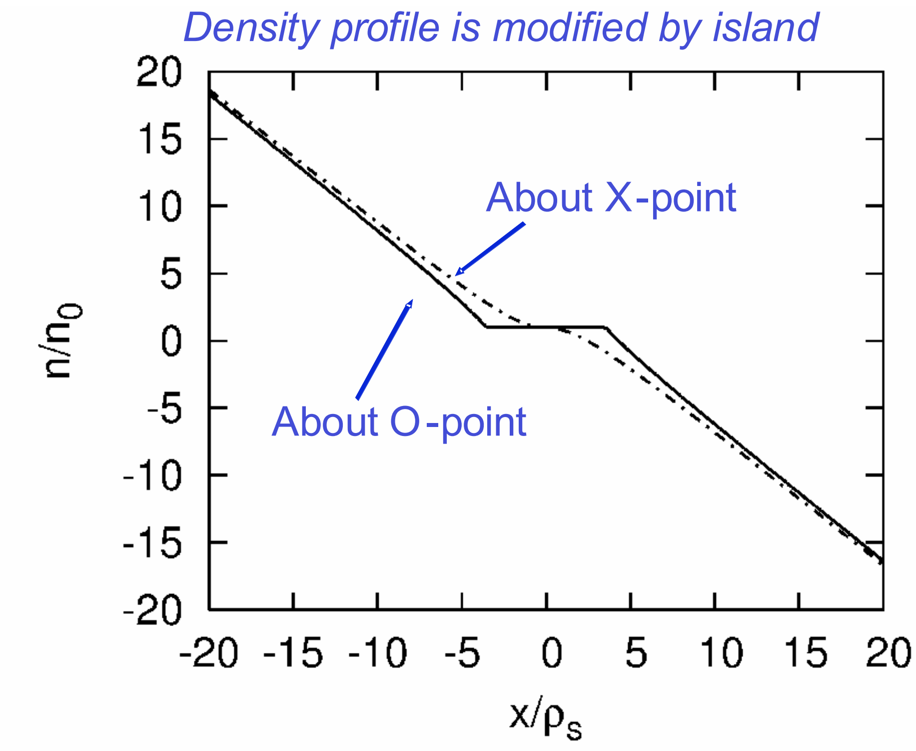
- The function K is arbitrary: determined by a model for momentum transport, for example
  – we choose the particular solution K=0
  – Interestingly, this linear solution is a particular solution of the non-linear equation

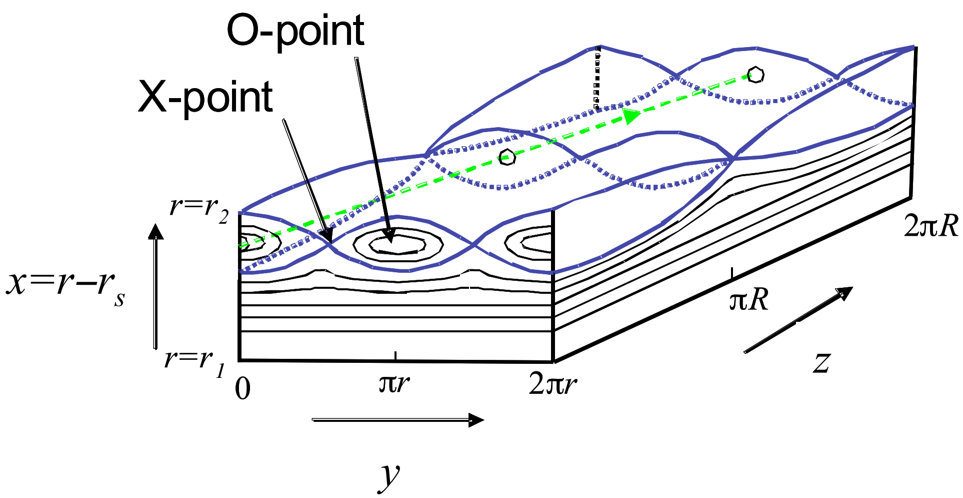
- Neglecting FLR effects, we equate the resulting ion density perturbation with the time-independent electron density response to derive the potential:

$$\overline{\Phi} = -\frac{\omega_E w}{L_n}\left(\frac{x}{w} - h(\chi) \right)$$

- Note that this form for the potential implies a strongly sheared flow around the island through the function $h(\chi)$
  – flow varies on a length scale ~w

