Thematic Programme Conceptual Aspects of Turbulence **(CATURB-07)**

Workshop and Minicourse "Conceptual Aspects of Turbulence: Mean Fields vs. Fluctuations"

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LAGRANGE VERSUS EULER FOR TURBULENT FLOWS

and/or vice versa, what is the relation?

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This is a short and modified version of

LECTURES XIX-XXII

'LAGRANGIAN' ASPECTS/SETTING OF TURBULENT FLOWS AND ITS RELATION TO THE EULERIAN ONE

delivered as a part of the activity within the frame of the Marie Curie Chair "Fundamental and Conceptual Aspects of Turbulent Flows". PDF files available at

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Sugestions/critisisms, etc are welcome

FUNDAMENTAL AND CONCEPTUAL ASPECTS OF TURBULENT FLOWS

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Lectures series as a part of the activity within the frame of the *Marie Curie* Chair "Fundamental and Conceptual Aspects of Turbulent Flows". PDF files available at

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or/and http://www.eng.tau.ac.il/~tsinober/

We absolutely must leave room for doubt or there is no progress and no learning. There is no learning without posing a question. And a question requires doubt…Now the freedom of doubt, which is absolutely essential for the development of science, was born from a struggle with constituted authorities… FEYNMANN, 1964

One owes to Euler the first general formulas for fluid motion … presented in the simple and luminous notation of partial differences… By this discovery, all fluid mechanics was reduced to a single point analysis, and if the equations involved were integrable, one could determine completely, in all cases the motion of a fluid moved by any forces..

LAGRANGE *Mécanique analitique*, Paris, 1788, Sec X. p. 271

Of course, fluid mechanics can, in principle, be worked entirely in the Lagrangian frame …even neglecting viscous forces… yield awkward moment equations.

CORRSIN 1962.

…the use of the viscous Lagrangian equations in turbulence theory is still a matter for the future, MONIN AND YAGLOM 1971

INTRODUCTORY NOTES AND RELATED

Examples showing how extremely intricate is the L-E- relation

SAME FLOW - NOT THE SAME PATTERN

Seeing is not necessarily believing

All frames (i.e. four different Lagrangian fields) correspond to the same Eulerian flow.

What one sees is real

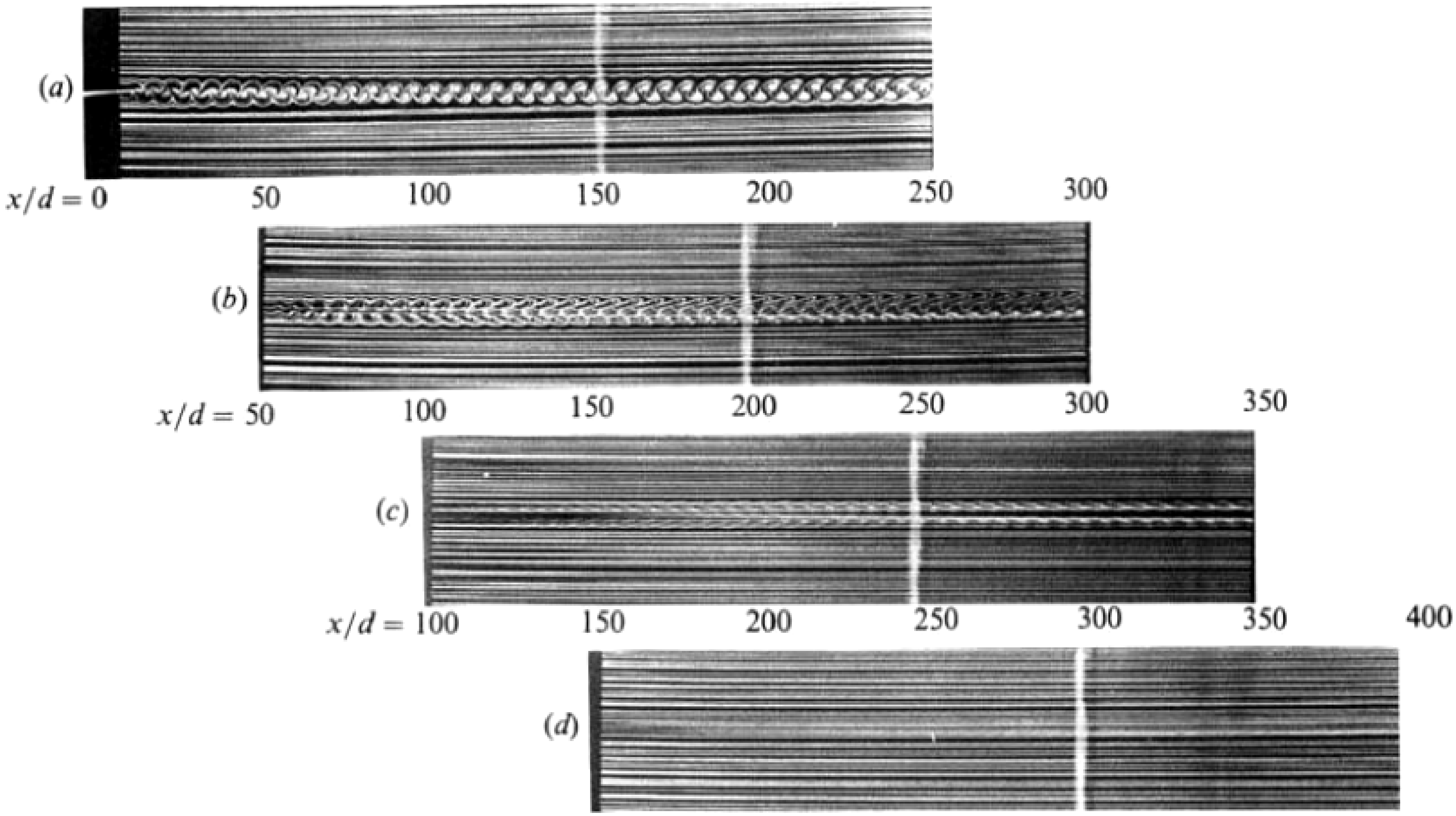


FIGURE 1. Circular-cylinder wake at $Re = 90$; smoke wire at (a) $x/d = 4$, (b) 50, (c) 100 and (d) 150.

The problem is interpretation

Cimbala, J.M., Nagib, H. M and Roshko, A. (1988) Large structures in the far wakes of two-dimensional bluff bodies, J. Fluid Mech., **190**, 265--298.

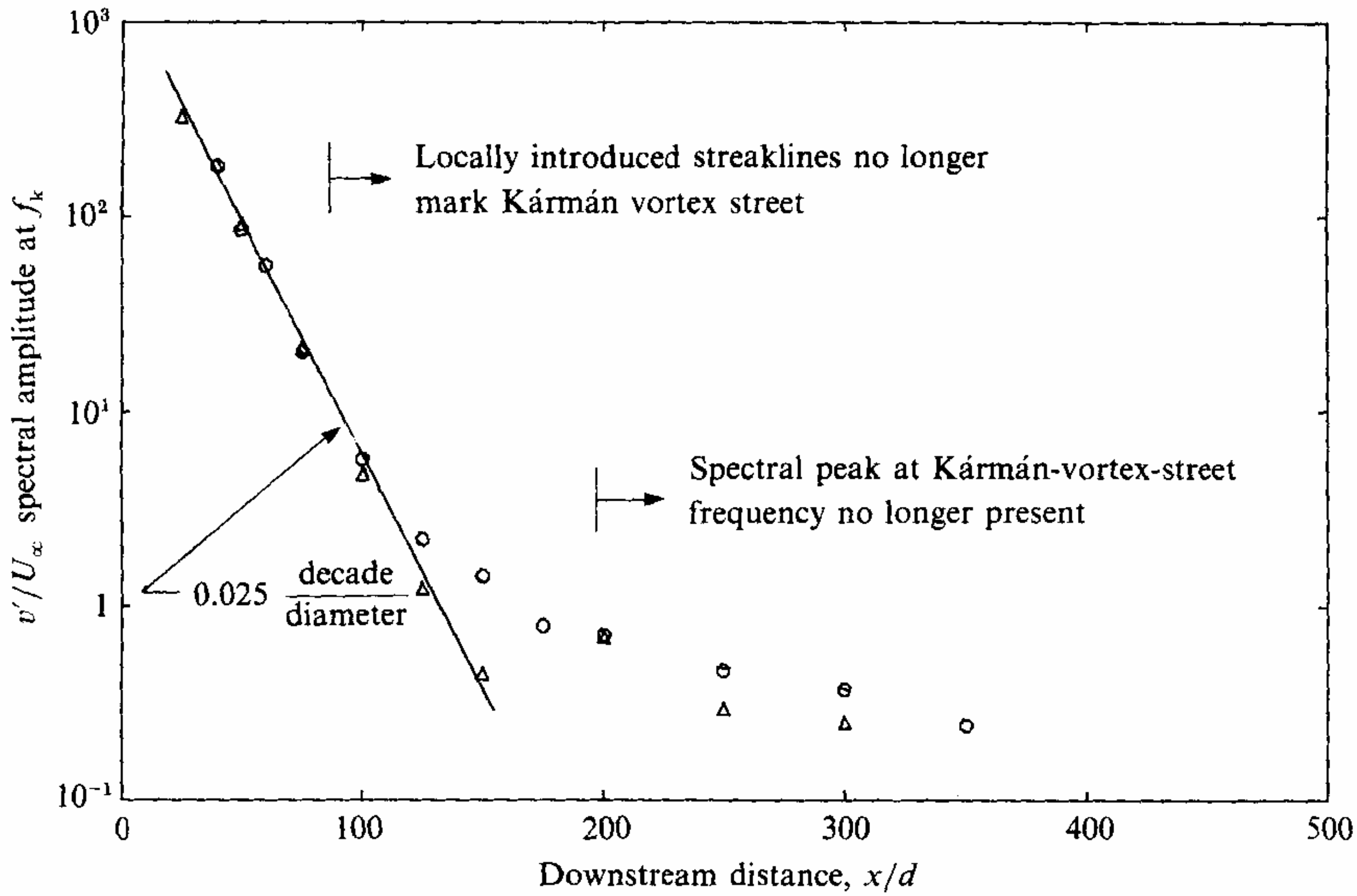
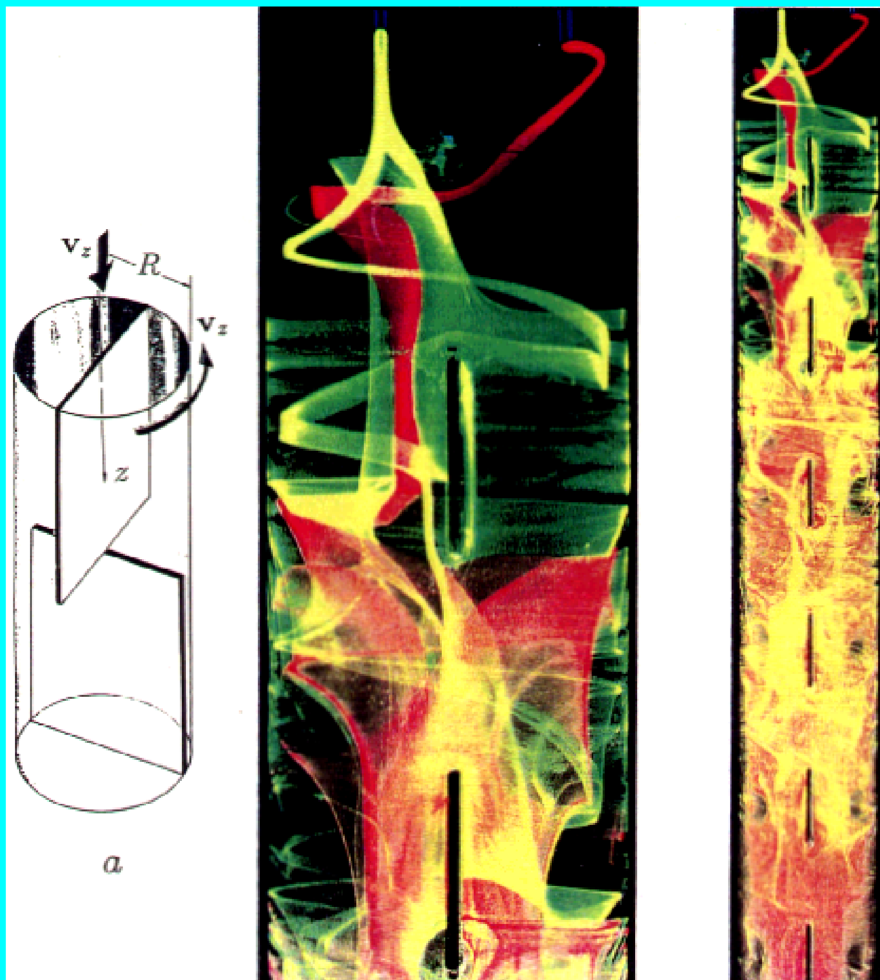


FIGURE 7. Exponential decay of Kármán vortex street; circular-cylinder wake at Re = 140 (○) and 150 (△).

LAMINAR EULERIAN FLOW AT Re~1 (E-LAMINAR)

CHAOTIC LAGRANGIAN (L-TURBULENT)



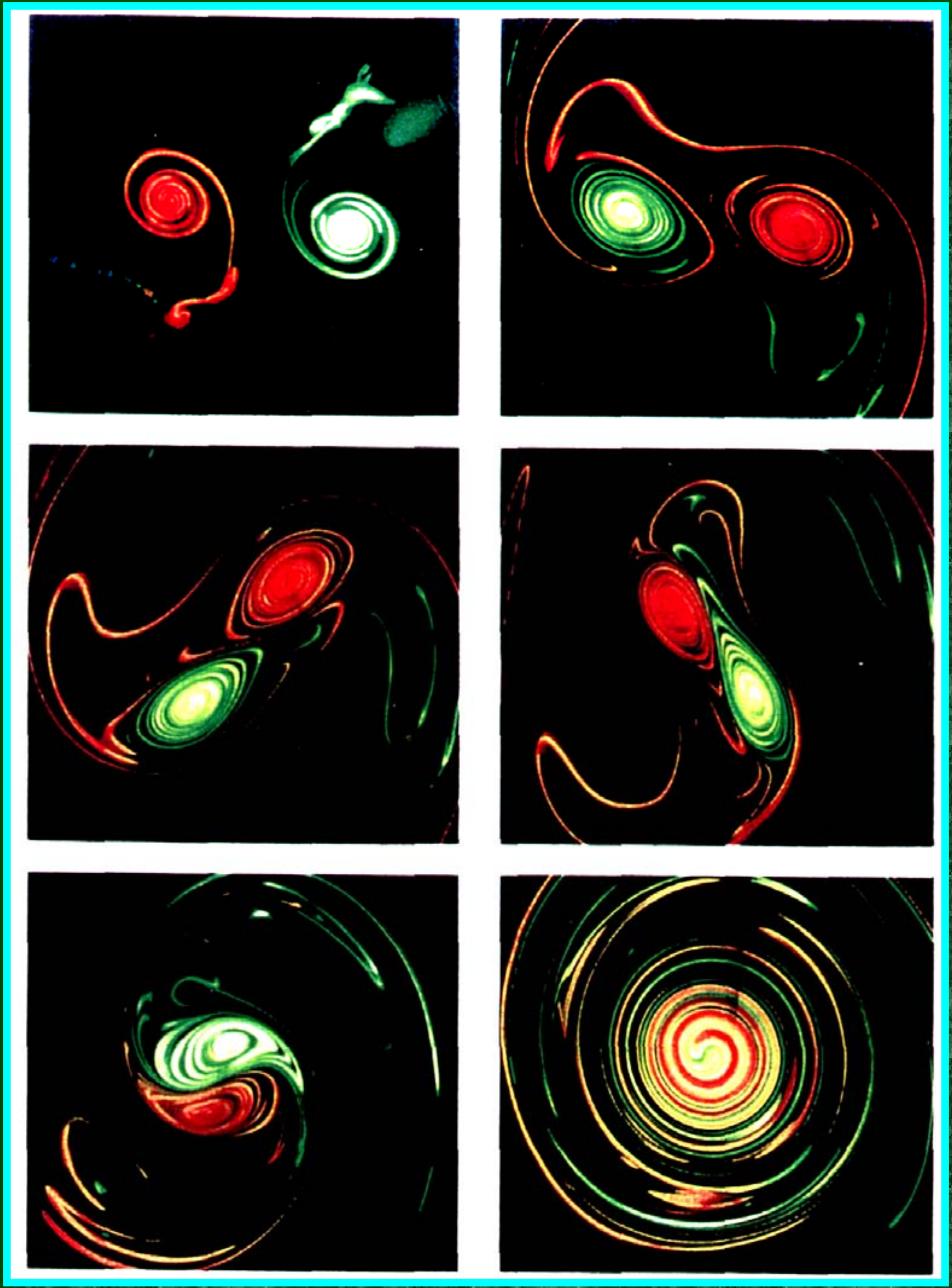
MIXING IN PMM, **Re ~ 1 (!)**

KUSH & OTTINO (1992)

RELEVANT TO MICROFLUIDICS with Re ~ 0 (!);

Linked twist maps (LTMs), Bernoulli mixing ...

The complexity and problematic aspects of the relation between the Lagrangian and Eulerian fields is seen in the example of Lagrangian (kinematic) chaos or Lagrangian turbulence (chaotic advection) with a priori prescribed and not random Eulerian velocity field (E-laminar). This is why Lagrangian description - being physically more transparent - is much more difficult than the Eulerian description. In such E-laminar but L-turbulent flows the Lagrangian statistics has no Eulerian counterpart, as in the flow shown at the left.



