Vlasov-Poisson with strong magnetic field: some mathematical results

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Motion of charged particles in a strong magnetic field

•Motion of a charged particle in a constant electromagnetic field (see for instance Lifshitz-Pitayevski *Physical kinetics* §60)

$$\begin{cases} \dot{x} = v \\ \dot{v} = \frac{q}{m}(E + \frac{1}{c} v \wedge B) \end{cases}$$

Notation:

$$\begin{cases} (x, v) \mapsto (x_{||}, v_{||}) \text{ projection in the direction of } B \\ (x, v) \mapsto (x_{\perp}, v_{\perp}) \text{ projection on the plane orthogonal to } B \end{cases}$$

•Parallel projection of the motion equation:

$$\ddot{x}_{||} = \frac{q}{m} E_{||}$$

so that

$$x_{||}(t) = x_{||}(0) + t v_{||}(0) + \frac{q}{m} \frac{t^2}{2} \frac{E \cdot B}{|B|}$$

•Van der Pol transformation for the transverse motion:

$$\dot{w} = \frac{q}{m}\mathcal{R}(\omega t)E_\perp\,, \qquad \text{with } w(t) = \mathcal{R}(\omega t)v_\perp(t)$$

where $\mathcal{R}(\theta)$ is the rotation of an angle θ around the axis oriented by B

$$\dot{\mathcal{R}}(t) = A\mathcal{R}(t)\,, \qquad \text{with } Av = v \wedge \frac{B}{|B|}$$

One finds that

$$x_\perp(t) = x_\perp(0) \qquad \underbrace{+ct\frac{E\wedge B}{|B|^2}}_{\text{slow secular drift}} \qquad \underbrace{+O\left(\frac{mc}{q|B|}\right) + O\left(\frac{c|E|}{|B|}\right)}_{\text{fast Larmor rotation}}$$

Transverse motion on a long time scale=slow drift in the direction $E_\perp$

•Hamiltonian perturbation methods for nontrivial field geometries: see for instance Littlejohn (1980s) for given electromagnetic field, more recently

•**Pbm:** handle a self-consistent electric field in a collisionless plasma

<u>Difficulty:</u> Hamiltonian perturbation methods may require a lot of regularity in the fields, uniformly in the high magnetic field limit

$\Rightarrow$ use only estimates propagated by the Vlasov equation that are uniform in that limit

$\boxed{\textbf{Mathematical toolbox}}$

a) Weak convergence in functional spaces:

•the strong magnetic field limit involves averaging out fast Larmor rotation

•weak convergence corresponds roughly with averaging out fast variables locally

b) Van der Pol transform:

•Pbm: to understand the asymptotic behavior of $X_\epsilon(t)$ for $\epsilon \ll 1$, where

$$\dot{X}_\epsilon = B(t, X_\epsilon) + \frac{1}{\epsilon} A X_\epsilon$$

Difficulty: X_ϵ contains high frequencies since $\dot{X}_\epsilon(t) = O(1/\epsilon)$

Idea: filter these high frequencies by solving EXPLICITLY the leading order in the equation:

$$Y_\epsilon := S\left(-\frac{t}{\epsilon}\right) X_\epsilon \,, \text{ where } S(t) = e^{tA}$$

(Think of A as a skew-adjoint matrix, so that $S(t)$ is a unitary transform.) Then Y_ϵ satisfies

$$\dot{Y}_\epsilon(t) = S\left(-\frac{t}{\epsilon}\right) B\left(t, S\left(\frac{t}{\epsilon}\right) Y_\epsilon(t)\right) = F\left(t, \frac{t}{\epsilon}, Y_\epsilon\right) = O(1)$$

so that Y_ϵ does not contain any more high frequencies since $\dot{Y}_\epsilon = O(1)$ $\Rightarrow$ one expects that $Y_\epsilon \to Y$ as $\epsilon \to 0^+$, where

$$\dot{Y}(t) = \langle F\rangle(t, Y) \text{ where } \langle F\rangle(t, Z) = \lim_{T\to+\infty} \frac{1}{T}\int_0^T F(t, s, Z)ds$$

and hence

$$X_\epsilon(t) \simeq S\left(\frac{t}{\epsilon}\right) Y(t)$$

Vlasov-Poisson with strong magnetic field

•**Pbm 1:** to derive the leading order, longitudinal particle motion with self-consistent electric field and strong, non constant magnetic field