The Modified Density Profile





- For the fluctuating response (ie the ITG piece), we neglect non-linearities in fluctuating quantities, but retain non-linearities with the time-independent perturbations
  – the impact of the sheared flow on the ITG mode is included through the non-linear ExB convective derivative

Ion response: time-dependent perturbations

- After much algebra:
  - perturbatively treat FLR and the parallel dynamics for the time-dependent ion response
  - impose quasi-neutrality to derive:

$$\rho_s^2 \frac{\partial^2 \tilde{\varphi}}{\partial x^2} + \left[\frac{L_n^2 \omega_{*e}^2}{L_s^2 \rho_s^2 (\Omega - \alpha_d S)^2} x^2 - \left[\frac{(\Omega - \alpha_d S) - (\omega_{*e} - \alpha_n S / \omega_E)}{(\Omega - \alpha_d S) + \tau^{-1}(1 + \eta_i)(\omega_{*e} - \alpha_n S / \omega_E)} + b \right] \right] \tilde{\varphi} = 0$$

FLR effects

Impact of shear

ITG drive

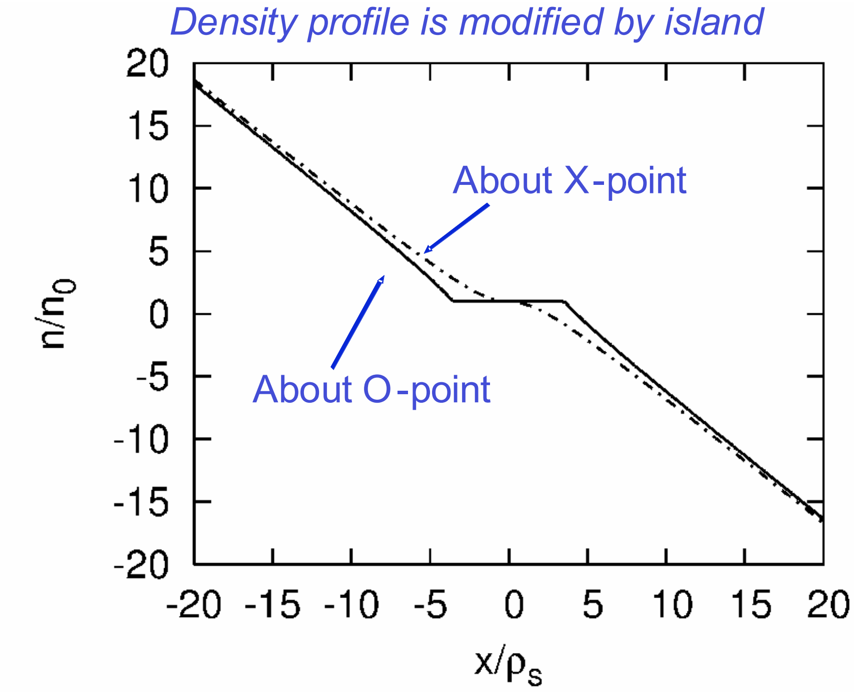
Shear flow: $S(x, y) = \omega_{*e} \omega_E \left(1 - w \frac{\partial h}{\partial x} \right)$

