Gyrokinetic simulations

Aim to describe turbulence in a tokamak plasma, thus giving information about heat and particle transport and confinement related phenomena

Several theoretical approaches and numerical methods (Vlasov, PIC, full-f, delta-f, etc.) have been and are still used

PIC method and ELMFIRE

Key ingredients for PIC simulation model

- Equations of motion

$$\frac{d\mathbf{R}}{dt} = U\hat{\mathbf{b}}^* + \frac{\mu}{e}\frac{\hat{\mathbf{b}}\times\nabla B}{B_\parallel^*} + \frac{\hat{\mathbf{b}}\times\nabla\phi}{B_\parallel^*} + \frac{1}{\Omega}\frac{d}{dt}(-\nabla\phi)$$

$$\frac{dv_\parallel}{dt} = \frac{\hat{\mathbf{b}}^*}{m}\cdot(\mu\nabla B + e\nabla\phi)$$

- The Poisson equation

$$-\frac{1}{4\pi e}\nabla^2\phi = \int d^6Z\Bigg[F + \frac{\Omega}{B}(\phi - \langle\phi\rangle)\frac{\partial F}{\partial\mu} + \frac{c}{eBB_\parallel}\mathbf{F}\cdot\left[\nabla S_1 + (\mathbf{u}_E - \langle\mathbf{u}_E\rangle)\right]\cdot\nabla F - \nabla\cdot\frac{m\nabla\phi}{B}F\Bigg]\delta_{gc}^3 - n_e(x,t)$$

The equations above represent the ones ELMFIRE[1] is based on.

In the standard set[2] the polarization drift is not included in the equations of motion and the 4th and 5th terms in the Poisson equation are not present.

Both sets of equations should in principle form a consistent description of the electrodynamic system comprised of charged particles.

In both approaches $\Delta\phi \approx 0$ and the terms involving gradient of distribution function are also small due to long scale length of equilibrium density and temperature

In the standard set the potential is solved from the second term (polarization density) on the RHS of the Poisson equation

$$-\int d^6Z\left[\frac{\Omega}{B}(\phi - \langle\phi\rangle)\frac{\partial F}{\partial\mu}\right] = \int d^6ZF\delta_{gc}^3 - n_e(x,t)$$

In the alternative approach, applied in ELMFIRE, the fact that the second order terms cancel in the long wave length limit is used

$$\int d^6Z\left[\frac{\Omega}{B}(\phi - \langle\phi\rangle)\frac{\partial F}{\partial\mu} - \nabla\cdot\frac{m\nabla\phi}{B}F\right]\delta_{gc}^3 = 0$$

Potential is solved by linearizing the density change (in each grid point) caused by

1. the ion polarization drift
2. the electron parallel acceleration

in such a way that the calculated potential adjusts the system into quasineutrality

To give a picture of the applied implicit scheme, we first define

$$\tilde{n}_i(x,t) = \int d^6ZF\delta_{gc}^3$$

Next we write out the simplified Poisson equation and consider how the densities can be interpreted to change in time

$$\begin{aligned}0 &= \tilde{n}_i(x_j,t) - n_e(x_j,t)\\ &= \tilde{n}_i(x_j,t-\Delta t) + \delta n_{i,w/o\,pol.drift}(x_j,t-\Delta t)\\ &\quad + \delta n_{i,pol.drift}(x_j,\phi(x_{j1,\dots,jn},t) - \phi(x_{j1,\dots,jn},t-\Delta t))\\ &\quad - n_e(x_j,t-\Delta t) - \delta n_{e,w/o\,par.acc.}(x_j,t-\Delta t)\\ &\quad - \delta n_{e,par.acc.}(x_j,\phi(x_{j1,\dots,jn},t) - \phi(x_{j1,\dots,jn},t-\Delta t))\end{aligned}$$

N.B. The density change due to polarization drift is calculated particle-wise, the change affects surrounding grid cells and depends respectively on the potentials in all of them.

By defining short hand notation

$$\bar{n}_i(x_j,t) = \tilde{n}_i(x_j,t-\Delta t) + \delta n_{i,w/o\,pol.drift}(x_j,t-\Delta t)$$
$$\bar{n}_e(x_j,t) = n_e(x_j,t-\Delta t) + \delta n_{e,w/o\,par.acc.}(x_j,t-\Delta t)$$

the equation above can be formulated as

$$\begin{aligned}&A(x_j,\phi(x_{j1,\dots,j2},t)) + B(x_j,\phi(x_{j1,\dots,j2},t))\\ &= \bar{n}_i(x_j,t) - A(\phi(x_{j1,\dots,j2},t-\Delta t))\\ &\quad - \bar{n}_e(x_j,t) + B\phi(x_{j1,\dots,j2},t-\Delta t))\end{aligned}$$

This approach, based on the theory presented by Sosenko[3], has been used in order to study an alternative numerical scheme where the computation of the standard polarization density term is not required.