Scaling: set $\epsilon = \omega_p/\omega_c \ll 1$ where

$$\begin{cases} \omega_c = \frac{q|B|}{mc} \text{ cyclotron frequency} \\ \omega_p = \frac{q|E|}{mu} \text{ plasma frequency, where } m|u|^2 = \varepsilon_0|E|^2 \end{cases}$$

Vlasov-Poisson in $3D$ periodic box with constant neutralizing background

$$\begin{cases} \partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon - \nabla_x V_\epsilon \cdot \nabla_x f_\epsilon + \frac{1}{\epsilon}(v \wedge B) \cdot \nabla_v f_\epsilon = 0\,, \\ \\ -\Delta_x V_\epsilon = \int_{\mathbf{R}^3} f_\epsilon dv - \iint_{\mathbf{T}^3 \times \mathbf{R}^3} f_\epsilon dx dv\,, \qquad (x,v) \in \mathbf{T}^3 \times \mathbf{R}^3 \end{cases}$$

in the time scale $1/\omega_p$

$\boxed{\textbf{Weak convergence in } L^p}$

•If a sequence $f_n \equiv f_n(x)$ is bounded in L^p_x — meaning that

$$\sup_n \left(\int |f_n(x)|^p dx \right)^{1/p} < \infty$$

we say that

$$f_n \rightharpoonup f \text{ in } L^p_x \text{ weak if } 1 \le p < \infty, \text{ or } L^\infty_x \text{ weak-*}$$

to mean that

$$\int_A f_n(x)dx \to \int_A f(x)dx \text{ for each cube } A$$

•All frequencies in f_n that go to infinity with n are averaged out by this procedure.

Example: let $f_n \equiv f_n(x)$ be a sequence of periodic functions with period 1 and bounded in L^2_x;

$$f_n \rightharpoonup \widehat{f} \text{ weakly in } L^2 \text{ iff } \widehat{f}_n(k) \to \widehat{f}(k) \text{ for each } k$$

whereas $f_n \to f$ (strongly) in L^2 — i.e. in quadratic mean:

$$\|f_n - f\|^2_{L^2} = \sum_k |\widehat{f}_n(k)|^2 \to 0 \text{ as } n \to \infty$$

FACT 1. Weak convergence and nonlinear operations don't mix well, in general

$$\cos(nx) \rightharpoonup 0 \text{ but } \cos(nx)^2 = \tfrac{1}{2}((1 + \cos(2nx)) \rightharpoonup \tfrac{1}{2}$$

FACT 2. ... however, one can pass to the limit in products where all the terms but one converge strongy:

$$\text{if } \begin{cases} f_n \rightharpoonup f \text{ in } L^2 \text{ weak} \\ g_n \to g \text{ in } L^2 \text{ strong} \end{cases} \qquad \text{then } f_n g_n \rightharpoonup fg \text{ in } L^1 \text{ weak}$$

•However, the fact that the sequence of functions satisfy an elliptic PDE can help in controling (some) high frequencies

Example: the Poisson equation

$$\text{if } \begin{cases} u_n \equiv u_n(x) \rightharpoonup u \text{ in } L^2 \text{ weak} \\ -\Delta u_n = O(1) \text{ in } L^2 \end{cases} \qquad \text{then } u_n \to u \text{ in } L^2 \text{ strong}$$

(See this in Fourier space for periodic functions: the Laplacian wipes out all high frequencies in u_n uniformly in n).