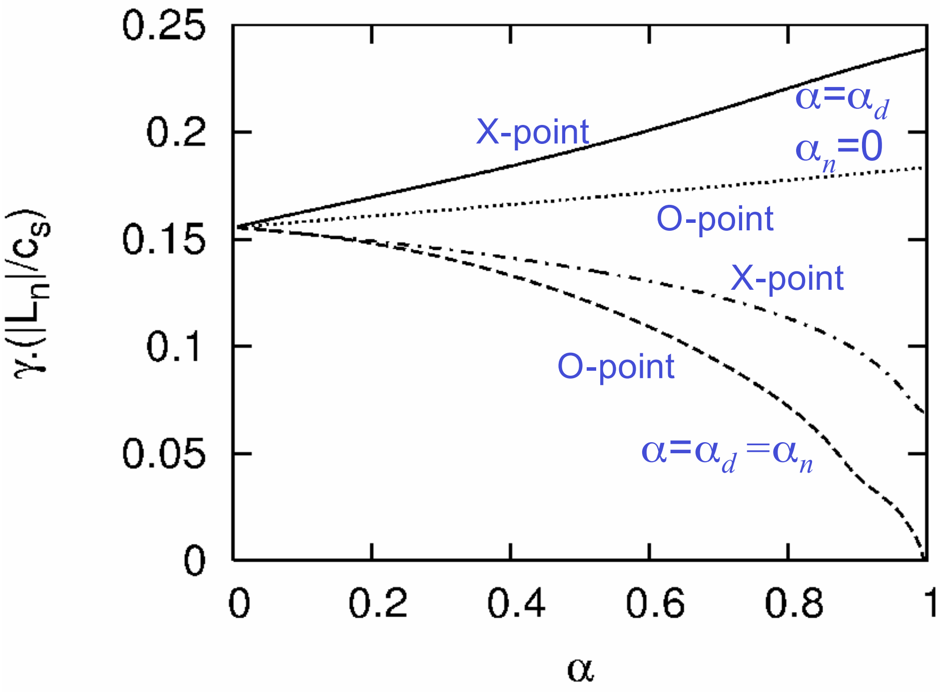
Doppler-shifted freq: $\Omega = \omega + \omega_E \omega_{*e}$

- Terms in α_d represent flow shear terms: a Doppler shift of mode frequency
- Terms in α_n represent density profile modification: modifies η_i and ω_{*e}
  - $\alpha_n = \alpha_d = 1$ for the case with the island
  - $\alpha_n = \alpha_d = 0$ for the case with no island

Local Stability Analysis (1)

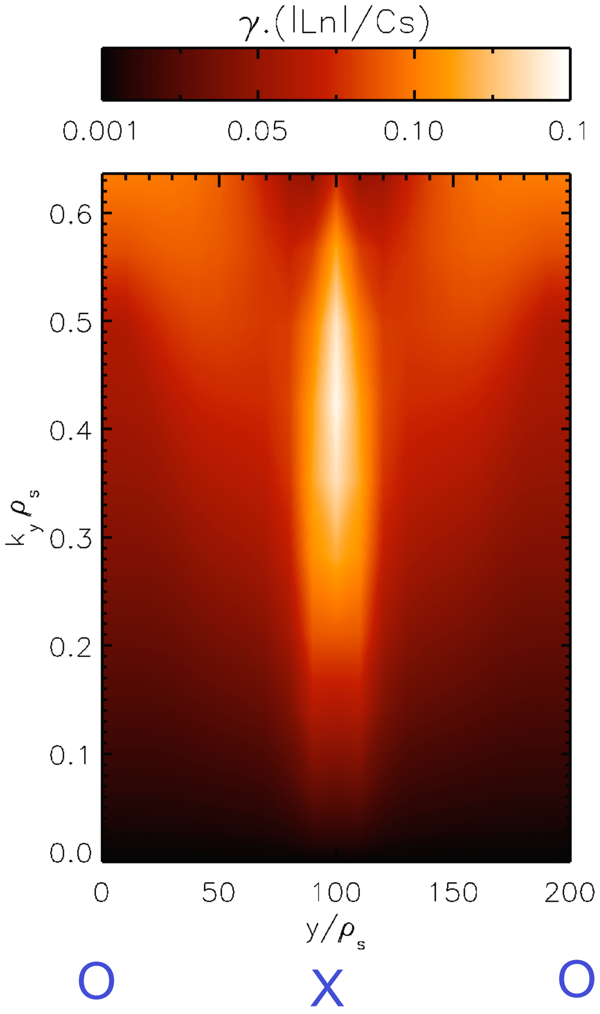
- Parameter set: η_i=10, τ=2, ρ_s/w=0.2, L_s/L_n=–15, ω_E=–0.5, b=0.12