Sosenko however did not consider if this system is consistent in the Hamiltonian sense.

Outline of the derivation - Lie transform perturbation method and action principle

The Lie transformation has been in a key role in gyrokinetic theory during the last few decades. Now the theory basis of ELMFIRE has been formulated in a way which endeavors to use the standard notation[2].

The near identity transform

$$Tz = \dots T_3T_2T_1z = Z \qquad T_n = e^{\varepsilon L_n}$$
$$L_nf = g_n^\mu\frac{\partial f}{\partial z^\mu} \qquad \frac{\partial Z^\mu}{\partial\varepsilon^n} = g_n^\mu(\mathbf{Z})$$
$$L_n\gamma = i_Gd\gamma = g_n^\nu\left(\frac{\partial\gamma_\mu}{\partial z^\nu} - \frac{\partial\gamma_\nu}{\partial z^\mu}\right)$$

transforms the particle Lagrangian γ into gyrocenter Lagrangian

$$\Gamma = T^{-1}\gamma + dS$$

Fundamental one form defines Poisson-Lagrange tensor and thus also the equations of motion

$$\hat{\omega}_{ij} = \frac{\partial\Gamma_j}{\partial Z^i} - \frac{\partial\Gamma_i}{\partial Z^j}$$
$$\hat{\omega}_{ij}\frac{dz^j}{dt} = \frac{\partial h}{\partial z^i} + \frac{\partial\Gamma_i}{\partial t}$$

The particle Lagrangian is

$$L = \mathbf{p}\cdot\dot{\mathbf{q}} - H = \frac{1}{2}m|\dot{\mathbf{x}}|^2 + \frac{e}{c}\mathbf{A}\cdot\dot{\mathbf{x}} - e\phi$$
$$\mathbf{q} = \mathbf{x} \qquad \mathbf{p} = \frac{\partial L}{\partial\dot{\mathbf{x}}} = \frac{e}{c}\mathbf{A} + m\dot{\mathbf{x}}$$

where

$$\mathbf{x} = \mathbf{X} + \mathbf{a}$$
$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = \mathbf{v} = U\hat{\mathbf{b}} + \dot{\mathbf{a}} + \mathbf{u}_E$$
$$\mathbf{u}_E = \frac{c\nabla\phi\times\mathbf{B}}{\mathbf{B}^2}$$

The Lagrangian divided to different orders is

$$\gamma_0 = \left(\frac{e}{c}\mathbf{A} + mU\mathbf{b}\right)\cdot d\mathbf{X} + \frac{\mu B}{\Omega}d\theta - H_0dt$$
$$H_0 = \left(\frac{1}{2}mU^2 + \mu B\right)$$

$$\gamma_1 = m\mathbf{u}_E\cdot d\mathbf{X} - e\phi dt$$
$$\gamma_2 = -\frac{1}{2}mu_E^2dt$$

In order to calculate the gyrocenter one form we proceed in a similar fashion as in the standard derivation.

The difference comes from the first order generating functions

$$g_1^U = \frac{1}{m}b_i^*(S_1)_{,i} + (\mathbf{u}_E - \langle\mathbf{u}_E\rangle)\cdot\frac{mc}{eB_\parallel}U\nabla\times\hat{\mathbf{b}}$$
$$g_1^{\mathbf{X}} = \hat{\mathbf{b}}^*\left(\frac{1}{m}\frac{\partial S_1}{\partial U}\right) + \frac{c}{eBB_\parallel}\mathbf{F}\cdot\nabla S_1 + \frac{c}{eBB_\parallel}m\mathbf{F}\cdot(\mathbf{u}_E - \langle\mathbf{u}_E\rangle)$$

Action principle was used to find the Poisson equation

$$\int d\mathbf{x}d\mathbf{v}dwdtF_pH_p \to \int d^8ZF_{gy}H_{gy}$$

varying the action with respect to the potential yields the Poisson equation

$$\frac{1}{4\pi}\int d^4x\varepsilon\nabla\delta\phi\cdot\nabla\phi - \int d^8Z\left[F\int d^3x\left(\delta\phi(\mathbf{x})\frac{\delta H}{\delta\phi(\mathbf{x})} + \nabla\delta\phi\cdot\frac{\delta H}{\delta\nabla\phi}\right)\right] = 0$$

$$\nabla^2\phi = -4\pi e\left[\int d^6ZF\langle T_{gy}^{-1}\delta_{gc}^3\rangle - \nabla\cdot\frac{m}{B}\nabla\phi\int d^6ZF\langle\delta_{gc}^3\rangle\right]$$

References

[1] Heikkinen et al., “Full f Gyrokinetic method for particle simulation of tokamak transport”, Journal of Comp. Phys., 227 (2008), 5582

[2] A.J. Brizard and T.S. Hahm, "Foundations of nonlinear gyrokinetic theory", Rev. Mod. Phys., 79 (2007), 421