•Case where the direction of B is constant:

Thm 1: [FG & L. StRaymond, JMPA 1999] Assume $B \equiv b(x_1, x_2)e_3$ with $b \in C(\mathbf{T}^2)$ and $b \neq 0$ on $\mathbf{T}^2$, and $f_\epsilon\big|_{t=0} = f^{in} \in L^1 \cap L^\infty_{x,v}$. In the limit as $\epsilon \longrightarrow 0$ and extracting subsequences if needed

$$f_\epsilon \rightharpoonup f \equiv f(t, x, \sqrt{v_1^2 + v_2^2}, v_3) \text{ in } L^\infty \text{ weak-*}$$

where

$$\begin{cases} \partial_t f + v_3 \partial_{x_3} f - \partial_{x_3} V \partial_{v_3} f = 0\,, \quad t, r > 0\,,\ x \in \mathbf{T}^3\,,\ v_3 \in \mathbf{R} \\ \\ -\Delta_x V = 2\pi \displaystyle\int_{\mathbf{R}^3} f r dr dv_3 - 2\pi \iint_{\mathbf{T}^3 \times \mathbf{R}^3} f dx r dr dv_3\,, \\ \\ f(0, x, r, v_3) = \frac{1}{2\pi} \displaystyle\int_{\mathbf{S}^1} f^{in}(x, r\omega, v_3) d\omega \end{cases}$$

•Case where the strength of B is constant:

Thm 2: [FG & L. StRaymond, JMPA 1999] Assume that $B \in C^1(\mathbf{T}^3)$ is s.t. $|B| = 1$ and $\operatorname{div} B = 0$, and $f_\epsilon\big|_{t=0} = f^{in} \in L^1 \cap L^\infty_{x,v}$. Let $\mathcal{R}(x,\theta)$ be the rotation of an angle θ around the oriented axis $\mathbf{R}B$, and define

$$g_\epsilon(t,x,w) := f_\epsilon(t,x,\mathcal{R}(x,-t/\epsilon)w)$$

Then, in the limit as $\epsilon \to 0$ and after extracting subsequences if needed

$$g_\epsilon \rightharpoonup g \quad \text{in } L^\infty_{t,x,w} \text{ weak-*}$$

and, denoting $D_u B = (u \cdot \nabla)B$ the covariant derivative along u, one has

$$\left\{\begin{array}{l} \partial_t g + (w \cdot B)B \cdot \operatorname{grad}_x g - (D_B V)B \cdot \operatorname{grad}_w g + (w \wedge X) \cdot \operatorname{grad}_w g = 0 \\ \quad \text{where } X = \frac{1}{2}(B \wedge D_w B + D_{B\wedge w}B - 3(w \cdot B)(B \wedge D_B B)) \\ \\ -\Delta_x V = \int_{\mathbf{R}^3} g dv - \iint_{\mathbf{T}^3 \times \mathbf{R}^3} f^{in} dx dv \,, \quad g\big|_{t=0} = f^{in} \end{array}\right.$$

Proof of Thm 1: use a priori uniform in ϵ a priori bounds on f_ϵ

$$0 \le f_\epsilon \le \sup_{x,v} f^{in}(x,v) \text{ (maximum principle for Vlasov)}$$

$$\iint (1+|v|^2) f_\epsilon(t,x,v)\,dx\,dv + \int |E_\epsilon(t,x)|^2 dx \le C \text{ (mass+energy conservation)}$$

Decomposing the number density into low- and high-speed components, one finds

$$\int \rho_\epsilon(t,x)^{5/3} dx \le C$$

so that, using Poisson's equation

$$\int |\nabla_x E_\epsilon(t,x)|^{5/3} dx + \int |\partial_t E_\epsilon(t,x)|^{5/4} dx \le C$$

so that

$$E_\epsilon \to E \text{ strongly in } L^\infty_t L^p_x \text{ for } 1 \le p < 2$$

Proof of Thm 2: observe that g_ϵ solves the nonautonomous equation

$$\begin{aligned}\partial_t g_\epsilon + \mathcal{R}(x,-t/\epsilon)w\cdot\nabla_x g_\epsilon + \mathcal{R}(x,t/\epsilon)E_\epsilon\cdot\nabla_w g_\epsilon \\ = \left((\mathcal{R}(x,-t/\epsilon)w\cdot\nabla_x)\mathcal{R}(x,t/\epsilon)\right)\mathcal{R}(x,-t/\epsilon)w\cdot\nabla_w g_\epsilon \\ \Rightarrow \text{no high frequencies in } t \text{ in } g_\epsilon\end{aligned}$$