Dye visualization of two simple corotating vortices merging into one vortex with simple velocity field, but not that simple filed of the passive scalar(s)

LEWEKE 2000

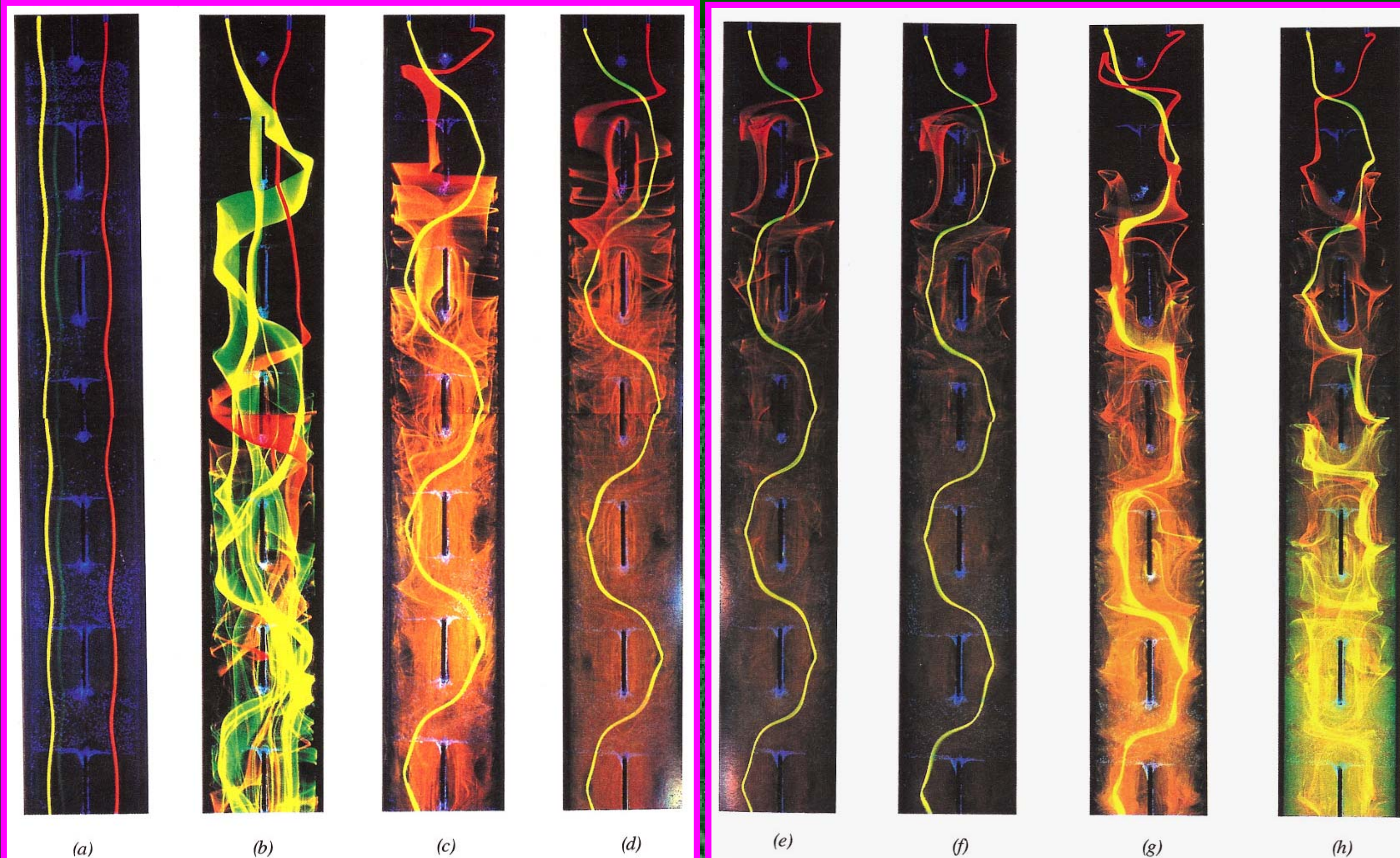


FIGURE 8. Mixing in the PPM as the mixing strength is increased. The mixing strength parameter and Reynolds numbers are *(a)* $\beta=0$, $Re_{\mathrm{PPM:axial}}=0.6$, and $Re_{\mathrm{PPM:cs}}=0$; *(b)* $\beta=4\pm0.1$, $Re_{\mathrm{PPM:axial}}=0.5$, and $Re_{\mathrm{PPM:cs}}=1.3$; *(c)* $\beta=10\pm0.1$, $Re_{\mathrm{PPM:axial}}=0.6$, and $Re_{\mathrm{PPM:cs}}=3.5$; *(d)* $\beta=15\pm0.2$, $Re_{\mathrm{PPM:axial}}=0.5$, and $Re_{\mathrm{PPM:cs}}=4.1$; *(e)* $\beta=20\pm0.3$, $Re_{\mathrm{PPM:axial}}=0.5$, and $Re_{\mathrm{PPM:cs}}=5.5$; *(f)* $\beta=25\pm0.4$, $Re_{\mathrm{PPM:axial}}=0.5$, and $Re_{\mathrm{PPM:cs}}=7.3$; *(g)* $\beta=30\pm0.6$, $Re_{\mathrm{PPM:axial}}=0.3$, and $Re_{\mathrm{PPM:cs}}=5.9$; and *(h)* $\beta=40\pm0.9$, $Re_{\mathrm{PPM:axial}}=0.3$, and $Re_{\mathrm{PPM:cs}}=7.5$. KUSH & OTTINO (1992)

The structure of a passive marker (L) can be (and usually is) very complicated, whereas the corresponding velocity field (E) is rather simple. The passive tracer may have a nontrivial structure (and statistics!) whereas the velocity field has none.

Visualizations. E- versus L-

Flow visualizations used for studying the structure of dynamical fields (velocity, vorticity, etc.) of turbulent flows may be quite misleading, making the question "what do we see?" extremely nontrivial. The meaning of 'seeing' turbulent flow is not so simple as the Eulerian flow structure is different from the Lagrangian one: watching the evolution of material 'coloured bands' (as suggested by Reynolds 1884) in a flow may not reveal the nature of the underlying motion, and even in the case of right qualitative observations the right result may come not necessarily for the right reasons. The famous verse by Richardson belongs to this kind of observation (which is not necessarily right either).

WEATHER PREDICTION

BY

NUMERICAL PROCESS

BY

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On the other hand we find that convectional motions are hindered by the formation of small eddies resembling those due to dynamical instability. Thus O. K. M. Douglas writing of observations from aeroplanes remarks : "The upward currents of large cumuli give rise to much turbulence within, below, and around the clouds, and the structure of the clouds is often very complex." One gets a similar impression when making a drawing of a rising cumulus from a fixed point; the details change before the sketch can be completed. We realize thus that: **big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity— in the molecular sense...**

Thus, because it is not possible to separate eddies into clearly defined classes according to the source of their energy; and as there is no object, for present purposes, in making a distinction based on size between cumulus eddies and eddies a few metres in diameter (since both are small compared with our coordinate chequer), therefore a single coefficient is used to represent the effect produced by eddies of all sizes and descriptions.

CAMBRIDGE
AT THE UNIVERSITY PRESS
1922

Do you see here a cascade?

.... *even wrong theories may help in designing machines.*

(FEYNMAN, 1996) Feynmann R., 1996 Lectures on Computation, Addison-Wesley.

This is a part of a broader question. Namely, what can be learnt about the properties and especially dynamics of genine turbulence (NSE, Euler) from studies of passive objects (particles, scalars, vectors)? In particular, what can be learnt about the velocity field and other dynamical variables in real turbulence from comparison of the behaviour of passive objects in real and some 'synthetic' turbulence?

We are again back with (some aspect of) the L-E relation

The Lagrangian description of fluid flows is physically more natural than the Eulerian one, since it is related most directly to the motion of fluid elements. Nevertheless, mostly technical difficulties (both in physical and numerical experiments) strongly hindered use of the Lagrangian approach in most of fluids dynamical problems. The traditional problems for which Lagrangian description is considered especially appropriate are *transport and mixing* in diverse applications, e.g. geophysical and environmental, cloud formation, chemical technology, combustion and material processing, sedimentation, bio-medical and recently microfluidics, and many others. In most of the above issues the concern is with with the *kinematic* aspects, i.e. with what is called today “passive turbulence”.

The term kinematic(s) is associated with several issues, all of which have in common things which are not directly related to the (Euler Navier-Stokes) dynamics of turbulence. In other words the dynamics of fluid motion, except incompressibility, does not enter into the problems in question: the velocity field is assumed to be known *a priory*. These issues include the following: *i)* kinematic (statistical) properties/structure of real turbulent and artificial (in some sense, e.g. Gaussian, synthetic, KS, low Re with random excitation) random flows such as (an)isotropy, (in)homogeneity, etc. *ii)* passive objects in random flow fields including artificial ones. *iii)* - Kinematic (Lagrangian) chaos.

In other words the main concern is in the *evolution of passive objects* (fluid particles, passive scalars such as dispersing contaminants, chemical species, temperature, moisture; passive vectors such as material lines, (weak) magnetic field in an electrically conducting fluid; passive surfaces such as material surfaces, and in some cases reacting surfaces and turbulent flames; material volumes) *in random fluid flows.* An essential point is that the evolution of passive objects obeys linear equations in which the velocity field does not `know' anything about the presence of these objects and therefore the velocity field is considered as given a priori be it a real fluid flow field or some artificial one. There is no involving phenomenon as pressure*. This does not mean that the problems of the evolution of passive objects are simple. The main complication and simultaneously rich variety of phenomena comes from the fact that the velocity field enters as a coefficient in front of the spatial derivatives, i.e. it is due its multiplicative character, so that statistical problems become in a sense nonlinear.

*Hence `shocks' in the form of ramp-cliff structures just like in the Burgers equation.

Another aspect is associated with the dynamics of inviscid fluids, such as theoretical problems of Euler equations, inviscid vortex dynamics and vortex methods, stability, dynamics of interfaces and surface waves, compressible flows. Though these issues seem to have little to do with genuine turbulence, there are views/beliefs that such things like possible singularity formation and collapse in Euler flows and that the infinite Reynolds number limit of turbulent flow is described by singular solutions of Euler equations . Some people regard these as "very attractive scenarios". They are definitely very attractive and mathematically beautiful (since Onsager 1949), but it is more than not clear whether they have anything to do with real turbulence at whatever large Reynolds numbers. One cannot take seriously claims like "*The existence of such near singularities for turbulent velocity fields at high Reynolds number has been confirmed by data from experiments and simulations*" or *"Observations from experiments and simulations suggest that material objects advected by such a rough velocity become fractal…", since* all the experimental and numerical evidence is obtained at moderate Re_λ at which no singularities, fractal structure, etc. are expected and observed (if such exist at all). This evidence cannot be used as supporting any models at infinite Re_λ, which in principle cannot be confirmed or disproved by experimental or numerical evidence. We will return to these issues in a later lecture on mathematical issues including the question " how much mathematics does help to understand turbulence?" We will make a relevant *ad hoc* reminding below

There is little (if any) treatment of dynamical aspects of turbulent flows (e.g. those corresponding to those described by NSE in Eulerian setting) in Lagrangian setting (one of our main concerns here). One of the reasons is the view that

A principal objective of any theory of fluid motion is the prediction of the spread of matter or "tracer" within the fluid. BENNET 2006

But the main reasons take their origin in the difficulties to handle the Lagrangian equations and related issues.

Is it true that dynamical issues *per se* can be treated satisfactory in Eulerian setting only? Is there any need to use for this purpose the Lagrangian setting too? Are there problems which require such an approach?

SOME BASIC INFORMATION

Eulerian and Lagrangian descriptions

In what is called Eulerian description the observation of the system is made in a *fixed* frame as the fluid goes by. In this case the motion is characterized by the velocity field $\mathbf{u}(\mathbf{x},t)$ as a function of position vector, $\mathbf{x}$, and time t. In the Lagrangian description the observation is made *following* the fluid particles(wherever they move)*. Here the dependent variable is the position of a fluid particle $\mathbf{X}(\mathbf{a},t)$, as a function of the particle label, $\mathbf{a}$, (usually it's initial position, i.e. $\mathbf{X}(\mathbf{a},0) \equiv \mathbf{a}$, and time t. The relation between the two ways of description is given by the following equation **

$$\frac{\partial \mathbf{X}(\mathbf{a},t)}{\partial t} = \mathbf{u}[\mathbf{X}(\mathbf{a},t);t] \qquad \text{(E-L)}$$

i.e. the Lagrangian velocity field, $\mathbf{v}(\mathbf{a},t) = \frac{\partial \mathbf{X}(\mathbf{a},t)}{\partial t}$ is related to the Eulerian velocity field, $\mathbf{u}(\mathbf{x},t)$ as $\mathbf{V}(\mathbf{a},t) \equiv \mathbf{u}[\mathbf{X}(\mathbf{a},t);t]$.

If the Eulerian velocity field is known/given – as in all problems of kinematic nature – then the above equation serves for determination of the trajectory of a fluid particle with the initial position $\mathbf{X}(0) \equiv \mathbf{a}$. This equation is *nonlinear* (for almost all) even for very simple fluid flows and is generically *non-integrable* for (again almost) all such flows, i.e. the fluid particle trajectories are chaotic. We will call (as in Tsinober 2001) these Lagrangian chaotic flows as L-turbulent which may be both E-laminar or E-turbulent. This chaotic property of the trajectories of the fluid particles makes it more difficult to follow them, i.e. much more difficult to utilize the Lagrangian description of even the simplest fluid flows which exhibit Lagrangian chaos.

In other words it can be claimed that

Lagrangian description =

NSE (i.e Euler) + the equation

$$\frac{\partial \mathbf{X}(\mathbf{a},t)}{\partial t} = \mathbf{u}[\mathbf{X}(\mathbf{a},t);t] \qquad \text{(E-L)}$$

Alternatively one can wrirte up the equations directly in the Lagrangian variables

LAGRANGIAN	EULERIAN
Observation following the system wherever it moves	Observation of the system in a fixed frame as fluid goes by
INDEPENDENT	*VARIABLES*
$\mathbf{a},\ t$	$\mathbf{x},\ t$
DEPENDENT	*VARIABLES*
$\mathbf{X}(\mathbf{a},t)$	$\mathbf{U}(\mathbf{x},t)$
GOVERNING	*EQUATIONS*
$\mathcal{D}(X_i)/\mathcal{D}(a_j) \equiv [X_1,\ X_2,\ X_3] = 1$	$\partial U_i/\partial x_i = 0$
$\partial^2 X_i/\partial^2 t = [X_j,\ X_k,\ p] +$ $\nu[X_n,\ X_{n+1},\ [X_n,\ X_{n+1},\ \partial X_i/\partial t],$ $(3+1 \Rightarrow 1)$	$Du_i/Dt \equiv \partial U_i/\partial t + U_k \partial U_i/\partial x_k =$ $-\partial p/\partial x_i + \nu \nabla^2 U_i$
$\partial\vartheta/\partial t = \chi[X_n,\ X_{n+1},\ [X_n,\ X_{n+1},\ \vartheta],$	$\partial\theta/\partial t + u_k \partial\theta/\partial x_k = \chi\nabla^2\theta^*$
Viscous terms are strongly nonlinear (fifth order nonlinearity)	Pressure is inherently 'non-Lagrangian' due to its nonlocality

LAGRANGIAN	EULERIAN
Observation following the system wherever it moves	Observation of the system in a fixed frame as fluid goes by
INDEPENDENT	*VARIABLES*
$\mathbf{a},\ t$	$\mathbf{x},\ t$
DEPENDENT	*VARIABLES*
$\mathbf{X}(\mathbf{a}, t)$	$\mathbf{U}(\mathbf{x}, t)$
GOVERNING	*EQUATIONS**
$\mathcal{D}(X_i)/\mathcal{D}(a_j) \equiv [X_1,\ X_2,\ X_3] = 1$	$\partial U_i/\partial x_i = 0$
$\partial^2 X_i/\partial^2 t = [X_j,\ X_k,\ p] +$ $\nu[X_n,\ X_{n+1},\ [X_n,\ X_{n+1},\ \partial X_i/\partial t],$ $(3+1 \Rightarrow 1)$	$Du_i/Dt \equiv \partial U_i/\partial t + U_k \partial U_i/\partial x_k =$ $-\partial p/\partial x_i + \nu \nabla^2 U_i$
$\partial \vartheta/\partial t = \chi[X_n,\ X_{n+1},\ [X_n,\ X_{n+1},\ \vartheta],$	$\partial \theta/\partial t + u_k \partial \theta/\partial x_k = \chi \nabla^2 \theta$**
Viscous terms are strongly nonlinear (fifth order nonlinearity)	Pressure is inherently 'non-Lagrangian' due to its nonlocality
Rate of strain is nonlocal in some sense	

$\ast$ see, e.g. Corrsin 1962; Monin & Yaglom 1971, ch 5, section 9.1.

$\ast\ast$The 'solution' to this equation with $\chi = 0$ and with an intial condition $\theta(\mathbf{x},0) = \delta(\mathbf{x})$ is $\theta(\mathbf{x},t) = \delta\{\mathbf{x} - \mathbf{X}(0,t)\}$,i. e. the PDF of $\theta(\mathbf{x},t)$ coinsides with the one of $\mathbf{X}(0.t)$. This means if the scalar field marked particle by particle so that $\theta(\mathbf{x},t) = \int \theta(\mathbf{a})\delta\{\mathbf{x} - \mathbf{X}(\mathbf{a},t)\}d\mathbf{a}$ then the equations $\partial\theta/\partial t + (\mathbf{U}\cdot\mathbf{\nabla})\theta = 0$ and $\frac{\partial \mathbf{X}(\mathbf{a},t)}{\partial t} = \mathbf{U}[\mathbf{X}(\mathbf{a},t); t]$ are equivalent.

However, in pure Lagrangian setting the equations are intractable (so far) and in order to obtain true (not modelling!) Lagrangian information one has to solve the problem in Eulerian setting and using this information toghetre with the equation (E-L) one can obtain the Lagrangian dynamics of any fluid particle. As the Euler information is defined on the computational grid it is necessary to use an appropriate/adequate interpolation scheme.

* The Lagrangian description is the analog of the classical particle mechanics where individual particles are labeled and tracked. It is also analog to Heisenberg representation in quantum mechanics. The Eulerian description is the analog to tge Schrodinger representation in quantum mechanics.

**The relation between the two ways of description can be seen also by looking at any conservative property of fluid particles (i.e. a nondiffusive passive scalar) such as nondiffusive 'dye' or any other (e.g. radioactive) label. Due its conservative character it is time independent in the Lagrangian description, i.e. has the form $\vartheta(\mathbf{a})$, but is time dependent in some fixed point of space, $\mathbf{x}$, i.e. in the Eulerian description, and has the form $\theta(\mathbf{x},t)$. Hence, both are related via $\vartheta(\mathbf{a}) = \theta[\mathbf{X}(\mathbf{x},t),t]$. Since $\frac{\partial\vartheta(\mathbf{a})}{\partial t} = \frac{D\theta}{Dt} = 0$ it follows that $\frac{\partial\theta}{\partial t} + u_k\frac{\partial\theta}{\partial x_k} = 0$, which is just an expression of the fact that the material derivative of any Lagrangian conservative property should vanish (see Monin and Yaglom, 1971).