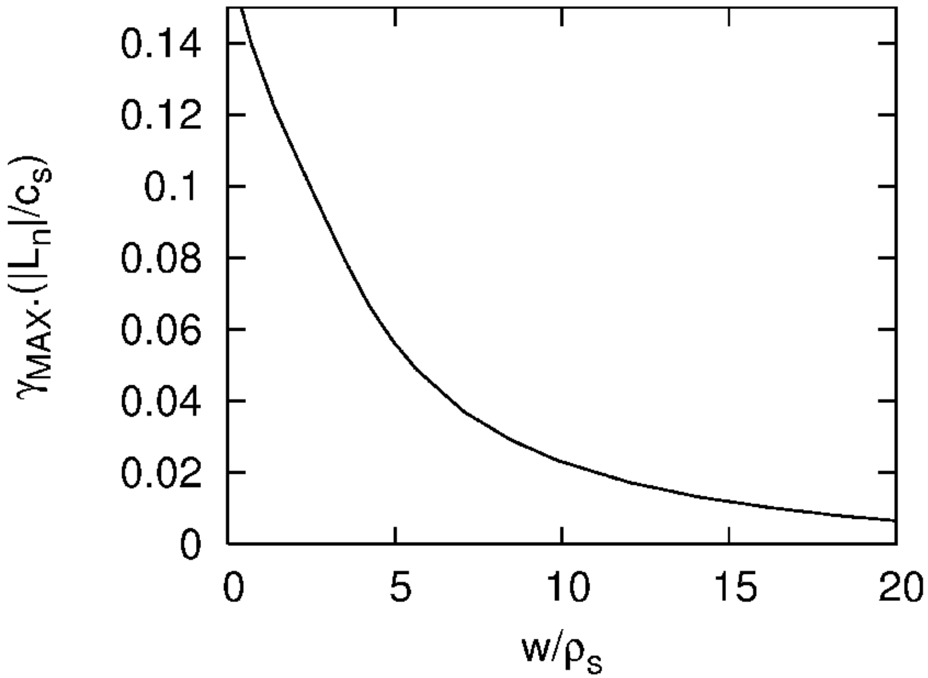


- Density profile flattened across island
- Doppler shift effect slightly destabilising
- Pressure profile effects substantially stabilise
    - ITG mode is most unstable in vicinity of X-point

Local Stability Analysis (2)