Therefore, by nonstationary phase, for each C^1 function $\psi\equiv\psi(x,w)$ and each smooth, mean-zero periodic function $a\equiv a(t)$, one has

$$a(t/\epsilon)\psi(x,v)\begin{pmatrix}1\\ E_\epsilon(t,x)\end{pmatrix}g_\epsilon(t,x)\rightharpoonup 0$$

so that

$$\mathcal{R}(x,-t/\epsilon)w\cdot\nabla_x g_\epsilon\rightharpoonup(w\cdot B)B\cdot\mathrm{grad}_x g$$

$$\mathcal{R}(x,-t/\epsilon)E_\epsilon\cdot\nabla_w g_\epsilon\rightharpoonup(E\cdot B)B\cdot\mathrm{grad}_w g$$

$$\left((\mathcal{R}(x,-t/\epsilon)w\cdot\nabla_x)\mathcal{R}(x,t/\epsilon)\right)\mathcal{R}(x,-t/\epsilon)w\cdot\nabla_w g_\epsilon\rightharpoonup(X\wedge w)\cdot\mathrm{grad}_w g$$

Guiding center for Vlasov-Poisson with strong magnetic field

•**Pbm 2:** to derive the next to leading order, transverse particle motion with self-consistent electric field and a strong, constant magnetic field

Scaling: set $\epsilon = \omega_p/\omega_c \ll 1$ where ω_p is the plasma frequency and ω_c the cyclotron frequency.

•Guiding center motion = secular dynamics with speed $c|E|/|B|$ on a long time scale T defined by

$$T\omega_p = \frac{\omega_c}{\omega_p} = \frac{1}{\epsilon} \gg 1$$

•Magnetic field of the form

$$B = |B|e_3\,, \quad \text{WLOG } |B| = 1$$

•Guiding center motion is observed in the plane orthogonal to B: for simplicity, restrict the charged particle motion to that plane, with constant neutralizing background.

Scaled Vlasov equation: denoting $v^{\perp} = v \wedge e_3$, one has

$$\left\{\begin{aligned} &\partial_t f_\epsilon + \frac{1}{\epsilon}(v \cdot \nabla_x f_\epsilon + E_\epsilon \cdot \nabla_v f_\epsilon) + \frac{1}{\epsilon^2} v^{\perp} \cdot \nabla_v f_\epsilon = 0, \quad x \in \mathbf{T}^2, \, v \in \mathbf{R}^2 \\ &E_\epsilon = -\nabla_x V_\epsilon \, , \qquad -\Delta_x V_\epsilon = \int_{\mathbf{R}^2} f_\epsilon dv - \iint_{\mathbf{T}^2 \times \mathbf{R}^2} f_\epsilon dx dv \\ &f_\epsilon \Big|_{t=0} = f_\epsilon^{in} \end{aligned}\right.$$

Thm 3:[FG & LS-R JMPA 1999, LS-R JMPA 2002] Assume that

$$\lim_{\epsilon\to 0^+}\epsilon\|f_\epsilon^{in}\|_{L^\infty_{x,v}}=0 \text{ and } \sup_{\epsilon>0}\left(\|(1+|v|^2)f_\epsilon^{in}\|_{L^1_{x,v}}+\|E_\epsilon^{in}\|^2_{L^2_x}\right)<\infty$$

(i) Modulo extraction of a subsequence, there exist

$$\begin{cases}\text{a radial distribution function } F\in L^\infty_t(\mathcal{M}_+(\mathbf{T}^2\times\mathbf{R}_+))\\ \text{and a defect measure } \nu\in L^\infty_t(\mathcal{M}_+(\mathbf{T}^2\times\mathbf{S}^1)) \text{ such that}\end{cases}$$

$$f_\epsilon\rightharpoonup F(t,x,|v|) \text{ in } L^\infty_t(\mathcal{M}_+(\mathbf{T}^2\times\mathbf{R}^2)) \text{ weak-*}, \text{ while}$$

$$\int_{\mathbf{R}^2}(f_\epsilon(t,x,v)-F(t,x,|v|))\,\phi(v/|v|)dv\to\int_{\mathbf{S}^1}\phi d\nu\,,\quad \phi\in C(\mathbf{S}^1)\,.$$