There are mixed, i. e. Eulerian-Lagrangian descriptions, which date back to 19th century, see references in Lamb 1932 and Cartes et al 2007, *Phys Fluids,* 19, 077101/1-7

LAGRANGIAN	EULERIAN
Observation following the system wherever it moves	Observation of the system in a fixed frame as fluid goes by
INDEPENDENT	*VARIABLES*
$\mathbf{a}, t$	$\mathbf{x}, t$
DEPENDENT	*VARIABLES*
$\mathbf{X}(\mathbf{a}, t)$	$\mathbf{U}(\mathbf{x}, t)$
GOVERNING	*EQUATIONS**
$\mathcal{D}(X_i)/\mathcal{D}(a_j) \equiv [X_1, X_2, X_3] = 1$	$\partial U_i/\partial x_i = 0$
$\partial^2 X_i/\partial^2 t = [X_j, X_k, p] +$ $\nu[X_n, X_{n+1}, [X_n, X_{n+1}, \partial X_i/\partial t],$ $(3+1 \Rightarrow 1)$	$Du_i/Dt \equiv \partial U_i/\partial t + U_k \partial U_i/\partial x_k =$ $-\partial p/\partial x_i + \nu \nabla^2 U_i$
$\partial \vartheta/\partial t = \chi[X_n, X_{n+1}, [X_n, X_{n+1}, \vartheta],$	$\partial \theta/\partial t + u_k \partial \theta/\partial x_k = \chi \nabla^2 \theta^*$
Viscous terms are strongly nonlinear (fifth order nonlinearity)	Pressure is inherently non-Lagrangian due to its nonlocality
Rate of strain is nonlocal in some sense	

There is an elegant version of the Lagrangian equations for an inviscid (!) flows using instead of the Lagrangian Variables $X_i(\mathbf{a},t)$ it is using the Jacobi matrix $\partial(X_i)/\partial(a_j)$ which completely describes the fluid motion (Yakubovich & Zenkovich 2001; Bennet 2006)

EULERIAN	LAGRANGIAN
Observation of the system in a fixed frame as fluid goes by	Observation following the system wherever it moves

INDEPENDENT	*VARIABLES*
$\mathbf{x},\ t$	$\mathbf{a}$(particle label, e.g. initial position), t

DEPENDENT	*VARIABLES*
$\mathbf{U}(\mathbf{x}, t)$	$\mathbf{X}(\mathbf{a}, t)$ - particle position at time t

GOVERNING	EQUATIONS
$\partial u_i/\partial x_i = 0$	$\mathcal{D}(X_i)/\mathcal{D}(a_j) \equiv [X_1,\ X_2,\ X_3] = 0$
$Du_i/Dt \equiv \partial u_i/\partial t + u_k \partial u_i/\partial x_k =$	$\partial^2 X_i/\partial^2 t = [X_j,\ X_k,\ p] +$
$-\partial p/\partial x_i + \nu \nabla^2 u_i$	$\nu[X_n,\ X_{n+1},\ [X_n,\ X_{n+1},\ \partial X_i/\partial t],$
$\partial\theta/\partial t + u_k \partial\theta/\partial x_k = \chi \nabla^2 \theta$	$\partial\vartheta/\partial t = \chi[X_n,\ X_{n+1},\ [X_n,\ X_{n+1},\ \vartheta],$
Pressure is inherently non-Lagrangian due to its nonlocality	Viscous terms are strongly nonlinear (fifth order nonlinearity)

Conservation laws in ideal fluid flows.

Definition 1.6 *The* **circulation** *(or* **flux of vorticity***) is defined by*

$$\kappa = \oint_C \mathbf{u} \cdot d\mathbf{l} = \int_S \boldsymbol{\omega} \cdot \hat{\boldsymbol{\nu}} \, d\sigma \,, \tag{5}$$

where C (of elementary directional length $d\mathbf{l}$) is a simple unknotted, closed circuit $C \equiv \partial S$, and S (of elementary area $d\sigma$) is a simply connected two-dimensional surface of unit normal $\hat{\boldsymbol{\nu}}$, pointing in the positive direction induced by the $\boldsymbol{\omega}$-field.

The two integrals are related by Stokes's theorem and in ideal conditions (Euler's equations) the common value $\kappa =$ constant is an *invariant* of fluid motion (Helmholtz's III law and Kelvin's theorem; see [30]). Moreover,

Definition 1.7 *The vorticity field $\boldsymbol{\omega}$ is said to be* **frozen** *in $\mathcal{D}$ if and only if it satisfies the transport (Helmholtz) equation*

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) \,. \tag{6}$$

Ricca, 2001 , *Geometric and topological aspects of vortex motion,* in RL Ricca (ed.), An introduction to the the Geometry and Topology of Fluid Flows, pp. 203-228 Kluwer

Conservation laws in ideal fluid flows.

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) \ . \tag{6}$$

A formal solution to (6) is represented by the Cauchy equations

$$\omega_i(\mathbf{x}, t) = \omega_j(\mathbf{a}, t_0) \frac{\partial x_i}{\partial a_j} \ , \tag{7}$$

that encapsulate both convection of the $\boldsymbol{\omega}$-field from $\mathbf{a}$ to $\mathbf{x}$, and rotation and distortion of the fluid elements by the deformation tensor $\partial x_i / \partial a_j$ (incompressibility is given by the condition $\det(\partial x_i / \partial a_j) = 1$). Since the tensor is a time-dependent diffeomorphism of position, it maps continuously (i.e. without cuts or reconnections) the initial field $\boldsymbol{\omega}(\mathbf{a}, t_0)$ to $\boldsymbol{\omega}(\mathbf{x}, t)$, thus establishing a *topological equivalence* between initial and final configuration (Figure 3). We write

$$\boldsymbol{\omega}(\mathbf{a}, t_0) \sim \boldsymbol{\omega}(\mathbf{x}, t) \ , \tag{8}$$

and we regard equation (6) as a master equation for frozen fields and equation (7) as a topological equivalence statement for the initial and final configuration fields.

2.1. LOCAL INVARIANTS AND DIFFERENTIAL FORM CONSERVATION LAWS

Local fluid invariants can be classified in four categories:

I type: *conserved quantity* ρ (e.g. mass per unit volume).
Governing equation for scalar quantities as a balance conservation law:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \ .$$

II type: *Lagrangian invariant* $\Im$ (e.g. a passive scalar, like ink).
Governing equation for scalar quantities advected Lagrangian invariantly by the flow:

$$\frac{d\Im}{dt} = \frac{\partial \Im}{\partial t} + (\mathbf{u} \cdot \nabla)\Im = 0 \ .$$

III type: *frozen-in vector field* $\boldsymbol{\omega}$ (e.g. vorticity).
Governing equation for vector quantities advected along the flow streamlines:

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\mathbf{u} \times \boldsymbol{\omega}) \ .$$

IV type: *Frobenius invariant* $\mathbf{S}$ (e.g. momentum of a vortex ring).
Governing equation for vector quantities advected by Frobenius-type surfaces frozen in the flow (see Figure 4):

$$\frac{d\mathbf{S}}{dt} = (\mathbf{S} \times \nabla) \times \mathbf{u} \ .$$

All local fluid invariants can be classified in these four categories. These four types of invariants can be expressed in terms of differential forms, each one corresponding to an invariant ω^p-form ($p = 0, 1, 2, 3$) obeying the conservation law

$$\mathcal{L}_{t,\mathbf{u}}\omega^p \equiv \frac{\partial \omega^p}{\partial t} + \mathcal{L}_{\mathbf{u}}\omega^p = 0 \ . \qquad (9)$$

As an important addition (at the enxt slide) it is reminded that the modified helicity (discussed in the lecture on helicity last year) with a special choice of the gauge is is also alocal (pointwise) Lagrangian invariant

MODIFIED HELICITY

Helicity is a global quantity which in many cases is not well defined. It appears that one can choose the gauge φ in such a way that the helicity density is a Lagrangian (non-dissipative) invariant, i.e. it is conserved (pointwise) along the paths of fluid particles and therefore for any fluid volume. Such a choice is possible both for magnetic field (ELSASSER, 1956; CHILDRESS & GILBERT, 1995) and for nonconducting fluid flows (KUZMIN, 1983; OSELEDETS, 1989). It is possible to do so also for a viscous flow (OSELEDETS, 1989) chosing φ obeying the equation

$$D\varphi/Dt = p - u^2/2 + \nu\nabla^2\varphi$$

Then the modified helicity density $h_m = \boldsymbol{\omega}\cdot\mathbf{v}$, with $\mathbf{v} = \mathbf{u}+\nabla\varphi$ satisfies the equation

$$Dh_m/Dt = \nu\{\nabla^2 h_m - 2(\partial\omega_i/\partial x_k)(\partial v_i/\partial x_k)\}$$

i.e. is a Lagrangian invariant if $\nu = 0$.

Conservation laws in ideal fluid flows.

In absence of dissipative and diffusive effects the invariance of circulation is of course just one manifestation of the ideal conditions of fluid motion. In this context it is natural to expect the existence of families of such quantities (not all necessarily scalars). One possible classification of invariants is based on their nature:

(*i*)	*local*	(*metric*)	$\longleftrightarrow$	*pointwise*
(*ii*)	*global*	(*metric*)	$\longleftrightarrow$	*integral*
(*iii*)	*topological*	(*non-metric*)	$\longleftrightarrow$	*algebraic*

Ricca, R.L. 2001 , *Geometric and topological aspects of vortex motion*, in RL Ricca (ed.), An introduction to the the Geometry and Topology of Fluid Flows, pp. 203-228 Kluwer.
Tur, A. and Yanovsky, V. 1993 , Invariants in dissipationless hydrodynamic media, *J. Fluid Mech.*, 248, 67-106.

The Lagrangian conservation laws are a consequence of Noether's theorem due to the so called relabelling symmetry (Salmon 1988, Bennet 2006). In fact, there are many such invariants all frequently called Cauchy invariants and related in some way to the conservation of circulation.

We are now ready for a historical digression.

A REMINDING ON WHY IDEAL CONSERVATION LAWS DO NOT HOLD IN TURBULENT FLOWS AT ANY REYNOLDS NUMBERS

Vorticity is not "approximately" frozen if fluid flows at any Reynolds number and the Kelvin theorem is violated. Turbulence is not a slightly viscous phenomenon.

The main (but not the only) point of concern here is that vorticity is not “approximately” frozen if fluid flows at any Reynolds number and the Kelvin theorem is violated: turbulence is not a slightly viscous phenomenon. So as in Euler approach one has to deal with equations explicitly containing viscosity. In pure Lagrangian setting these equations are intractable (so far) and in order to obtain true (not modelling!) Lagrangian information one has to solve the problem in Eulerian setting. Using this information one can obtain the Lagrangian dynamics of any fluid particle. As the Euler information is defined on the computational grid it is necessary to use an appropriate/adequate interpolation scheme.

Experimentally direct partice tracking velocimetry allows to access Lagrangian information at low to moderate Re_{λ}. We will get to the issue of methods (with the emphasis on the experimental ones) in a later lecture.

In an inviscid flow

$$D\boldsymbol{\omega}/Dt = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u}; \quad D\mathbf{l}/Dt = (\mathbf{l}\cdot\nabla)\mathbf{u}$$

$$D(\boldsymbol{\omega}-\mathbf{l})/Dt = \{(\boldsymbol{\omega}-\mathbf{l})\cdot\nabla\}\mathbf{u}$$

So $\boldsymbol{\omega} = \mathbf{l}$ at all times if initially $\boldsymbol{\omega} - \mathbf{l} = 0$;

However, in a flow with $\nu \neq 0$ whatever small an approximate balance (Tennkes & Lumley) holds $\langle \omega_i \omega_k s_{ik} \rangle \approx \nu \langle \omega_i \nabla^2 \omega_i \rangle$, holds, i.e. the vortex lines are not frozen into the fluid at whatever high Reynolds number (vortex lines are loosing their identity as material lines at any Reynolds number) – otherwise how the enstrophy production can be approximately balanced by viscous terms again at any – whatever large – Reynolds number.

In other words in slightly viscous flows frozennes is meaningless with a consequent (not slight!) violation of the Kelvin circulation theorem - just like the claim that turbulence is slightly viscous at whatever large Re. In this context the question: what happens with enstrophy and strain production as $\nu \to 0$ is of special interest.

The above is not entirely new (at least in part) ..

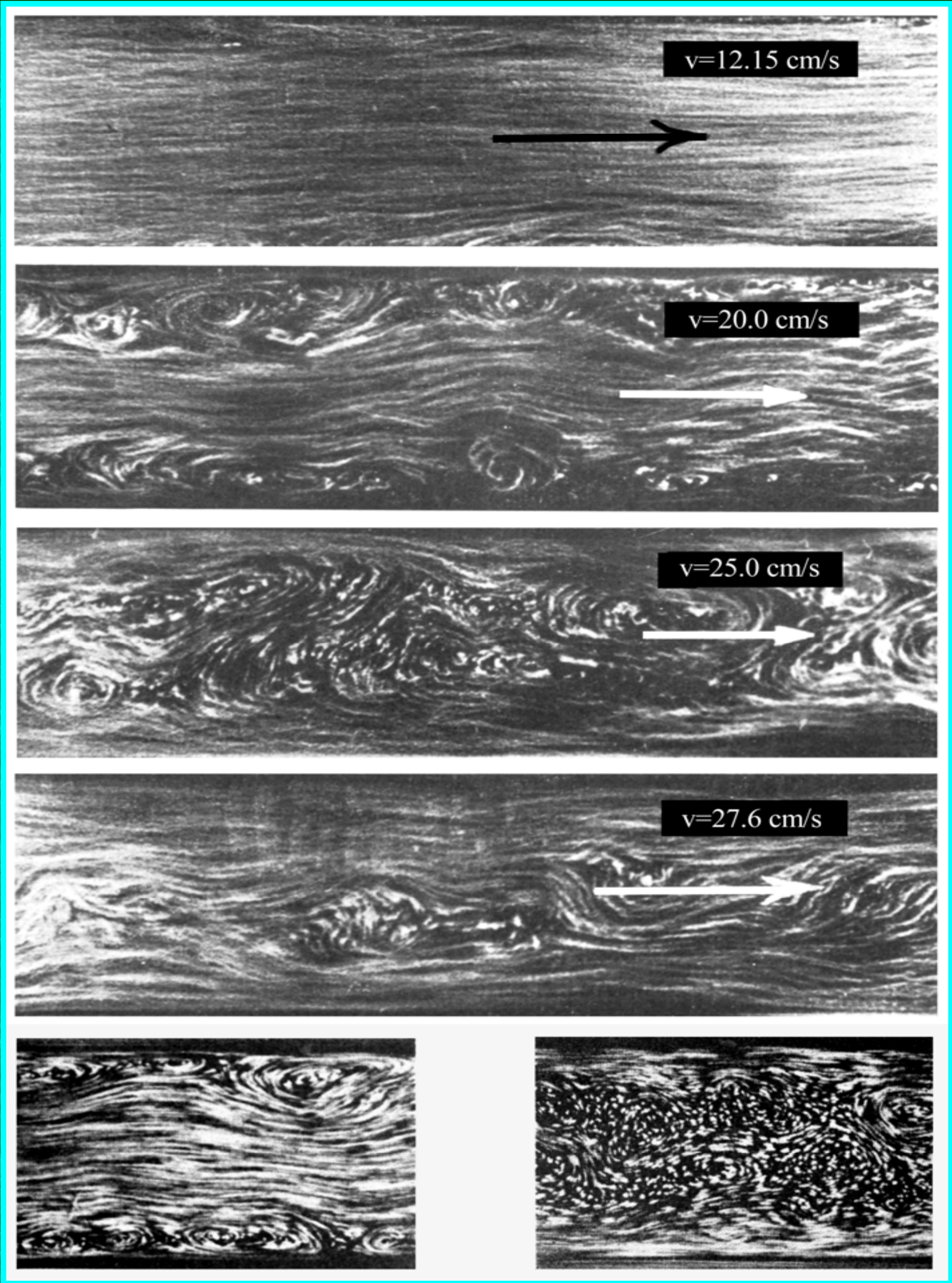
. *a material line which is initially coinsiding with a vortex line continues to do so. It is thus possible and convenient to regard a vortex-line as having a continuing identity and as moving with the fluid (In a viscous fluid it is, of course, possible to draw the pattern of vortex lines at any instant, but there is no way in which particular vortex-line can be identified at different instants).* BATCHELOR, 1967, p. 274

Some Issues of Conceptual Nature

Accelerations, Random Taylor Hypothesis (RTH) or sweeping decorrelation hypothesis. Accelerations. In what sense is RTH valid?

The issue concerns a commonly used concept known as the *random Taylor hypothesis* or the sweeping decorrelation hypothesis which is an important generalization of the common Taylor hypothesis. It was suggested by TENNEKES 1975 that in turbulence at high Reynolds number *the dissipative eddies flow past an Eulerian observer in a time frame much shorter than the time scale which characterizes their own dynamics,* i.e. $\tau_E / \tau_L \sim \mathrm{Re}^{-1/4}$ (hence in the Lagrangian setting the correlations are expected to be much larger which is mostly - but not always - the case).

In turn this implies that Taylor's "frozen turbulence" approximation would be valid (at least qualitatively) for the analysis of the advection of the turbulence microstructure by the large-scale motions. Already at the very beginning one encounters an ambiguity (not the only one) as the time scale (like many other things) of the *Eulerian observer* depends on the velocity of the Eulerian frame in which the *Eulerian observer* lives:



The four upper pictures, from TOLLMIEN 1931 correspond to the visualization of a turbulent water flow in an open 6 cm wide channel photographed by a moving camera at various speeds. The mean velocity of the flow is 16.7cm/s.

The two lower pictures are from PRANDTL AND TIETJENS 1934. In the right picture, the camera moves with the speed equal to the velocity of water in the center of the channel. In the left picture, the speed of the camera is small and close to the velocity of the water near the walls.

CONVENTIONAL TAYLOR HYPOTHESIS

$$\frac{\partial}{\partial x_1} = -U^{-1}\frac{\partial}{\partial t}$$

Works (roughly) when

$$\left\langle u^2 \right\rangle^{1/2} / U << 1$$

The explicit dynamic conditions are more complex, see

Uberoi, MS. and Corrsin, S 1952, Diffusion of heat from a line source in isotropic turbulence, *NACA Rep.,* 1142, 1953 {originally *NACA, TN* 2710, 1(52)}.

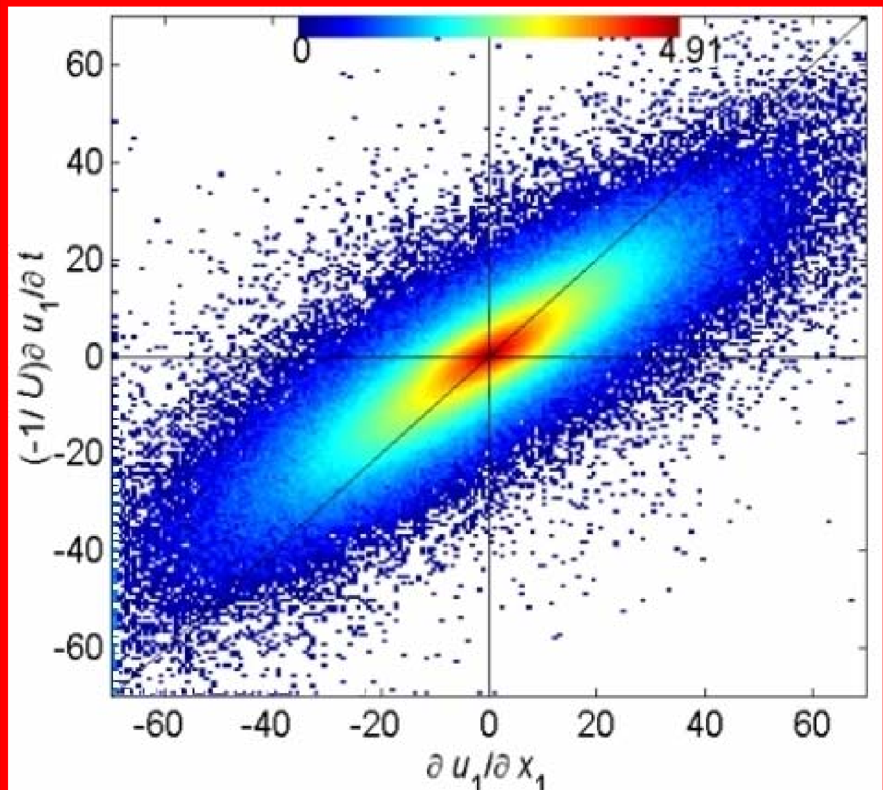
Lin, CC 1953, On Taylor Hypothesis and the acceleration terms in the Navier-Stokes Equations, *Quart. app. Math.* 10, 4.