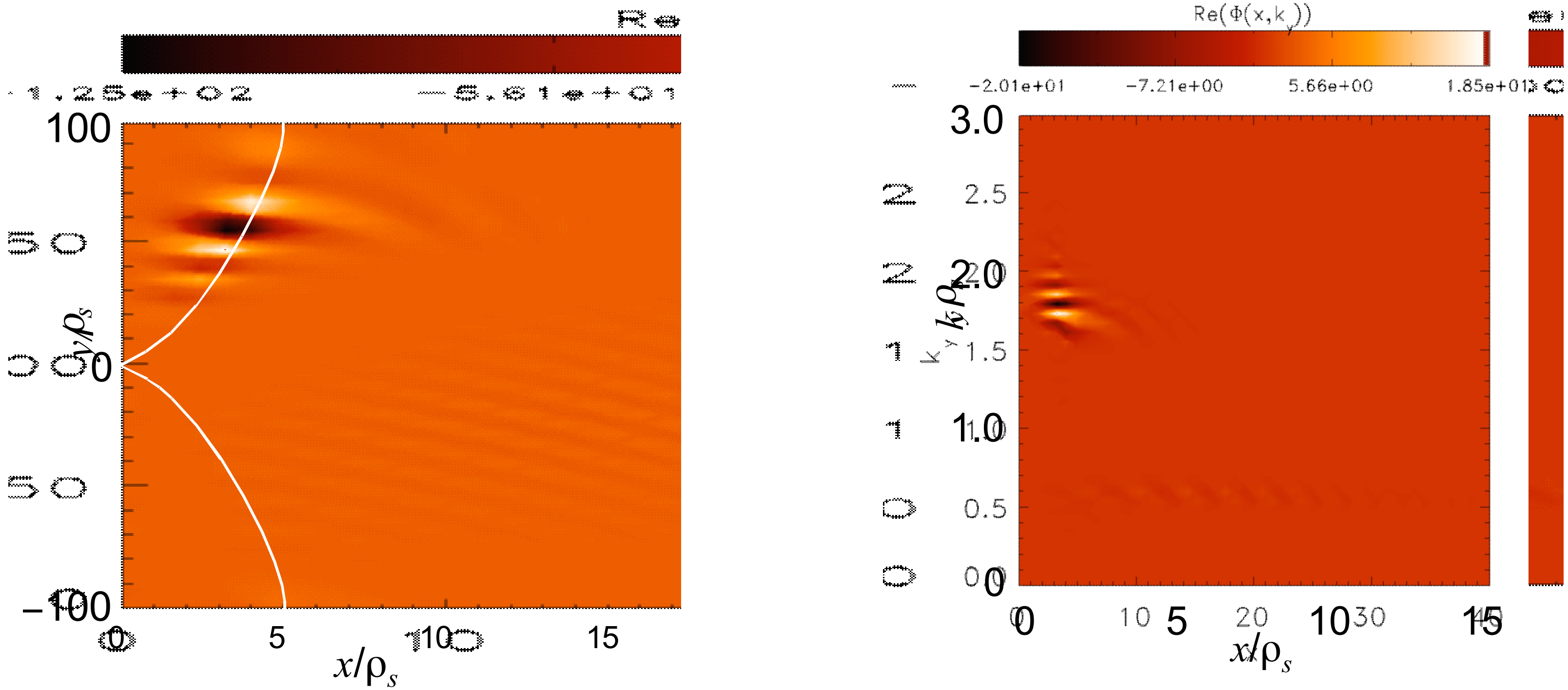
Local growth rate has a maximum at $k_y\rho_s$=0.42



Taking the k_y that maximises growth, larger islands have bigger influence on stability

2-D mode structure

- Transforming back to real space: $ik_y \rightarrow \partial/\partial y$ to give 2-D eigenmode equation
- Numerical solution yields localised, unstable mode



$\Rightarrow$ More unstable than prediction of local theory
$\Rightarrow$ Mode is not localised about position of maximum instability

A simple model

- One can construct a simple model
  - Set $\alpha_n=\alpha_d=0$ to return to standard slab
  - Instead introduce a sinusoidal y-variation to η_i
  - Model equation is then analytically tractable:

$$\rho_s^2 \frac{\partial^2 \tilde{\varphi}}{\partial x^2} + \left[\frac{\sigma^2 k^2}{\rho_s^2 \Omega^2} x^2 + \left(\frac{\Omega - k}{\Omega + \eta} \right) + k^2 \right] \tilde{\varphi} = 0$$

- We have introduced $\eta=(1+\eta_i)/\tau=\eta_0(1+\varepsilon\cos K_y y)$; $k=k_y\rho_s$

- Solve analytically (cf WKB), to derive eigenvalue equation

$$\left[-\eta \rho_*^3 k^3 + \rho_*^2 \left(\Omega \mp \frac{i\sigma\eta}{\Omega} \right) k^2 + \rho_* (1 \mp i\sigma) k^2 + \Omega \right] \tilde{\varphi} = 0$$

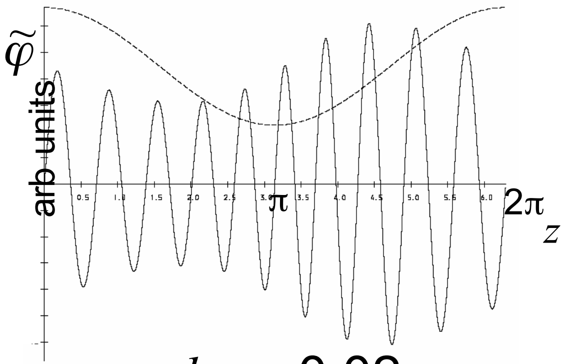
where $\rho_*=K\rho_s$

- Returning to real space, and defining $z=K_y y$, we have

$$\left[i\eta \rho_*^3 \frac{\partial^3}{\partial z^3} - \rho_*^2 \left(\Omega \mp \frac{i\sigma\eta}{\Omega} \right) \frac{\partial^2}{\partial z^2} + i\rho_* (1 \mp i\sigma) \frac{\partial}{\partial z} + \Omega \right] \tilde{\varphi} = 0$$