(ii) The limiting macroscopic density $\rho(t,x)=\int_{\mathbf{R}^2}F(t,x,|v|)dv$ satisfies

$$\begin{cases}\partial_t\rho+\operatorname{div}_x(\rho E^\perp)=0\,, \qquad E=\nabla_x\Delta_x^{-1}\left(\rho-\int_{\mathbf{T}^2}\rho dx\right)\\ \rho\Big|_{t=0}=\text{weak-}\lim_{\epsilon\to 0}\int_{\mathbf{R}^2}f_\epsilon^{in}dx\end{cases}$$

Remarks:

a) analogy with 2D incompressible, inviscid fluid mechanics (2D Euler)

$$\partial_t \omega + \mathrm{div}_x(\omega u) = 0\,, \quad \mathrm{div}_x\, u = 0\,, \quad \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} = \mathrm{curl}_x \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix}$$

Here

$$\begin{cases} \text{the velocity field } u \text{ corresponds with } E^{\perp} \\ \text{the vorticity } \omega \text{ corresponds with } \rho - \int_{\mathbf{T}^2} \rho dx \end{cases}$$

b) in the statement of Thm 3, the term $\mathrm{div}_x(\rho E^{\perp})$ is to be understood as

$$\mathrm{div}_x(\rho E^{\perp}) := \partial_{x_1}\partial_{x_2}(E_2^2 - E_1^2) + (\partial_{x_1}^2 - \partial_{x_2}^2)E_1 E_2$$

●case of Euler-Poisson with strong magnetic field proved by E. Grenier ($\sim$ 1996)

●similar result obtained by Y. Brenier ($\sim$ 2000) for well-prepared initial data, by using some modulated energy method

●gyrokinetic limit (with finite Larmor radius effect) done by E. Frenod and E. Sonnendrucker ($\sim$ 2001), completed by D. Han-Kwan (see poster in this workshop)

Thm 4:[LS-R JMPA 2002] Assume that

$$(1+|v|^2)^r f^{in} \in W^{s,\infty}(\mathbf{T}^2 \times \mathbf{R}^2) \text{ with } r > 3\,,\ \ s \geq 3$$

and let g be the solution of

$$\begin{cases} \partial_t g + E^\perp \cdot \nabla_x g + \frac{1}{2}(m - \rho v)^\perp \cdot \nabla_v g = 0 \\ \begin{pmatrix} \rho \\ m \end{pmatrix} = \int_{\mathbf{R}^2} \begin{pmatrix} 1 \\ v \end{pmatrix} g dv\,, \quad E = \nabla_x \Delta_x^{-1} \left(\rho - \int_{\mathbf{T}^2} \rho dx \right) \\ \rho\Big|_{t=0} = \int_{\mathbf{R}^2} f^{in} dv \end{cases}$$

Then, for each $p \in [1, +\infty)$ one has

$$f_\epsilon(t,x,v) - g(t,x,\mathcal{R}(-t/\epsilon^2)v) \to 0 \text{ in } L^\infty_{loc}(dt; L^p_{x,v})$$

as $\epsilon \to 0^+$.

Ideas in the proof of Thm 3:

1) write the evolution of density and current:

$$\partial_t \rho_\epsilon + \mathrm{div}_x \frac{1}{\epsilon} \int v f_\epsilon dv = 0$$

$$\epsilon \partial_t \int v f_\epsilon + \mathrm{div}_x \int v \otimes v f_\epsilon dv - \rho_\epsilon E_\epsilon - \frac{1}{\epsilon} \int v^\perp f_\epsilon dv = 0$$

eliminating the current leads to

$$\partial_t \rho_\epsilon + \mathrm{div}_x(\rho_\epsilon E_\epsilon^\perp) = (\partial_{x_1}^2 - \partial_{x_2}^2) \int v_1 v_2 f_\epsilon dv$$
$$+ \partial_{x_1}\partial_{x_2} \int (v_2^2 - v_1^2) f_\epsilon dv + \epsilon \partial_t \, \mathrm{div}_x \int v^\perp f_\epsilon$$

Last term in r.h.s. $\rightharpoonup 0$; the other terms satisfy

$$(\partial_{x_1}^2 - \partial_{x_2}^2) \int v_1 v_2 f_\epsilon dv + \partial_{x_1}\partial_{x_2} \int (v_2^2 - v_1^2) f_\epsilon dv$$
$$\rightharpoonup (\partial_{x_1}^2 - \partial_{x_2}^2)\langle \nu, \omega_1 \omega_2 \rangle + \partial_{x_1}\partial_{x_2} \langle \nu, v_2^2 - v_1^2 \rangle$$