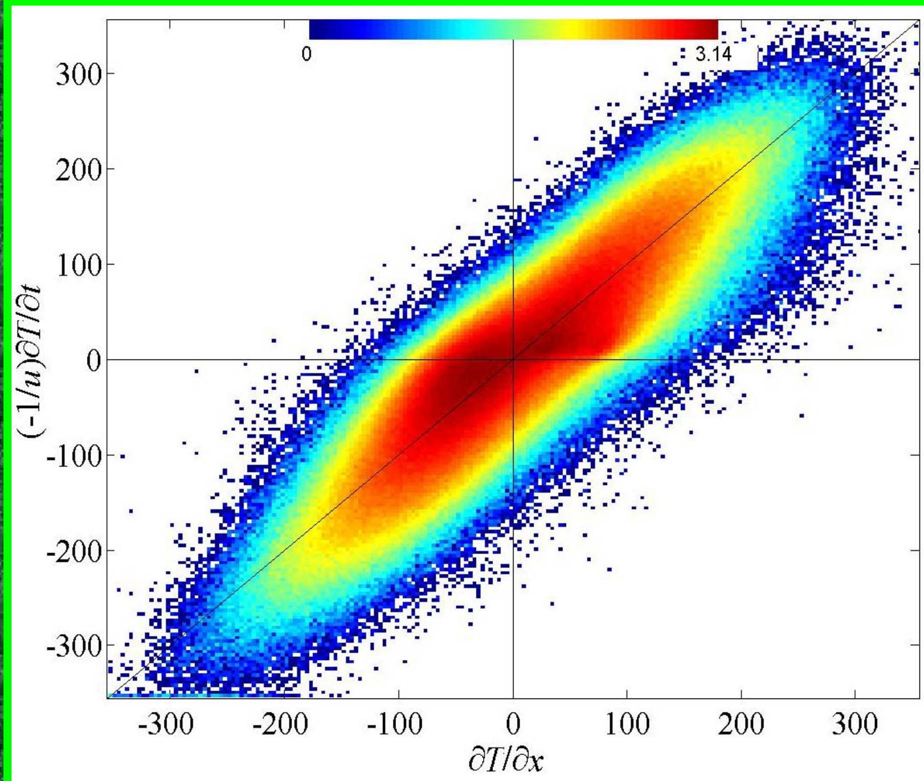
and references in Tsinober, A. Yeung, P.K. and P. Vedula, P. (2001) Random Taylor hypothesis and the behavior of local and convective accelerations in isotropic turbulence, *Physics of Fluids,* 13, 1974-1984.

JOINT PDFS FOR CONVENTIONAL TAYLOR HYPOTHESIS

velocity

temperature





To assess its validity it should be recognized that, in fact, Tennekes' hypothesis consists of two ingredients. ***First***, it is proposed that the Lagrangian acceleration $\mathbf{a}$ of fluid particles is in some sense small, such that time scales measuring Eulerian and Lagrangian rates of change could be estimated by simply setting $\mathbf{a} = 0$ which is good for the purpose of getting at least qualitaively correct estimate. However, it is obvious, that $\mathbf{a} = 0$ cannot be perfectly true (this has far more serious consequences in a number of conceptual issues (see below) than usually appreciated (can one imagine that the flow is governed by the equation like $\partial\mathbf{u}/\partial t + (\mathbf{u}\cdot\nabla)\mathbf{u} = 0$? or $\nabla p + \nu\nabla^2\mathbf{u} = 0$?)

It is noteworthy that the above assumption was formulated for the turbulent fluctuations (which are local pointwise in space and time) instead of for statistical measures of these quantities.

The ***second*** assumption made by Tennekes is of statistical nature, namely, that *the microstructure is statistically independent of the energy containing eddies*. This assumption appears also too strong and should be replaced by a more limited interpretation of this hypothesis in the sense that the microstructure is *statistically decorrelated* from the energy containing eddies as there is a growing body of experimental evidence (see refs in Tsinober 2001) showing that large and small scales are not statistically independent though weakly correlated (see the lecture on nonlocality). In this connection it is important to stress that the RTH is frequently misinterpreted (following the original Taylor hypothesis of frozenness) in the sense that the small scales are just swept as a kind of passive objects and in the dynamical context (at best) are just a passive sink of energy.

In the above issues one already encounters the grand and multi-faceted problem (as we will see below it is **the** problem) of the relationship between the Eulerian and Lagrangian settings (see also next slide). Indeed the **Lagrangian** acceleration $\mathbf{a}$ is represented via its **Eulerian** components $\mathbf{a}_l = \partial \mathbf{u}/\partial t$ and $\mathbf{a}_c = (\mathbf{u}\cdot\nabla)\mathbf{u}$ as $\mathbf{a} = \mathbf{a}_l + \mathbf{a}_c$. So if the RTH holds the two vectors have to be strongly anti-aligned (i.e. the angle θ between the two should $\sim \pi$) resulting in strong mutual cancellation.

$$\left|\mathbf{a}_l\right|, \left|\mathbf{a}_c\right| >> \left|\mathbf{a}\right|$$

This was really observed both in DNS and laboratory (field) experiments.

Lagrangian acceleration and its Euler 'components'

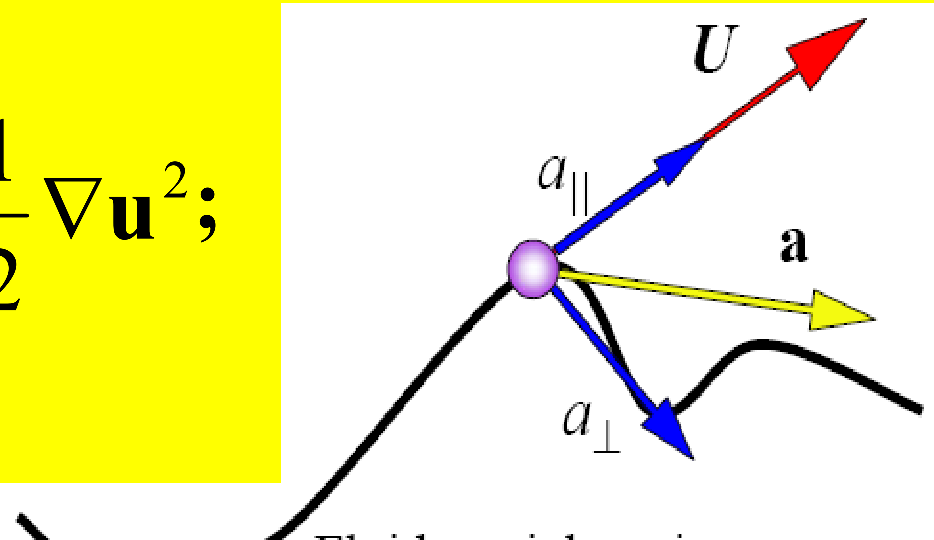
$$\mathbf{a} = \mathbf{a}_l + \mathbf{a}_c = \mathbf{a}_l + \mathbf{a}_L + \mathbf{a}_B = \mathbf{a}_\perp + \mathbf{a}_\parallel$$

$$\mathbf{a}_l = \frac{\partial \mathbf{u}}{\partial t}; \quad \mathbf{a}_c = (\mathbf{u}\cdot\nabla)\mathbf{u}; \quad \mathbf{a}_L = \boldsymbol{\omega}\times\mathbf{u};$$

$$\mathbf{a}_\perp = \mathbf{a} - \mathbf{a}_\parallel; \quad \mathbf{a}_B = \frac{1}{2}\nabla\mathbf{u}^2;$$

$$\mathbf{a}_\parallel = (\mathbf{a}\cdot\hat{\mathbf{u}})\hat{\mathbf{u}};$$

$$\hat{\mathbf{u}} = \mathbf{u}/u$$



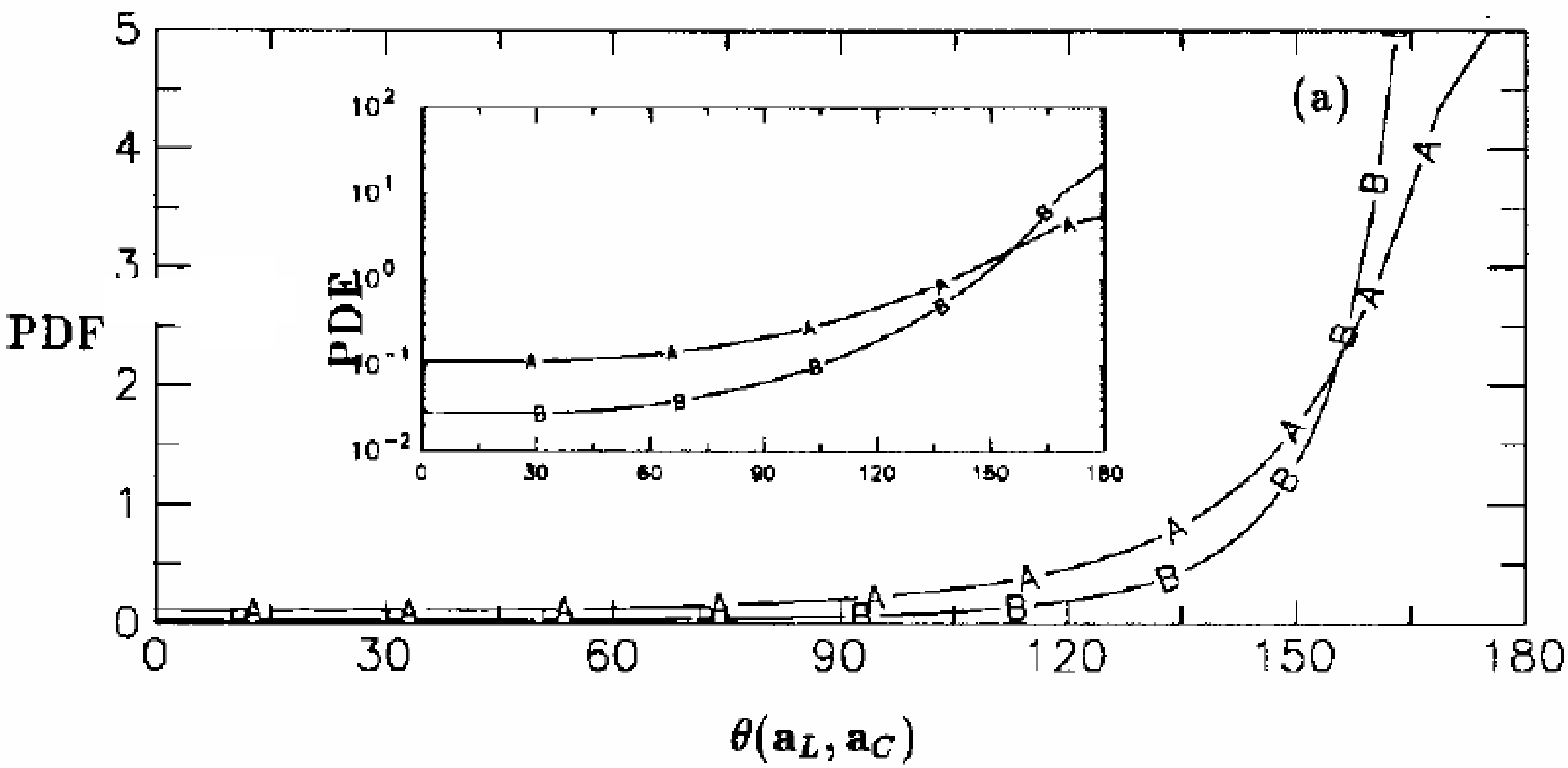
Fluid particle trajectory

Random Taylor Hypothesis Tennekes 1975

$$|\mathbf{a}_l|, |\mathbf{a}_c| >> |\mathbf{a}|$$

$$\mathbf{a} \equiv \frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}$$

$$\mathbf{a}_l = \frac{\partial \mathbf{u}}{\partial t}, \quad \mathbf{a}_c = (\mathbf{u} \cdot \nabla)\mathbf{u}$$



PDFs of $\theta(\mathbf{a}_L, \mathbf{a}_C)$ for R_λ 38 and 240 (lines A and B, respectively). The inset shows the same PDFs on a logarithmic scale.

1976 Phys. Fluids, Vol. 13, No. 7, July 2001

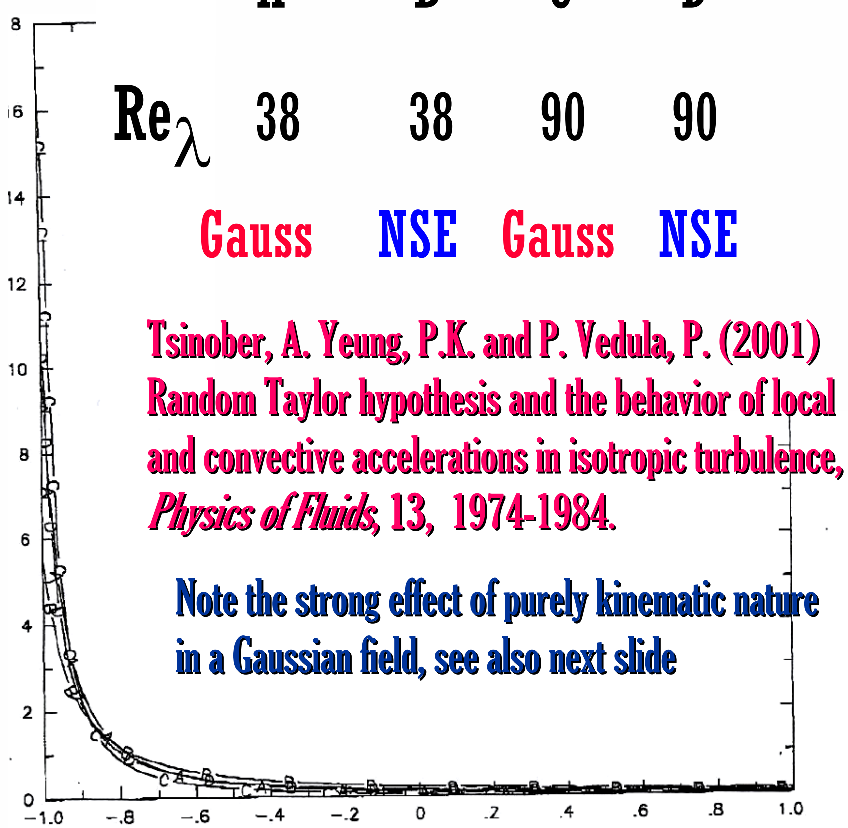
DNS

PDFS OF THE COSINE OF THE ANGLE BETWEEN

$$\mathbf{a}_l = \frac{\partial \mathbf{u}}{\partial t}$$

AND

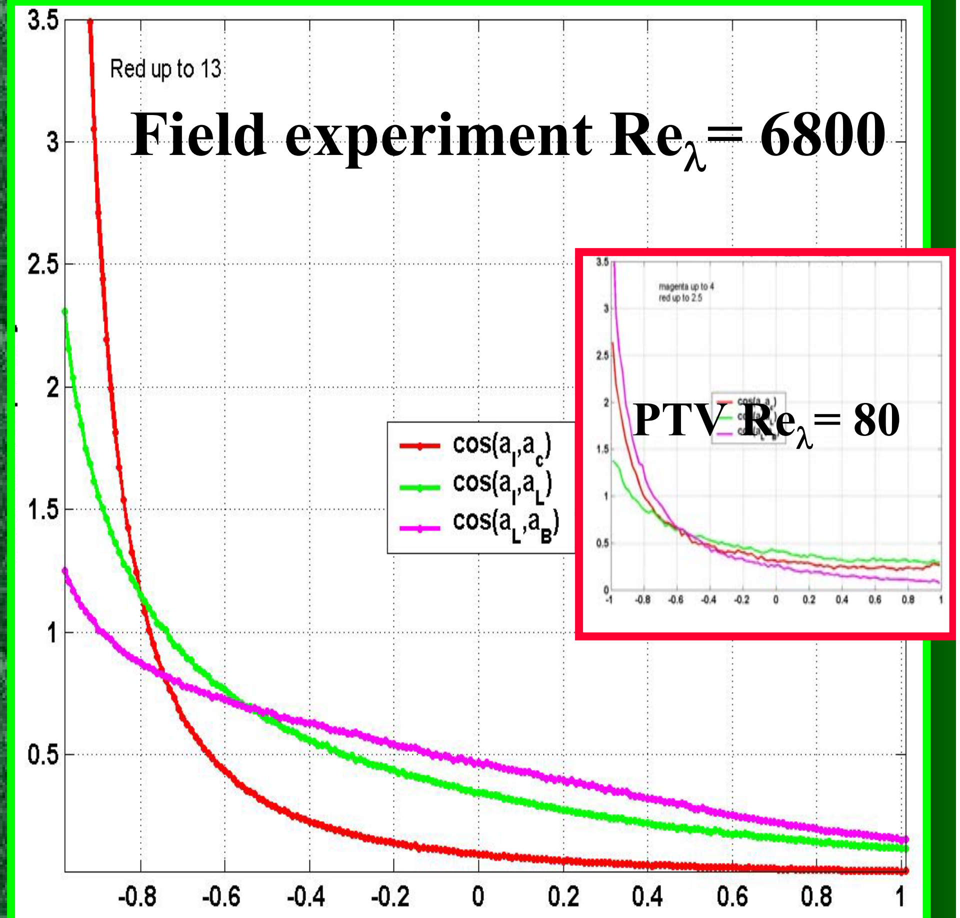
$$\mathbf{a}_c = (\mathbf{u}\cdot\nabla)\mathbf{u}$$



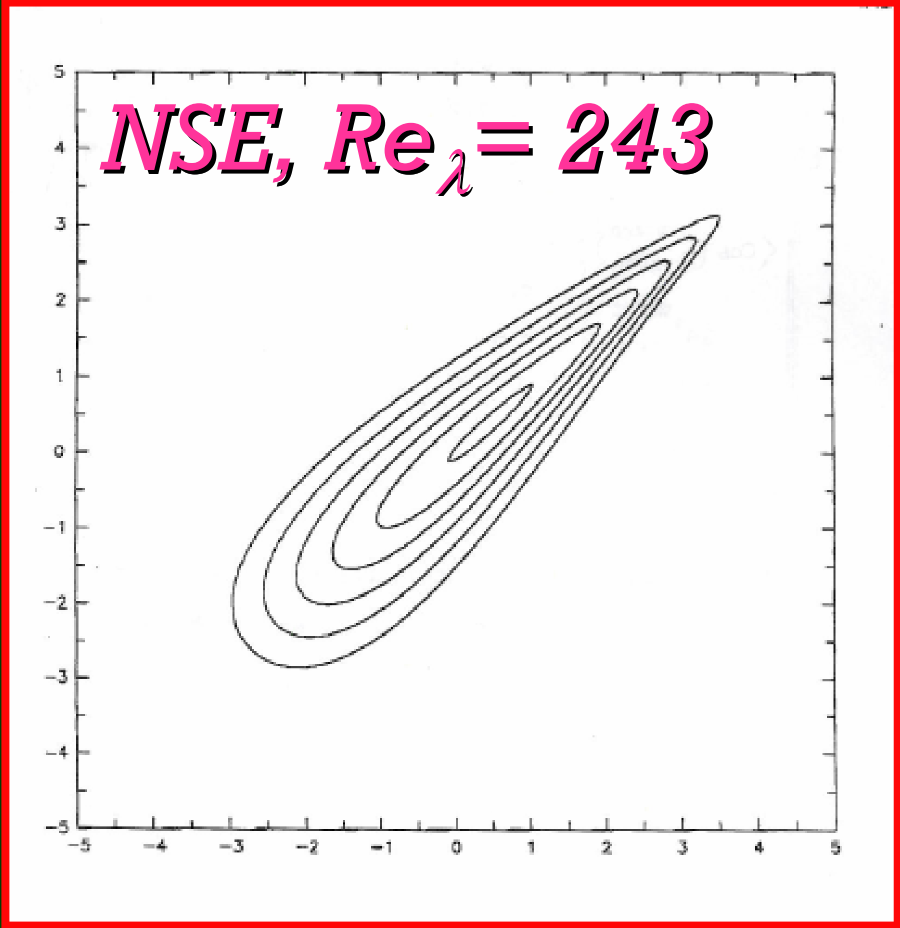
	A	B	C	D
Re_λ	38	38	90	90
	Gauss	NSE	Gauss	NSE