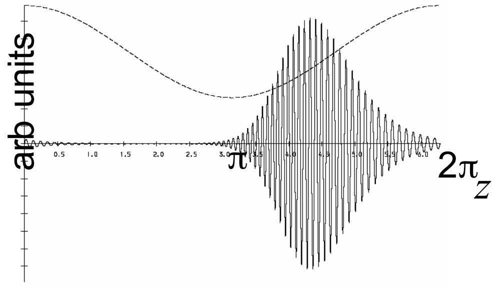
Localised solns exist: not localised at pos^n of max drive

- Numerical solution for ρ_*=0.002



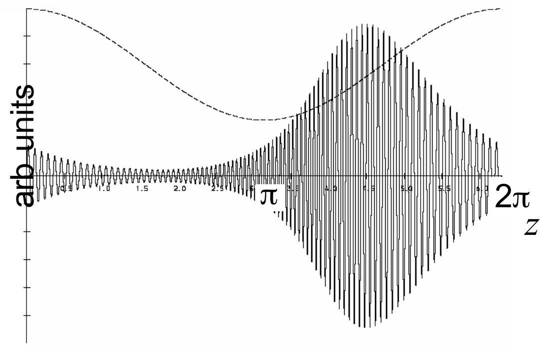
$k_y\rho_s$=0.02

Essentially slab-like



$k_y\rho_s$=0.14

Localised mode

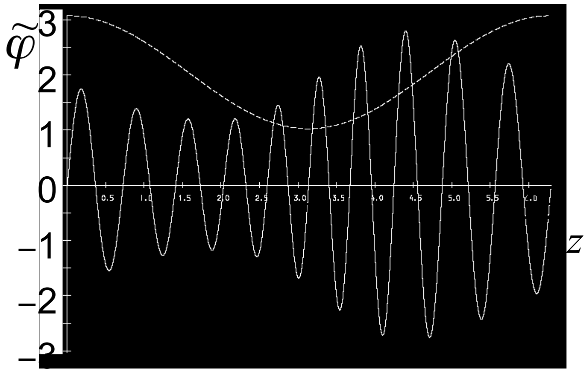


$k_y\rho_s$=0.18

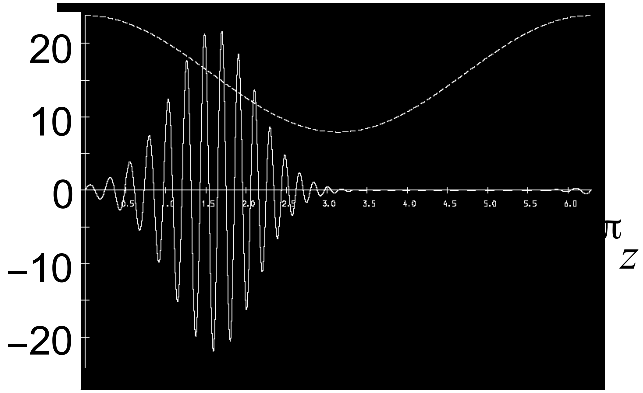
Mode starts to leak out

More slab-like?