2) write

$$\mathrm{div}_x(\rho_\epsilon E_\epsilon^\perp) = \partial_{x_1}\partial_{x_2}(E_{\epsilon,1}^2 - E_{\epsilon,2}^2) + (\partial_{x_2}^2 - \partial_{x_1}^2)(E_{\epsilon,1}E_{\epsilon,2})$$

Lemma[J.-M. Delort, 1991] Assume that

$$\sup_\epsilon \int |E_\epsilon|^2 dx < \infty \text{ and } \mathrm{div}_x E_\epsilon = a_\epsilon + b_\epsilon$$

with

$$a_\epsilon \geq 0\,, \quad \sup_\epsilon \int a_\epsilon dx < \infty \text{ and } \sup_{x,\epsilon} |b_\epsilon(x)| < \infty$$

If $E_\epsilon \rightharpoonup E$ in L_x^2 weak, one has

$$E_{\epsilon,1}^2 - E_{\epsilon,2}^2 \rightharpoonup E_1^2 - E_2^2 \quad \text{and} \quad E_{\epsilon,1}E_{\epsilon,2} \rightharpoonup E_1 E_2$$

(Used in the context of vortex sheets for $2D$ incompressible Euler.)

3) **Observation 1:** the defect measure may exist. For instance, assume

$$\iint |v|^2 f_\epsilon^{in} dx dv \to 1 \text{ and } 0 \leq f_\epsilon^{in} \leq C\epsilon^3$$

Then $\nu \neq 0$ (for any subsequence extracted from f_ϵ^{in} as $\epsilon \to 0$.)

A priori, one has the following limiting equation for the macroscopic density

$$\partial_t \rho + \mathrm{div}_x(\rho E^\perp) = (\partial_{x_1}^2 - \partial_{x_2}^2)\langle \nu, \omega_1\omega_2\rangle + \partial_{x_1}\partial_{x_2}\langle \nu, v_2^2 - v_1^2\rangle$$

and it may happen that $\nu \neq 0$. On the other hand, if

$$(\partial_{x_1}^2 - \partial_{x_2}^2)\langle \nu, \omega_1\omega_2\rangle + \partial_{x_1}\partial_{x_2}\langle \nu, v_2^2 - v_1^2\rangle = 0$$

this defect measure will not affect the dynamics of ν.

4) **Observation 2:** assume that

$$0 \leq f^{in} \leq C\,, \quad \text{and} \iint |v|^3 f^{in} dx dv < \infty$$

a) If

$$\int_0^T \iint |v|^3 f_\epsilon dt dx dv = o\left(\frac{1}{\epsilon}\right)$$

then the defect measure ν is independent of the angle variable ω (rotation invariant), so that in particular

$$\langle \nu, \omega_1 \omega_2 \rangle = \langle \nu, v_2^2 - v_1^2 \rangle = 0$$

b) One always has

$$\int_0^T \iint f_\epsilon dt dx dv = O\left(\frac{\sqrt{|\ln \epsilon|}}{\epsilon}\right)$$

$\Rightarrow$ to get rid of this defect measure in the equation for the charge density amounts to controling particles with speed of $O(1/\epsilon)$

4) Going back to step 1 (the equations for the charge and current densities) and replacing the original particle distribution function f_ϵ with its truncation

$$\tilde{f}_\epsilon(t,x,v)\chi(\tfrac{1}{2}\epsilon^\alpha|v|^2) \text{ for } \alpha \in (\tfrac{3}{2},2)$$

and χ a smooth truncation such that

$$0 \le \chi \le 1\,, \quad \chi = 1 \text{ on } [0,1]\,, \quad \chi = 0 \text{ on } [2,\infty)\,, \quad |\chi'| \le 2$$

L. StRaymond was able to show that

$$(\partial^2_{x_1} - \partial^2_{x_2})\langle \nu, \omega_1\omega_2\rangle + \partial_{x_1}\partial_{x_2}\langle \nu, v_2^2 - v_1^2\rangle = 0$$

Guiding center + quasineutral limit for Vlasov-Poisson

Scaling: assume that

$$\rho_e \sim \lambda_D \ll L \text{ where } \begin{cases} \rho_e = \text{Larmor radius of electrons} \\ \lambda_D = \text{ Debye length} \\ L = \text{ observation length scale} \end{cases}$$

What happens to the drift-kinetic regime when gradient lengths are comparable to the Larmor radius?