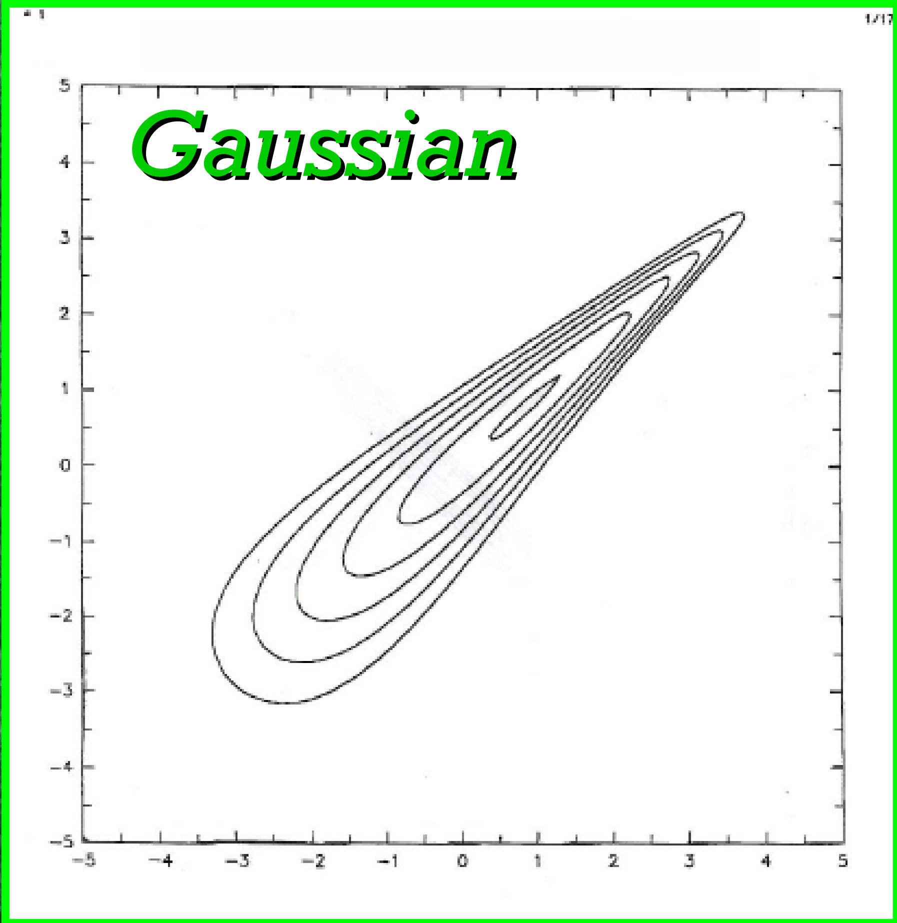
Tsinober, A. Yeung, P.K. and P. Vedula, P. (2001) Random Taylor hypothesis and the behavior of local and convective accelerations in isotropic turbulence, *Physics of Fluids*, 13, 1974-1984.

Note the strong effect of purely kinematic nature in a Gaussian field, see also next slide



JOINT PDFS OF $\mathbf{a}_l = \frac{\partial \mathbf{u}}{\partial t}$ AND $\mathbf{a}_c = (\mathbf{u} \cdot \nabla)\mathbf{u}$





MORE GENERALLY

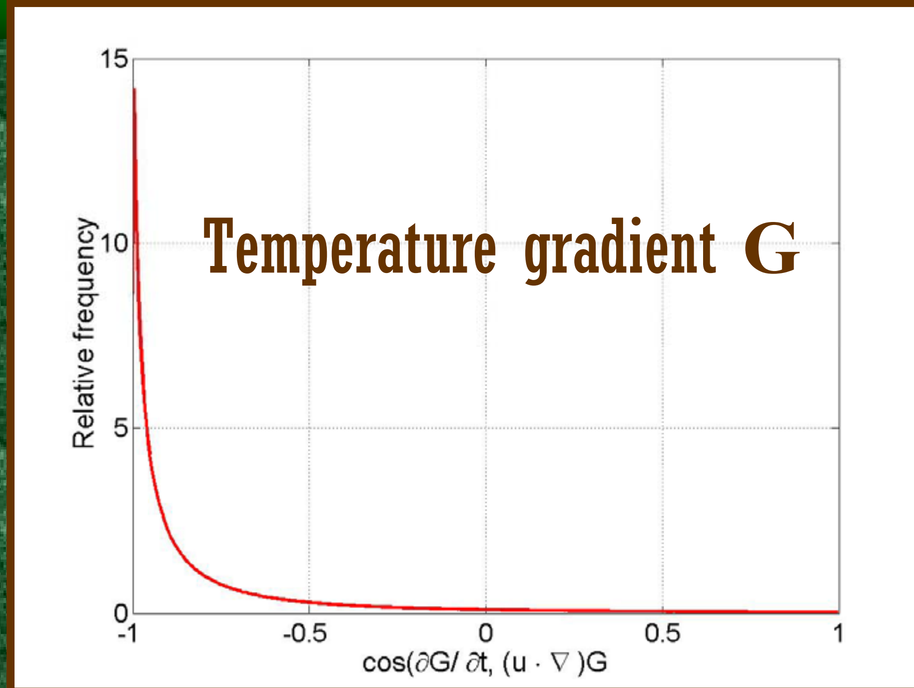
For any quantity (scalar, vector, tensor)

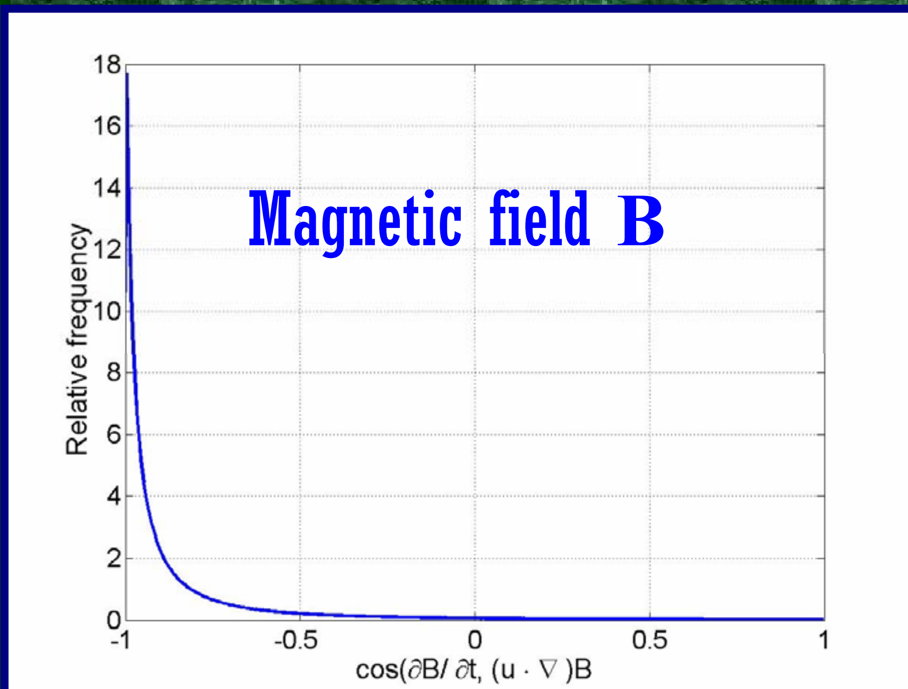
$$\left|\frac{DQ}{Dt}\right| \ll \left|\frac{\partial Q}{\partial t}\right|, \quad \left|(\mathbf{u}\cdot\nabla)Q\right|$$

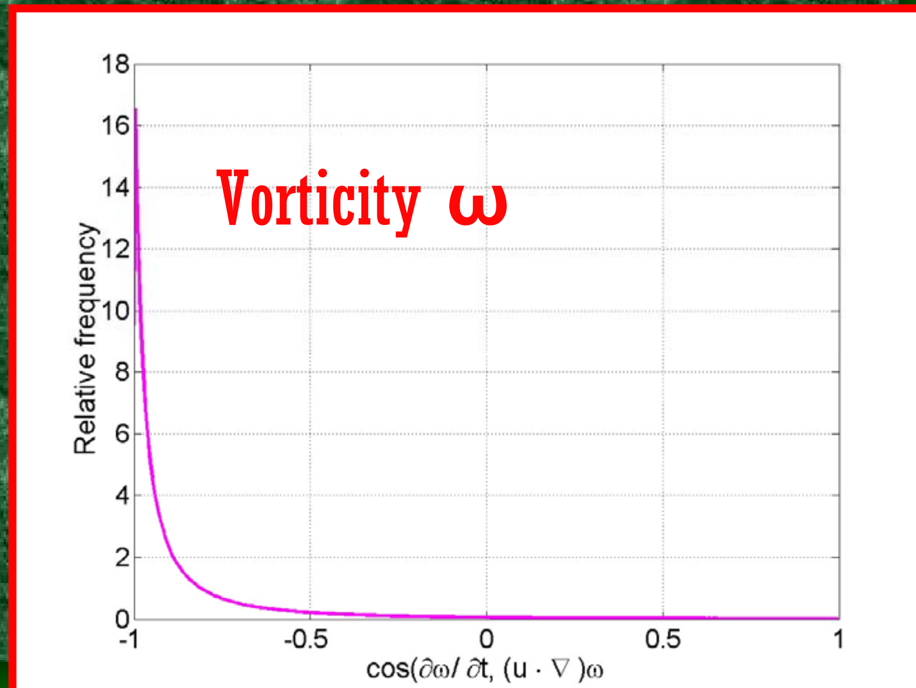
$$\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + (\mathbf{u}\cdot\nabla)Q$$

The Lagrangian derivative $DQ/Dt = \partial Q/\partial t + u_{\_k}\partial Q/\partial x_k$ is much smaller than its Eulerian components.

EXAMPLES FOR OTHER QUANTITIES from DNS of NSE







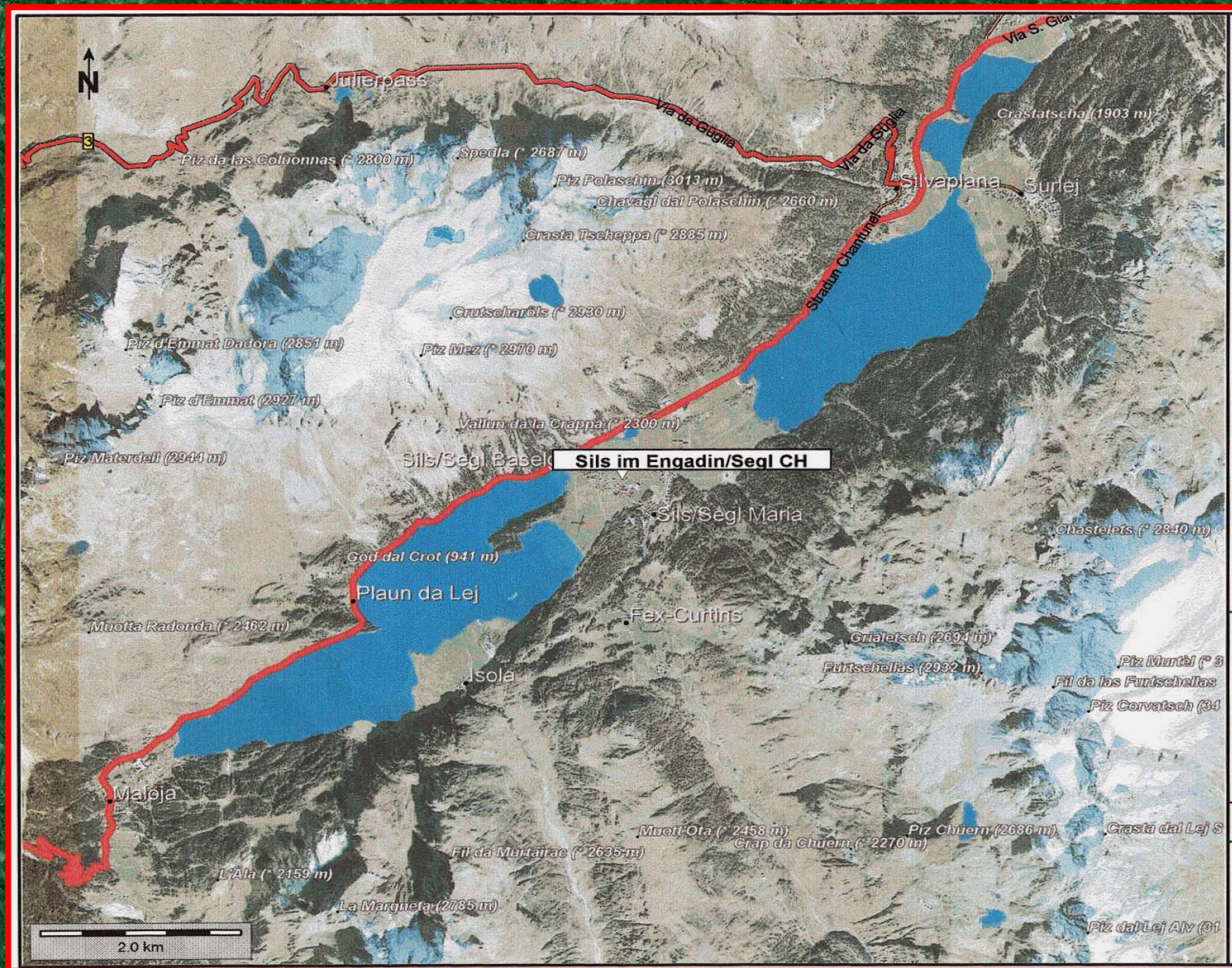
A SMALL EXCERCISE

Express the local convective accelerations in a frame moving with mean velocity $\mathbf{U}$ in terms of their values $\mathbf{a}_l = \dfrac{\partial \mathbf{u}}{\partial t}$ and $\mathbf{a}_c = (\mathbf{u} \cdot \nabla)\mathbf{u}$ in the frame with $\mathbf{U} = 0$.

AND A QUESTION

The question is whether the smallness of $DQ/Dt = \partial Q/\partial t + u_k \partial Q/\partial x_k$ as compared to both $\partial Q/\partial t$ and $u_k \partial Q/\partial x_k$ can be seen as an indication of the existence of a small parameter in turbulence theory.

KOLMOGOROV 4/5 LAW, NONLOCALITY AND SWEEPING DECORRELATION HYPOTHESIS


N
Julierpass
3
Via da Güglia
Via S. Gia
Piz da las Coluonnas (* 2800 m)
Spedla (* 2687 m)
Piz Polaschin (3013 m)
Chavagl dal Polaschin (* 2660 m)
Crasta Tscheppa (* 2885 m)
Crastatscha (1903 m)
Silvaplana
Surlej
Stradun Chantunel
Crutscharöls (* 2930 m)
Piz d'Emmat Dadora (2851 m)
Piz Mez (* 2979 m)
Piz d'Emmat (2927 m)
Vallun da la Crappa (* 2300 m)
Piz Materdell (2944 m)
Sils/Segl Baselg
Sils im Engadin/Segl CH
Sils/Segl Maria
Chastelets (* 2840 m)
God dal Crot (941 m)
Plaun da Lej
Fex-Curtins
Muotta Radonda (* 2462 m)
Grialetsch (2694 m)
Furtschellas (2932 m)
Piz Murtèl (* 3
Fil da las Furtschellas
Piz Corvatsch (34
Isola
Maloja
Muott'Ota (* 2458 m)
Crap da Chüern (* 2270 m)
Piz Chüern (2686 m)
Crasta dal Lej S
Fil da Murtairac (* 2635 m)
L'Ala (* 2159 m)
La Margneta (2785 m)
Piz dal Lej Alv (31
2.0 km

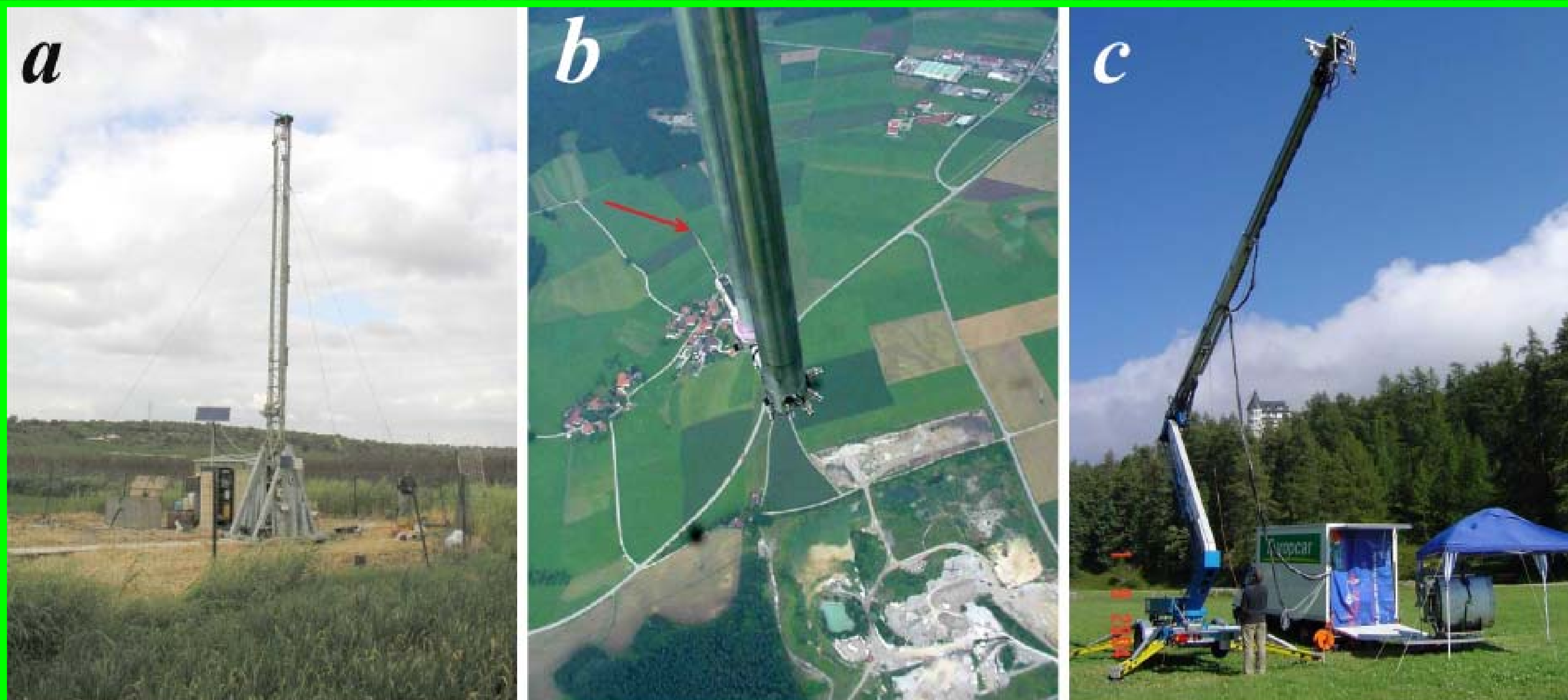
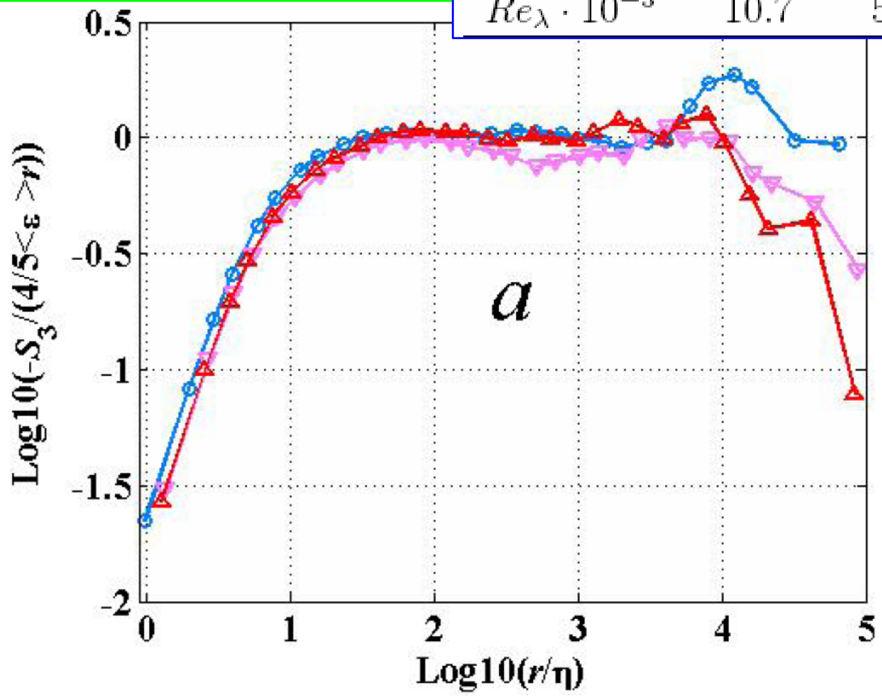