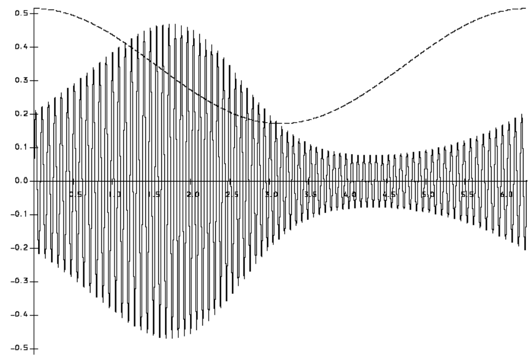
- Numerical solution for ρ_*=0.01



$k_y\rho_s$=0.1



$k_y\rho_s$=0.3



$k_y\rho_s$=0.8

A "local" expansion…not accurate, but interesting

- We seek solutions of the form $\varphi=\exp(-i\int k\,dz/\rho_*)f(z)$ by expanding for small ρ_*:
  - $f=f_0+\rho_* f_1+\ldots \qquad \Omega=\Omega_0(k_y,z)+\rho_*^2\delta\Omega+\ldots$

$$\left[i\eta\left(\rho_*\frac{\partial}{\partial z}-ik\right)^3-\left(\Omega\mp\frac{i\sigma\eta}{\Omega}\right)\left(\rho_*\frac{\partial}{\partial z}-ik\right)^2+i(1\mp i\sigma)\left(\rho_*\frac{\partial}{\partial z}-ik\right)+\Omega\right]\tilde{\varphi}=0$$

- To leading order, we have:

$$F(\Omega,k,z)f_0=\left[-\eta k^3+\left(\Omega_0\mp\frac{i\sigma\eta}{\Omega_0}\right)k^2+(1\mp i\sigma)k+\Omega_0\right]f_0=0 \qquad \Rightarrow \Omega_0(k,z)$$

- The next order yields $\quad iF_k\dfrac{\partial f_0}{\partial z}+Ff_1=0 \qquad F_k\equiv\left.\dfrac{\partial F}{\partial k}\right|_\Omega$

and so $dF/dk|_\Omega=0$; this is equivalent to the condition $d\Omega_0/dk=0$

- These two equations yield k and Ω_0
- Finally, at second order, we have $\quad F_{kk}\rho_*^2\dfrac{\partial^2 f_0}{\partial z^2}=F_\Omega[\Omega-\Omega_0(z)]f_0$

$\Longrightarrow$ expanding Ω_0 about extremum $z=z_0$, provides $\quad f_0=\exp\left[\pm\dfrac{i}{2}\left(-\dfrac{F_\Omega}{F_{kk}}\Omega_{zz}\right)^{1/2}\dfrac{(z-z_0)^2}{\rho_*}\right]$

Suggests shifted, localised modes, but…

- Combining f_0 with the eikonal we derive φ:

$$\varphi = \exp\left[\pm\frac{i}{2}\left(-\frac{F_\Omega}{F_{kk}}\Omega_{zz}\right)^{1/2}\frac{(z-z_0)^2}{\rho_*}+\frac{i}{\rho_*}kz\right]$$

- This provides a mode which is localised with width $\Delta y \sim (\rho_s/K_y)^{1/2}$ about a position

$$z_1 = z_0 + \mathrm{Im}(k_y\rho_s)\left\{\mathrm{Re}\left[i\left(-\frac{F_\Omega}{F_{kk}}\Omega_{zz}\right)^{1/2}\right]\right\}^{-1}$$

- If the shift is large ($\mathrm{Im}(k_y\rho_s) \sim \sqrt{\rho_*}$), then the ordering breaks down

- From the numerical solutions, the shift is indeed large
  - Note also that the width of the localised modes from the numerical solutions does not scale as $\sqrt{\rho_*}$
  - The standard approach (eg as used in ballooning theory) does not work

- Requires us to extend our formalism to Mathieu-type approach to address more extended modes….work in progress!

Summary

- The presence of a magnetic island chain stabilises ITG modes according to the local theory
- But a more complete 2-D model predicts a more unstable mode, localised in y (not allowed in the absence of the island)

- A local WKB theory qualitatively reproduces these trends
  - predicts a complex k_y which results in a shift in the mode peak amplitude relative to position of maximum instability drive
  - Unfortunately the local WKB theory then fails and needs to be extended
  - work in progress
  - a similar theory may apply for toroidal drift modes in tokamaks (could be very dangerous to do linear drift wave stability with θ_0=0 $k \leftrightarrow \theta_0$)