Scaled Vlasov-Poisson equation:

$$\begin{cases} \partial_t f_\epsilon + v \cdot \nabla_x f_\epsilon - \frac{1}{\epsilon}(\nabla_x V_\epsilon + v \wedge e_3) \cdot \nabla_v f_\epsilon = 0, \quad x \in \mathbf{T}^3, v \in \mathbf{R}^3 \\ \\ \epsilon \Delta_x^{-1} V_\epsilon = 1 - \int_{\mathbf{R}^3} f_\epsilon dv \\ \\ f_\epsilon \Big|_{t=0} = f_\epsilon^{in}, \end{cases}$$

Assume that

$$\iint_{\mathbf{T}^3\times\mathbf{R}^3} f_\epsilon^{in}dxdv = 1\,, \quad \iint_{\mathbf{T}^3\times\mathbf{R}^3} |v|^2 f_\epsilon^{in}dxdv + \int_{\mathbf{T}^3} |\nabla_x V_\epsilon|^2 dx \le C$$

- The small ϵ limit of the scaled Vlasov-Poisson system above is governed by the 2D-3C incompressible Euler equations

$$\left\{\begin{array}{l} \partial_t J + (J\nabla_x)J + \nabla_x \Pi = 0 \\ \\ \mathrm{div}_x\, J = 0\,, \qquad \partial_{x_3} J = 0 \\ \\ J\Big|_{t=0} = J^{in}\,, \end{array}\right. \qquad \text{i.e. } J(t,x) = \begin{pmatrix} J_1(t,x_1,x_2) \\ J_1(t,x_1,x_2) \\ J_1(t,x_1,x_2) \end{pmatrix}$$

Thm 5: [F.G. & L.StRaymond, M3AS 2003] Assume that f_ϵ^{in} satisfy

$$\int f_\epsilon^{in} dv \to 1 \text{ uniformly in } x \in \mathbf{T}^3$$

$$\iint |v - J^{in}|^2 f_\epsilon^{in} dv dx + \int |\nabla V_\epsilon^{in} + J^{in} \wedge e_3|^2 dx \to 0$$

for some smooth J^{in}. Then

$$\nabla_x V_\epsilon \to e_3 \wedge J \text{ in } L^2_{loc}(t,x)$$

$$\int (v - J_\epsilon) f_\epsilon dv \to 0 \text{ in } L^1_{loc}(t,x)$$

$$\int f_\epsilon \rightharpoonup 1 \text{ in } L^\infty_{loc}(t, \mathcal{M}_x) \text{ weak-*}$$

where J is the solution of the 2D-3C incompressible Euler system with initial data J^{in}

Roughly speaking, the initial distribution function converges to a “monokinetic" profile:

$$f_\epsilon^{in} \longrightarrow \delta_{v=J^{in}}$$

•Method of proof: compute the time derivative of the modulated energy

$$\iint |v-\mathcal{J}|^2 f_\epsilon dx dv + \int |\nabla_x V_\epsilon - \nabla_x (-\Delta)^{-1/2}\Phi| dx$$

where $\mathcal{J}$ and Φ are given, smooth functions, and apply Gronwall's inequality to show that this quantity vanishes iff

$$\mathcal{J} = J \text{ and } -\nabla_x (-\Delta)^{-1/2}\Phi = J \wedge e_3$$

Remark:

1) more generally, one can handle “non monokinetic" asymptotic initial profile, by replacing the term

$$\iint |v - \mathcal{J}|^2 f_\epsilon dx dv$$

in the modulated energy above with some relative entropy adapted to the desired initial profile

2) one can also handle more general initial data $\Rightarrow$ leads to fast oscillating modes that are governed by systems of linear equations driven by the 2D-3C Euler solution J