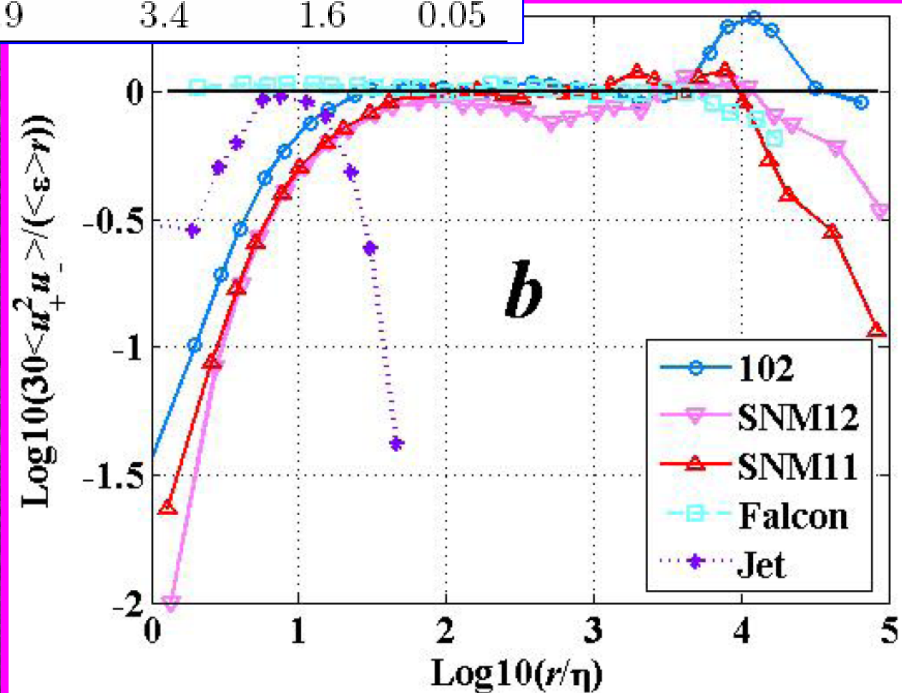
FIG. 1. Kfar Glikson measurement station, Israel, the probe on the mast (a). Airborne experiment, Germany, the probe in the flight (b). Sils-Maria experiment, Switzerland, the probe on the lifting machine (c).

Experiment	102,	SNM12	SNM11	Falcon	Jet
	Israel	Switzerland		Airborn	
$Re_\lambda \cdot 10^{-3}$	10.7	5.9	3.4	1.6	0.05

The Taylor micro-scale Reynolds numbers, Re_λ, for the experiments.

Experiment	102,	SNM12	SNM11	Falcon	Jet
$Re_\lambda \cdot 10^{-3}$	10.7	5.9	3.4	1.6	0.05





a **Conventional Kolmogorov 4/5 law 1941b**

$$\langle(\Delta u_1)^3\rangle = -4/5\varepsilon r$$

b **Equivalent Hosokawa's relation 2007**

$$\langle u_+^2 u_-\rangle = \varepsilon r/30$$

$$2u_+ = u_1(\mathbf{x}+\mathbf{r}) + u_1(\mathbf{x}),\ 2u_- = u_1(\mathbf{x}+\mathbf{r}) - u_1(\mathbf{x})$$

$u_1(\mathbf{x})$ is the longitudinal velocity component

Kolmogorov 4/5 law, non-locality and sweeping decorrelation hypothesis, M. Kholmyansky, and A. Tsinober (2007) submitted

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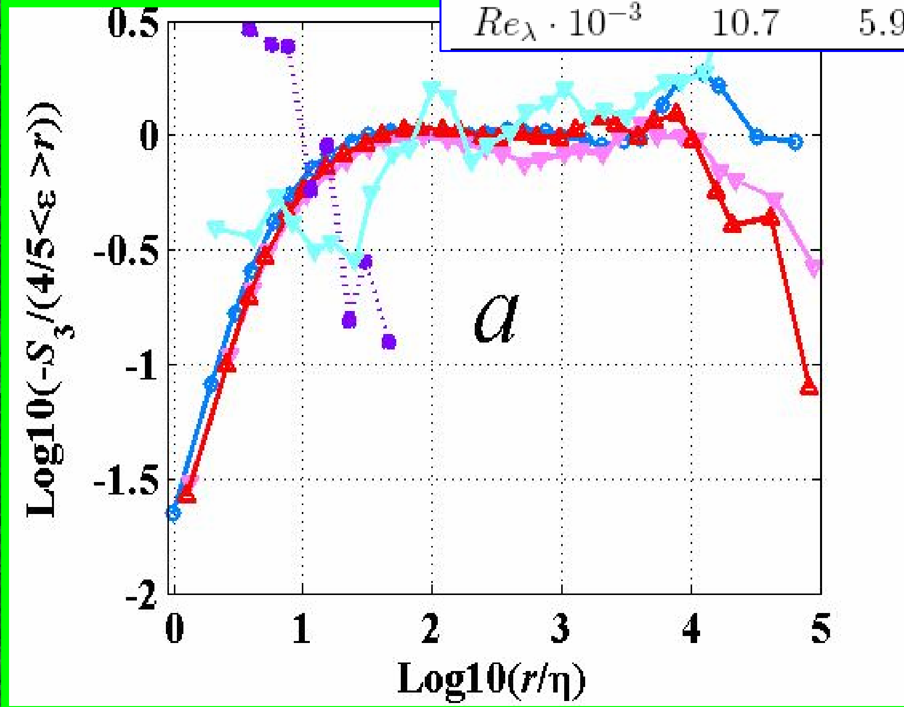

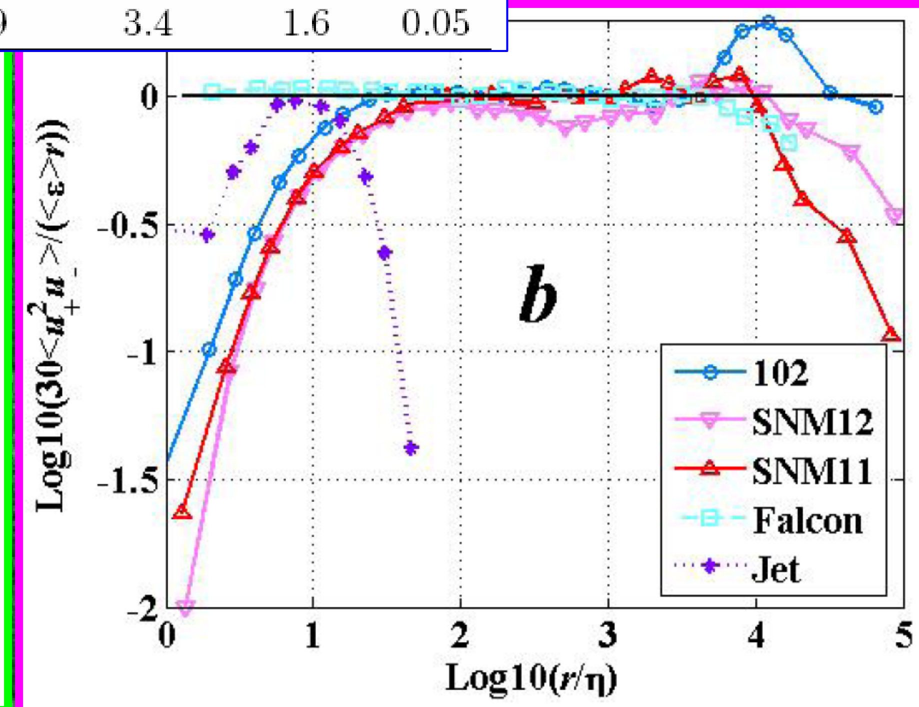

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Kolmogorov 4/5 law, non-locality and sweeping decorrelation hypothesis, M. Kholmyansky, and A. Tsinober (2007) submitted

For more on the experiments see

G. Gulitskii, M., Kholmyansky,W. Kinzlebach, B. Lüthi, A. Tsinober and S. Yorish (2007) Velocity and temperature derivatives in high Reynolds number turbulent flows in the atmospheric surface layer. Part I. Facilities, methods and some general results, *J. Fluid Mech*. 589, 57—81.

G. Gulitskii, M., Kholmyansky, W. Kinzlebach, B. Lüthi, A. Tsinober and S. Yorish (2007) Velocity and temperature derivatives in high Reynolds number turbulent flows in the atmospheric surface layer. Part II. Accelerarations andrelated matters, *J. Fluid Mech*. 589, 83—102.

G. Gulitskii, M., Kholmyansky,W. Kinzlebach, B. Lüthi, A. Tsinober and S. Yorish (2007) Velocity and temperature derivatives in high Reynolds number turbulent flows in the atmospheric surface layer. Part III. Temperature and jointstatistics of temperature and velocity derivatives, *J. Fluid Mech*. 589, 103- 123.

and references therein

Following KRAICHNAN 1964 there are two main ingredients in the (Eulerian) decorrelation: the mentioned sweeping of microstucture by the large scale motions (and associated kinematic nonlocality) and the local straining (which is roughly pure Lagrangian). It appears that this kind of “decomposition” is insufficient as it is missing an essential dynamical aspect - the interaction between the two as it is clearly demonstrated by the Hosokawa’s version of the Kolmogorov 4/5 law. As we have seen the random Taylor hypothesis (and, of course, the conventional Taylor hypothesis) lack/discard this aspect at the outset (this does not mean that these hypotheses are useless): both are ‘too kinematic’, while acceleration is a dynamic quantity in the first place

Relation between Lagrangian and Eulerian statistical properties of turbulent flow

This is a long-standing and most difficult problem posed by Corrsin in 1957. The general reason is because the Lagrangian field is an extremely complicated non-linear functional of the Eulerian field and vice versa (there is also a problem of invertibility). The complexity of this relation can be seen in the example of Lagrangian turbulence (chaotic advection) with a priori prescribed and not random Eulerian velocity field (E-laminar) among others. In this extreme example the Lagrangian statistics has no Eulerian counterpart. In other words, generally, it may be meaningless to look for such a relation.

Given the marker dispersion the problem is to determine the source(s) of agitation. In general, owing to chaotic advection, this inverse problem is impossible to solve. AREF 1984.

… the possession of such relationship would imply that one had (in some sense) solved the general turbulence problem. Thus it seems arguable that such an aim, although natural, may be somewhat illusory. Nevertheless attempts to realize tis aim can teach us about the subject… MCCOMB, 1990.

It is clear that some aspects of the fluid motion are easier to understand in in the Eulerian framework while others are easier to describe in the Lagrangian framework. FRIEDLANDER & LIPTON-LIFSCHITZ 2003

What one sees is real. The problem is interpretation

LAGRANGIAN CORRELATION AND SOME DIFFICULTIES IN TURBULENT DIFFUSION EXPERIMENTS

Advances in Geophysics, **6**, 441-448 !959; ATMOSPHERIC DIFFUSION AND AIR POLLUTION, Proceedings of a Symposium held at Oxford, August 24-29 ,1958

S. Corrsin

See also Corrsin 1957, Review of Surveys in Mechanics: G. I. Taylor 70th Anniversary Volume, edited by G. K. BATCHELOR & R. M. DAVIES, *J. Fluid Mechanics*, 2, 616-612;

Mechanical Engineering Department, Johns Hopkins University, Baltimore, Maryland, U.S.A.

1. The Problem of Relating Lagrangian and Eulerian Correlation

For turbulent diffusion we are interested in the statistics of at least single "fluid particle" displacement; sometimes we need the joint statistics of two or more. Since particle displacement is an integral function of its (Lagrangian) turbulent velocity $v_i(\boldsymbol{a}, t)$, i.e.

$$X_i(\boldsymbol{a}, t) = a_i + \int_0^t v_i(\boldsymbol{a}, t_1)\, dt_1, \qquad (1)\dagger$$

it follows that even the simple probability density function of displacement depends upon the full functional probability of v_i. $\boldsymbol{a}$ is the "initial" position of the fluid particle, e.g. $a_k \equiv X_k(\boldsymbol{a}, 0)$.

Evidently

$$v_i[\boldsymbol{a}, t] = u_i[\boldsymbol{X}(\boldsymbol{a}, t), t] \tag{2}$$

and an integral equation for displacement results from substitution of (2) into (1):

$$X_i(\boldsymbol{a}, t) = a_i + \int_0^t u_i[\boldsymbol{X}(\boldsymbol{a}, t_1), t_1]\, dt_1. \tag{3}$$

We are, of course, concerned with only the statistical properties of these random variables.

Since Taylor's demonstration of the connection between mean square particle displacement [e.g. $\overline{X_1^2}(\boldsymbol{a},t)$] and time auto-correlation of Lagrangian velocity [$\overline{v_1(\boldsymbol{a},t)v_1(\boldsymbol{a},t+\tau)}$], the problem of expressing this correlation in terms of the Eulerian properties has become a very important practical one.†

The Lagrangian two-point correlation tensor is

$$L_{ik} \equiv \overline{v_i(\boldsymbol{a}, t)\, v_k(\boldsymbol{a} + \boldsymbol{\alpha}, t + \tau)} = \overline{u_i[\boldsymbol{X}(\boldsymbol{a}, t), t]\, u_k[\boldsymbol{X}(\boldsymbol{a} + \boldsymbol{\alpha}, t + \tau), t + \tau]}, \tag{4}$$

where the average is over a suitable ensemble of realizations. For stationary, homogeneous fields this depends on $(\boldsymbol{\alpha}, \tau)$ only. The correlation introduced by Taylor is $L_{11}(0, \tau)$.

The Eulerian two-point correlation (in space-time) for a homogeneous, stationary field is

$$E_{jl}(\xi, T) \equiv \overline{u_j(\boldsymbol{x}, t)\, u_l(\boldsymbol{x}+\boldsymbol{\xi}, t+T)} \tag{5}$$

and, in general, there is no reason to expect that L_{ik} and E_{jl} will be uniquely related.

Equation (4) shows explicitly that L_{ik} is the average over an ensemble of random trajectories $\boldsymbol{X}(\boldsymbol{a}, t)$ in the ensemble of random Eulerian fields $\boldsymbol{u}(\boldsymbol{x}, t)$. Little mathematical work appears to have been done on the properties of such statistical functions, even in the degenerate case in which the trajectories are statistically independent of the fields.‡ The immense complexity of our problem is finally brought out by the realization that each trajectory in the ensemble is related to the field it traverses [Equation (3)].

$$L_{ik} \equiv \overline{v_i(\boldsymbol{a}, t)\, v_k(\boldsymbol{a}+\boldsymbol{\alpha}, t+\tau)} = \overline{u_i[\boldsymbol{X}(\boldsymbol{a}, t), t]\, u_k[\boldsymbol{X}(\boldsymbol{a}+\boldsymbol{\alpha}, t+\tau), t+\tau]}, \tag{4}$$

$$X_i(\boldsymbol{a}, t) = a_i + \int_0^t u_i[\boldsymbol{X}(\boldsymbol{a}, t_1), t_1]\, dt_1. \tag{3}$$

Independence approximation and many other

At the same meeting in 1958 Corrsin proposed (see next slide) the so called independence approximation to relate the Lagrangian and Eulerian velocity correlations assuming that at large times the probability distributions of i) particle displacemenets and of the Eulerian velocity field become statistically independent. Generally this hypothesis (as a host of others) is not correct as is shown , e.g. in recent experiments by Ott and Mann (*New Journal of Physics* 7 (2005) 142). Corrsin also proposed simple estimates for the relations between characteristic scales of tirbulent flows in 1963, an important estimate was given by Tennekes in 1975. Later references on request.

Corrsin, S. (1959) Progress Report on some turbulent diffusion reearch, *Advances in Geophysics*, 6, 161-164 ATMOSPHERIC DIFFUSION AND AIR POLLUTION, Proceedings of a Symposium held at Oxford, August 24-29 ,1958.

3. *Conjecture on an Asymptotic Connection between Lagrangian and Eulerian Correlations*

In terms of an ensemble of Eulerian fields, the Lagrangian time correlation for the velocity of a single fluid particle is an average in which the sample gap is different for each member of the ensemble:

$$L_{ik}(0,\tau) = \overline{u_i[\boldsymbol{a},0]\,u_k[\mathbf{X}(\boldsymbol{a},\tau),\tau]} \tag{4}$$

where $u_j(\boldsymbol{x},t)$ is the Eulerian field, assumed homogeneous in space-time. $\mathbf{X}(\boldsymbol{a},t)$ is the fluid particle displacement.

For very large time intervals, it may be possible to neglect the individual identities of displacements on the ensemble, using only an average weighted with the probability density of displacement, $\gamma(\Delta)$. Then

$$L_{ik}(0,\tau) \to \iiint_{-\infty}^{\infty} \gamma(\boldsymbol{\Delta})\, E_{ik}(\boldsymbol{\Delta},\tau)\, d\boldsymbol{\Delta} \tag{5}$$

$E_{ik}(\boldsymbol{\xi},\tau)$ is the Eulerian (space-time) correlation function.

MORE ON DIFFICULTIES

Lumley 1962 (at te Meeting in Marseiles in 1961) pointed to the general mathematical nature of the difficulties in relating the the two formulations. Roughly, there is a general relationship in terms of path (Feynman, functional) integrals, but is does not help much (if at all).

Apart of this 'formalistic' issue (as mentioned) there is one more important aspect associated with the 'more chaotic' nature of the Lagrangian setting. This can be seen as an indication that the pure Lagrangian dynamical equations (so far intractable for viscous flows) are more rich than their Navier Stokes counterpart as explained below.

SAME FLOW - NOT THE SAME PATTERN

All frames (i.e. four different Lagrangian fields) correspond to the same Eulerian flow.

What one sees is real

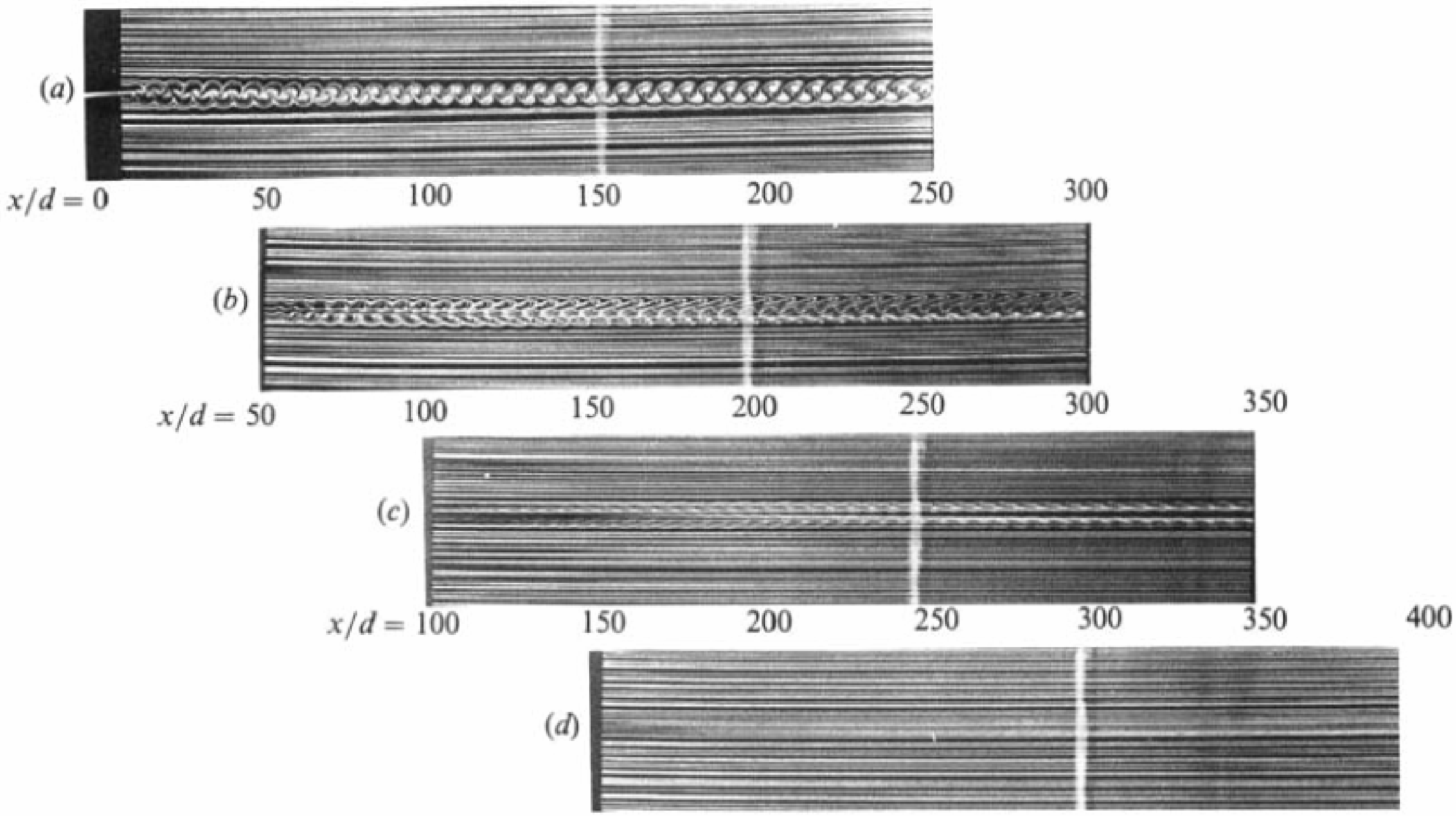


FIGURE 1. Circular-cylinder wake at $Re = 90$; smoke wire at (a) $x/d = 4$, (b) 50, (c) 100 and (d) 150.

The problem is interpretation

Cimbala, J.M., Nagib, H. M and Roshko, A. (1988) Large structures in the far wakes of two-dimensional bluff bodies, J. Fluid Mech., **190**, 265--298.

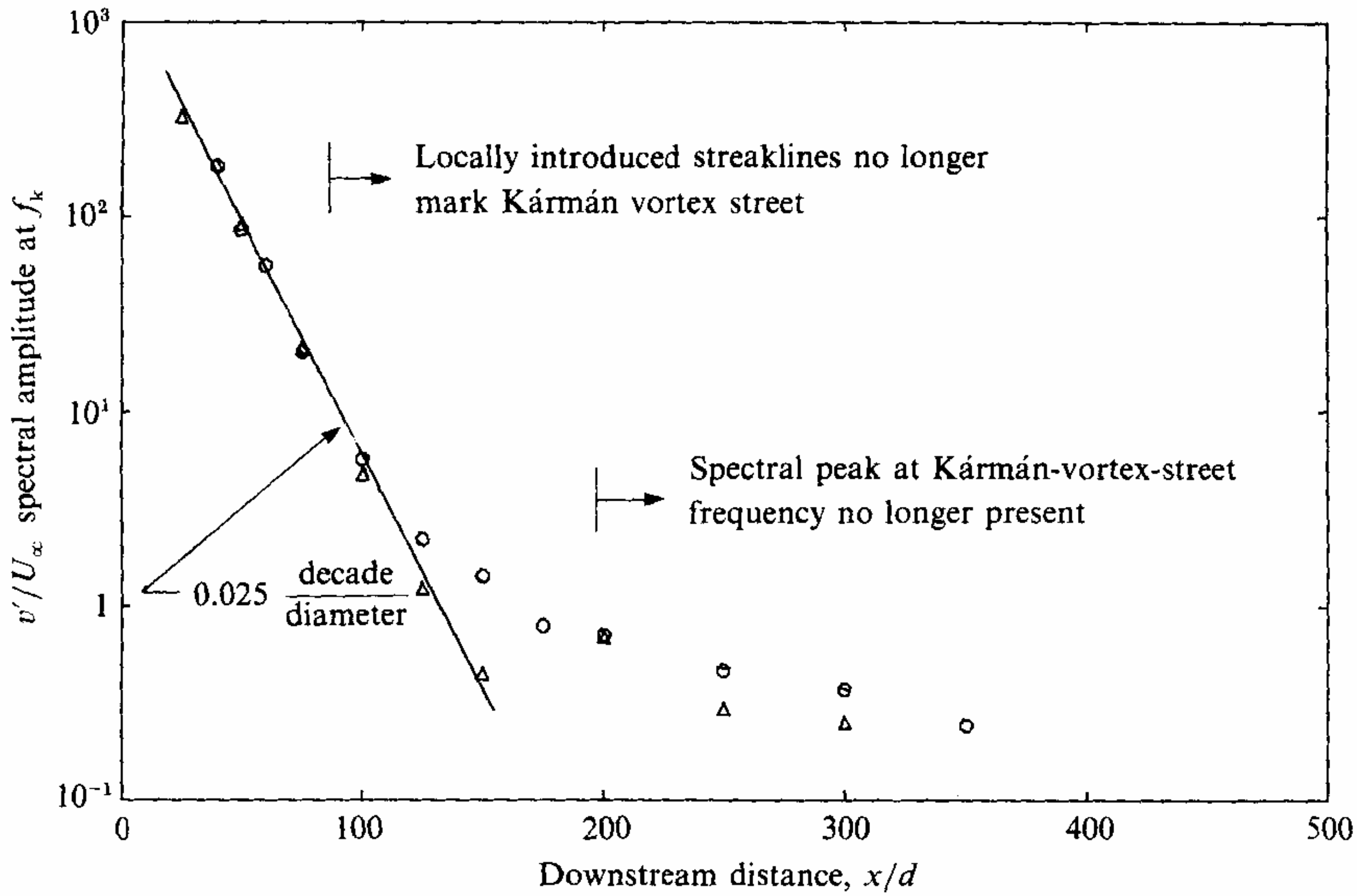
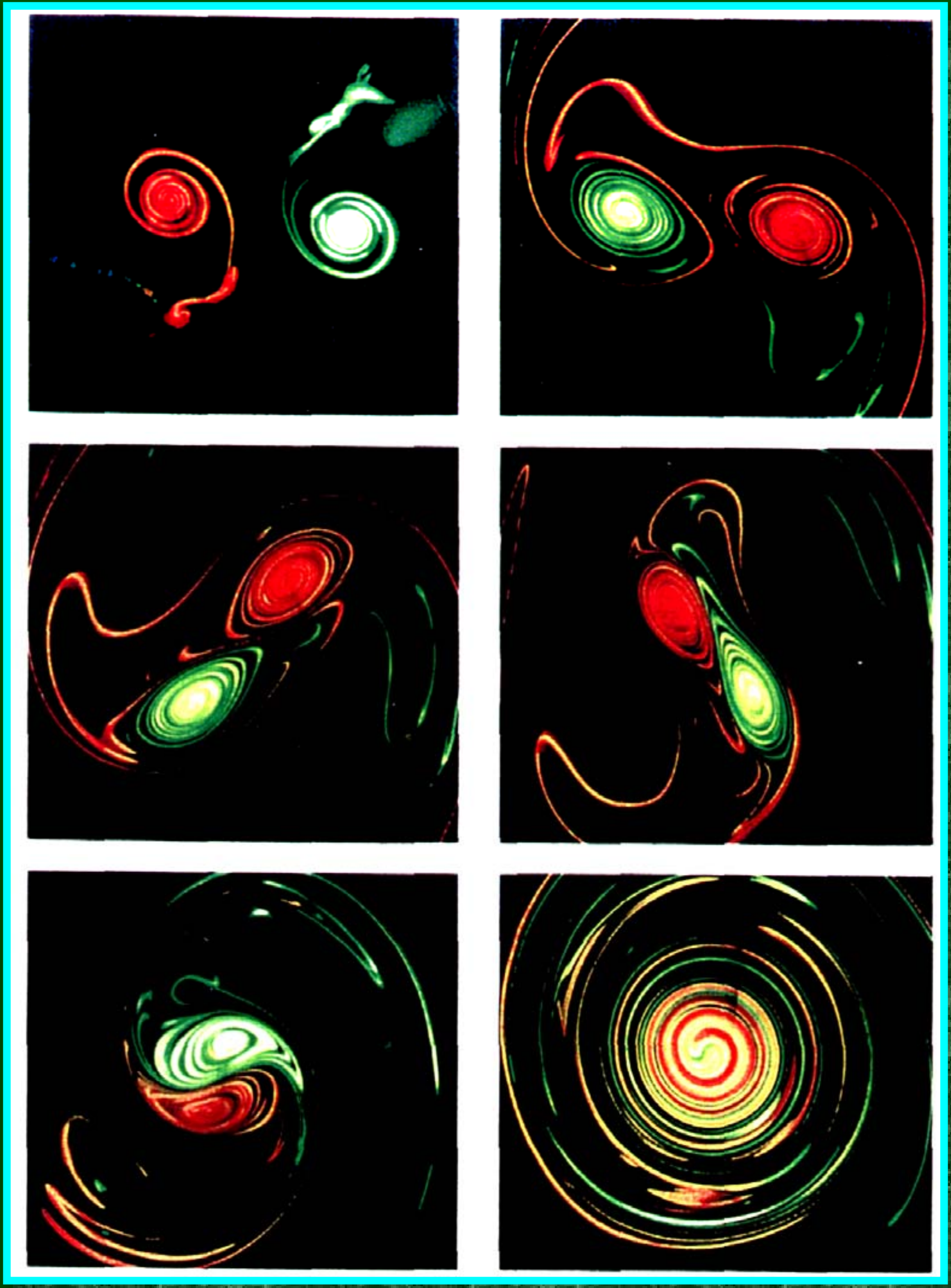


FIGURE 7. Exponential decay of Kármán vortex street; circular-cylinder wake at $Re = 140$ (○) and 150 (△).



Dye visualization of two simple corotating vortices merging into one vortex with simple velocity field, but not that simple filed of the passive scalar(s)

LEWEKE 2000

Kinematics versus Dynamics

E-Laminar but L-turbulent.

E-turbulent necessarily L-turbulent.

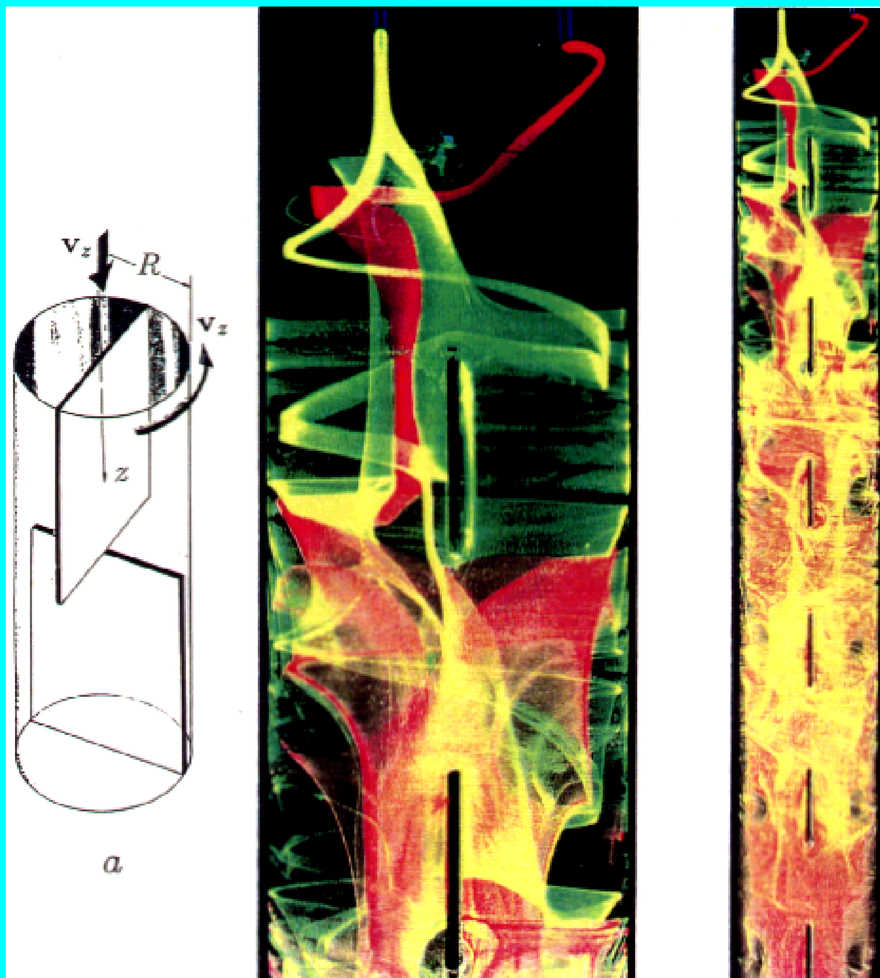
Since the equations describing the evolution of passive objects are linear, it may seem that there is no place for chaotic behaviour of passive objects if the velocity field is not random and is regular and fully laminar, because the chaotic behaviour appears/shows up in nonlinear systems. There is, however, no real contradiction or paradox. This apparent contradiction is resolved via looking at the the fluid flow in the Lagrangian setting in which the observation is made following the fluid particles wherever they move. Here the dependent variable is the position of a fluid particle, $X(a,t)$, as a function of the particle label, a, (usually it's initial position, i.e. $a \equiv X(0)$) and time, t. The relation between the two ways of description is given by the following equation

$$\partial X(a,t))/\partial t = u[X(a,t);\ t] \qquad \text{(E-L)}$$

i.e. the Lagrangian velocity field, $v(a,t) = \partial X(a,t))/\partial t$, is related to the Eulerian velocity field, $u(x,t)$, as $V(a,t) \equiv u\,[X(a,t);t]$.

$$\partial \mathbf{X}(\mathbf{a},t))/\partial t = \mathbf{u}[\mathbf{X}(\mathbf{a},t);\ t] \qquad \{E\text{-}L\}$$

i.e. the Lagrangian velocity field, $v(a,t) = \partial X(a,t))/\partial t$, is related to the Eulerian velocity field, $u(x,t)$, as $V(a,t) \equiv u\ [X(a,t);t]$. Though the Eulerian velocity field, $u(x;t)$ is not chaotic and is regular and laminar, the Lagrangian velocity field $v(a,t) \equiv u[X(a,t);t]$ is chaotic because $X(a,t)$ is chaotic: the equation (E-L) is not integrable even for simplest laminar Euler fields with the exception of very simple flows such as unidirectional ones. It has to be reminded that this chaotic behaviour is of purely kinematic nature resulting solely from the equation {E-L} (and various equations for passive objects - reminding again - linear in Euler setting) and has nothing to do with dynamics, i.e. genuine (as NSE) turbulence.



MIXING IN PMM, Re ~ 1 (!)

KUSH & OTTINO (1992)

RELEVANT TO MICROFLUIDICS with Re ~ 0 (!);

Linked twist maps (LTMs), Bernoulli mixing ...

The complexity and problematic aspects of the relation between the Lagrangian and Eulerian fields is seen in the example of Lagrangian (kinematic) chaos or Lagrangian turbulence (chaotic advection) with a priori prescribed and not random Eulerian velocity field (E-laminar). This is why Lagrangian description - being physically more transparent - is much more difficult than the Eulerian description. In such E-laminar but L-turbulent flows the Lagrangian statistics has no Eulerian counterpart, as in the flow shown at the left.

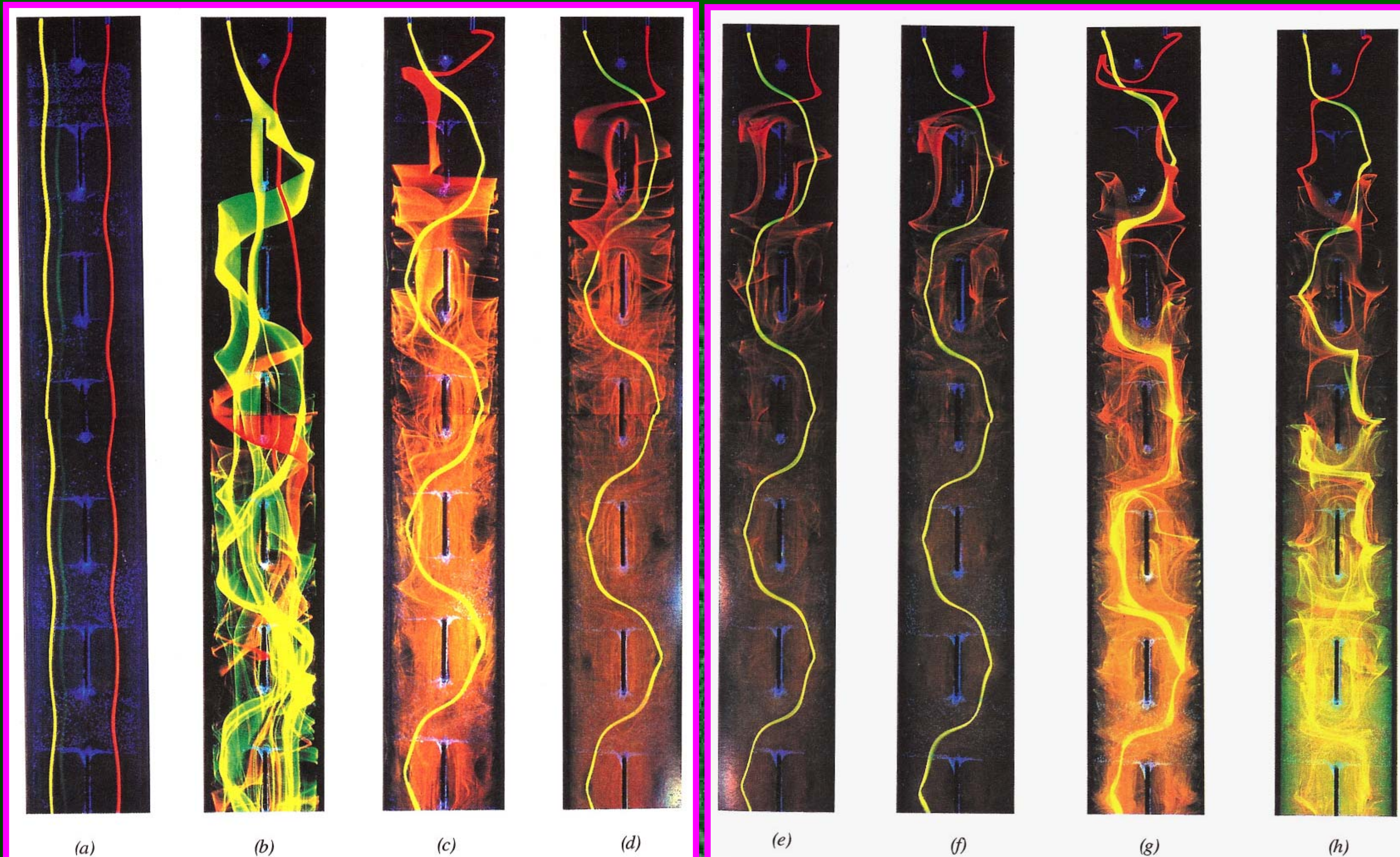
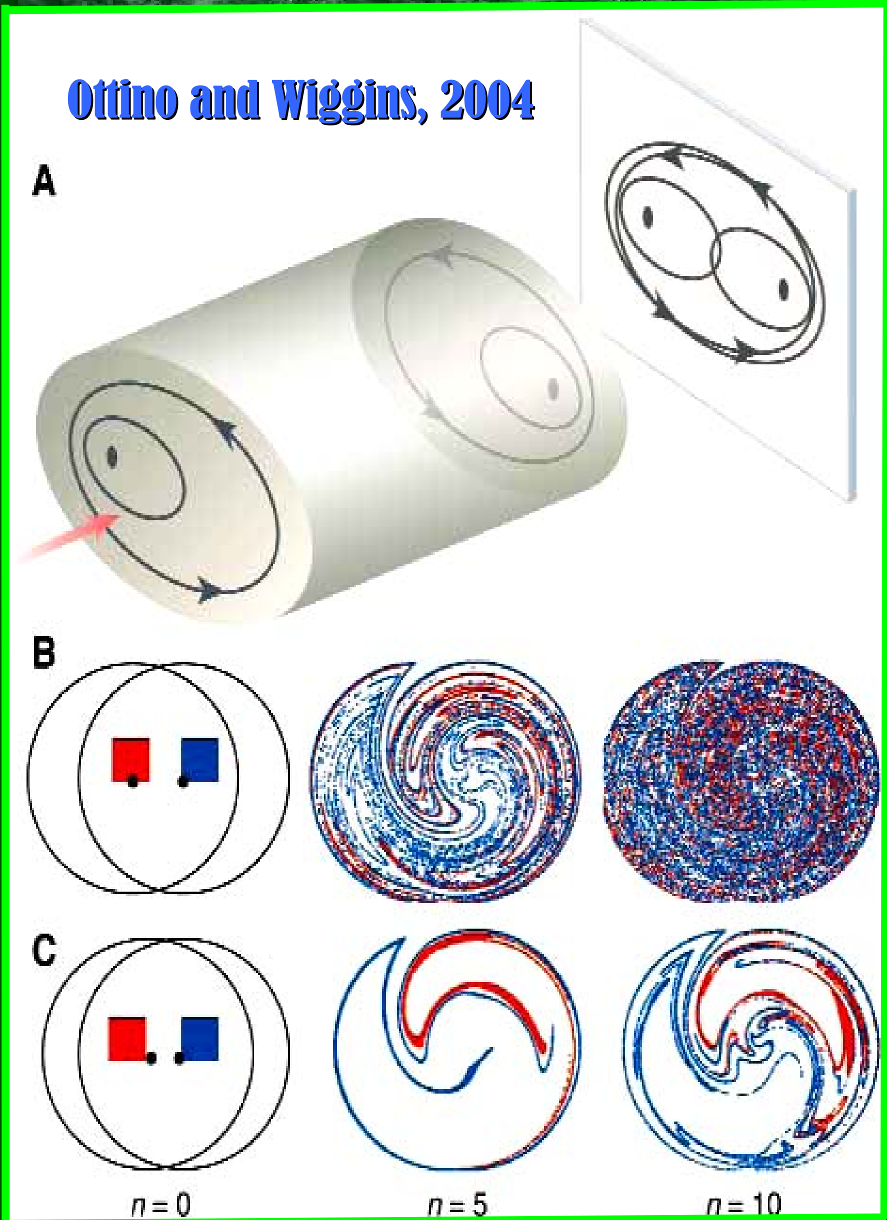


FIGURE 8. Mixing in the PPM as the mixing strength is increased. The mixing strength parameter and Reynolds numbers are *(a)* $\beta=0$, $Re_{\mathrm{PPM:axial}}=0.6$, and $Re_{\mathrm{PPM:cs}}=0$; *(b)* $\beta=4\pm0.1$, $Re_{\mathrm{PPM:axial}}=0.5$, and $Re_{\mathrm{PPM:cs}}=1.3$; *(c)* $\beta=10\pm0.1$, $Re_{\mathrm{PPM:axial}}=0.6$, and $Re_{\mathrm{PPM:cs}}=3.5$; *(d)* $\beta=15\pm0.2$, $Re_{\mathrm{PPM:axial}}=0.5$, and $Re_{\mathrm{PPM:cs}}=4.1$; *(e)* $\beta=20\pm0.3$, $Re_{\mathrm{PPM:axial}}=0.5$, and $Re_{\mathrm{PPM:cs}}=5.5$; *(f)* $\beta=25\pm0.4$, $Re_{\mathrm{PPM:axial}}=0.5$, and $Re_{\mathrm{PPM:cs}}=7.3$; *(g)* $\beta=30\pm0.6$, $Re_{\mathrm{PPM:axial}}=0.3$, and $Re_{\mathrm{PPM:cs}}=5.9$; and *(h)* $\beta=40\pm0.9$, $Re_{\mathrm{PPM:axial}}=0.3$, and $Re_{\mathrm{PPM:cs}}=7.5$. KUSH & OTTINO (1992)

Mixer with a twist

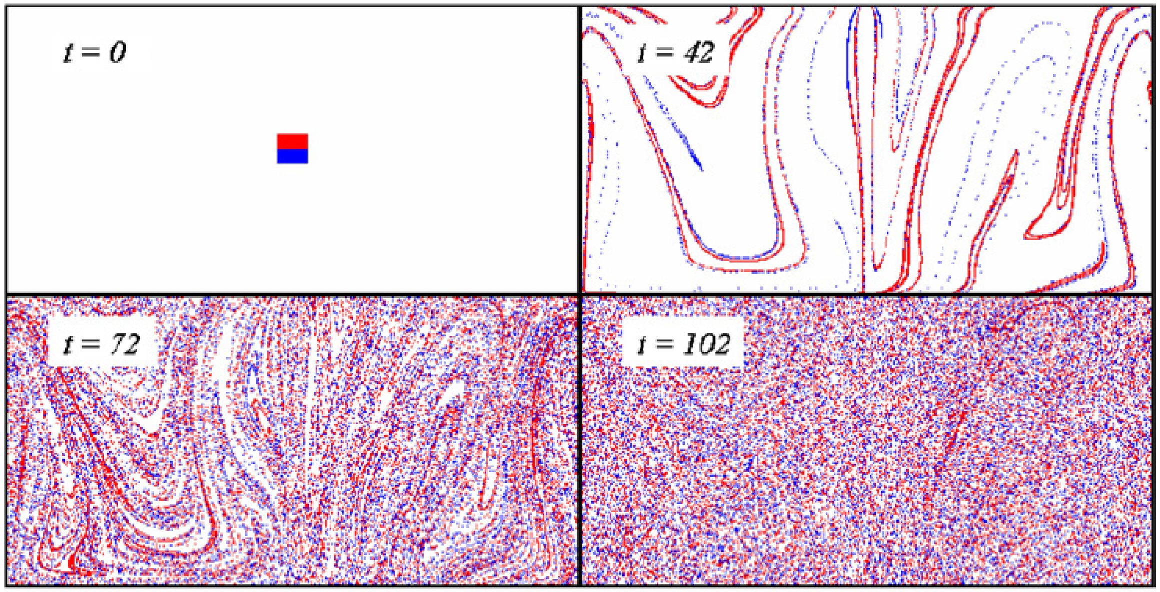


A . Schematic representation of a channel type micromixer. Streamline patterns are shown at the ends ofthe mixing element. The details of the shape and internal structure of the channel, the motion of boundaries, and the manner of driving are not shown; they can be anything that produces the desired cross-sectional flow (which defines the family of designs).

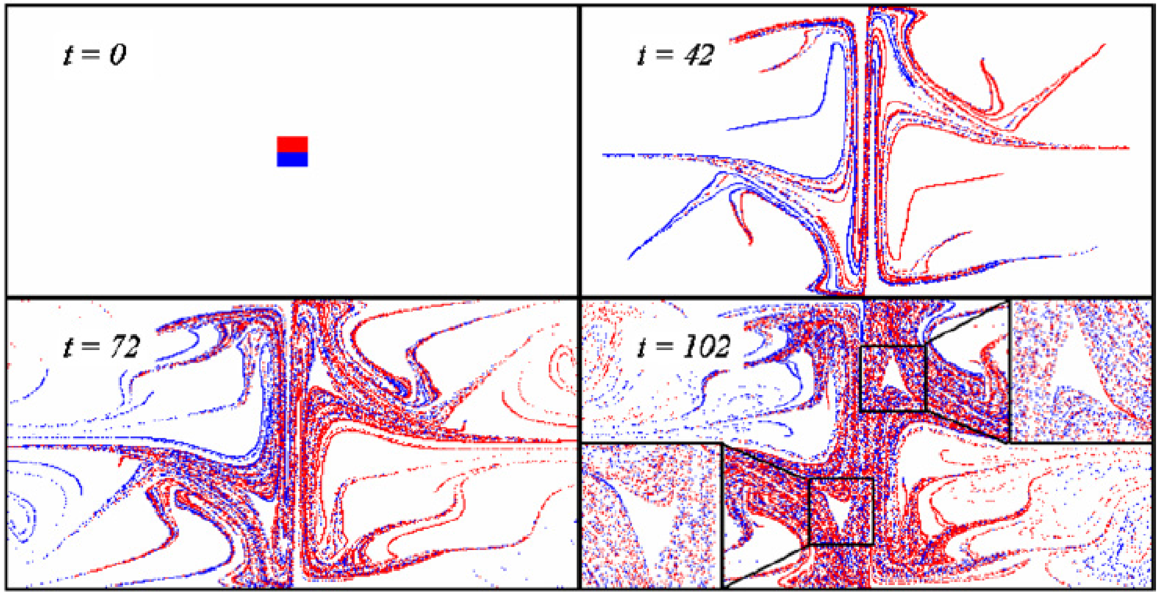
B. Two blobs shown in the superposition of the outer streamlines in the cross section at the end of each mixing element for a case where the flow features underlying the LTM theorems provide good mixing properties.The integer ndenotes the number of mixing segments (where a mixing segment is two concatenated mixing elements). The flow appears well mixed after 10 mixing elements.

C. The same blobs as in **B**, but for a case where the flow features underlying the LTM theorems fail to provide good mixing.

MICROMIXING



(*a*) case B&C at period $T = 6$



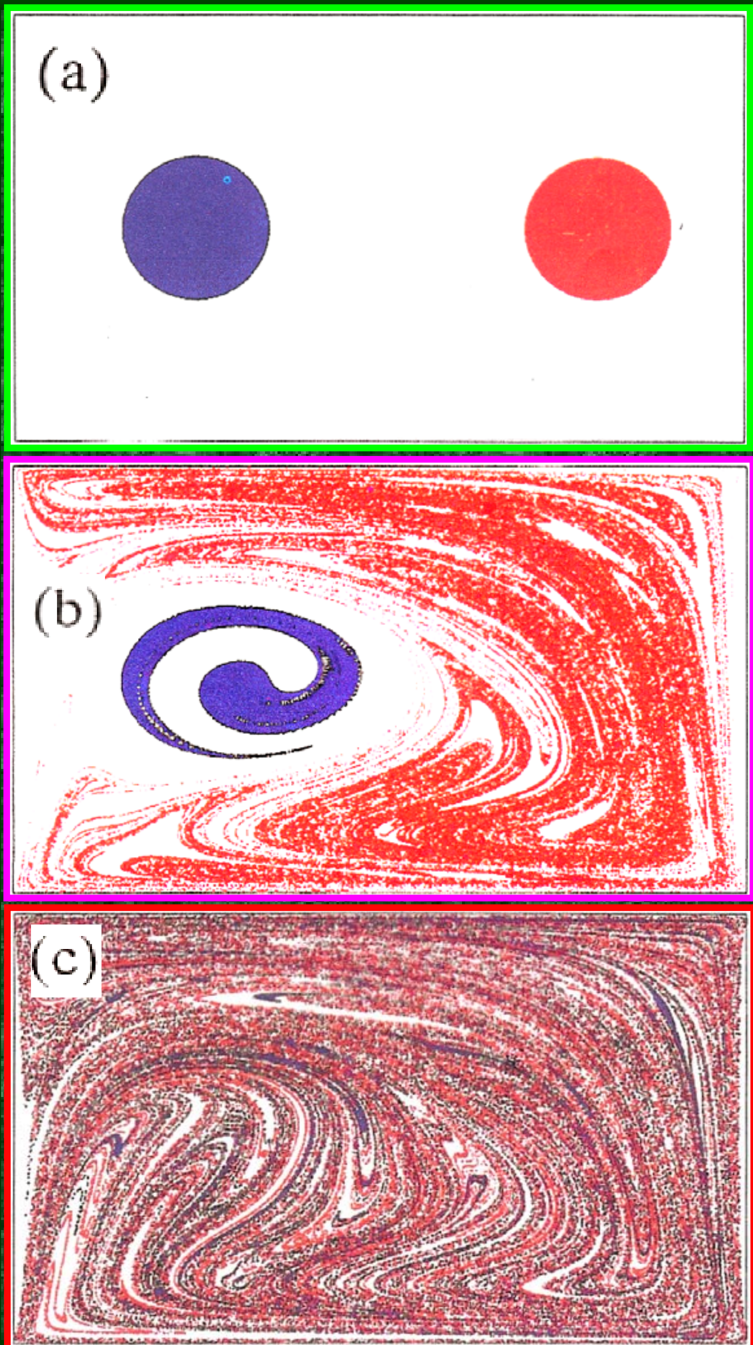
(*b*) case A&D at period $T = 6$

HJ Kim and A Beskok 2007 Quantification of chaotic strength and mixing in a micro fluidic system, *J. Micromech. Microeng.* 17 , 2197–2210.

Snapshots of spread of passive tracer particles for cases B&C at period $T = 6$ (*a*) and A&D at period $T = 6$ (*b*).

A total of 40 000 passive particles are tracked in time. Snapshots show the dispersing state at respective times. Figure (*a*) shows flow domain filled with passive tracer particles, and these particles are distributed uniformly at $t = 102$. Figure (*b*) shows presence of two small void regions that remain near the vertical centerline of the domain at $t = 102$. The insets show the geometric structure of the void region.

MIXING IN APERIODIC CHAOTIC FLOWS



LIU M, MUZZIO FJ, PESKIN RL 1994
Quantification of mixing in aperiodic chaotic flows, *Chaos, Solitons & Fractals*, 4 (6), 869-893 .1994

(a) Initial location of particles in the cavity flow.

(b) Positions of the particles after 8 periods of the periodic flow with *T* = 7.0. The red particles are located inside the chaotic regions and undergo rapid mixing. The blue particles are inside an island of regular motion and hardly mix at all.

(c) Mixing is greatly enhanced by using the SB aperiodic flow After the same 8 periods, particles of both colors are thoroughly mixed throughout the flow domain.

Tip for mixing of two components of epoxy at Re~0

MULTISCALE AND/OR RANDOM AND /OR E-LAMINAR BUT NOT E-GEINUINE

The above qualification includes all artificial velocity fields both random and/or multiscale or not . The field of particle trajectories is (can be seen) as a passive object: it is a Lagrangian signature of the underlying (and prescibed) velocity field of any nature be it genuinely turbulent, or Lagrangian chaotic such as E-Laminar, synthetic random or not, restricted Euler, kinematic simulations of Lagrangian chaotic evolution, turbulent-like multiscale fields, including real E-laminar flows at Re≈0 from linear Stokes equations with random forcing, flows in porous media, microdevices, to name some.

We stress that the E-turbulence is a **dynamical** phenomenon whereas this is not necessarily the case with the L-turbulence which may be a **purely kinematic** one . In other words the flow can be purely L-turbulent (i.e. E-laminar) as in the above examples with artificial velocity fields or real flows at very low Reynolds numbers

However, if the flow is E-turbulent (i.e. Re >> 1) it is L-turbulent as well. An important consequence is that the structure and evolution of passive objects in genuine turbulent flows arises from two (essentially and unfortunately inseparable) contributions: one due to the Lagrangian chaos and the other due to the random nature of the (Eulerian) velocity field itself.

Hence, one can expect adequate kinematic simulation or simulation in random and/or multiscale real E-laminar flows of those properties (Lagrangian) which are insensitive (or weakly sensitive) to the differences between the genuine turbulent velocity fields and those used for the purposes of modeling (quite a non-trivial issue). An important counterexample is the difference between backwards and forwards relative dispersion (with the mean square separation following particle pairs backwards in time being twice as large as forwards) in genuine turbulence. Another one is the qualitative difference in alignment properties of a passive vector in genuine (NSE) and Gaussian velocity field

Forwards and Backwards Relative Dispersion

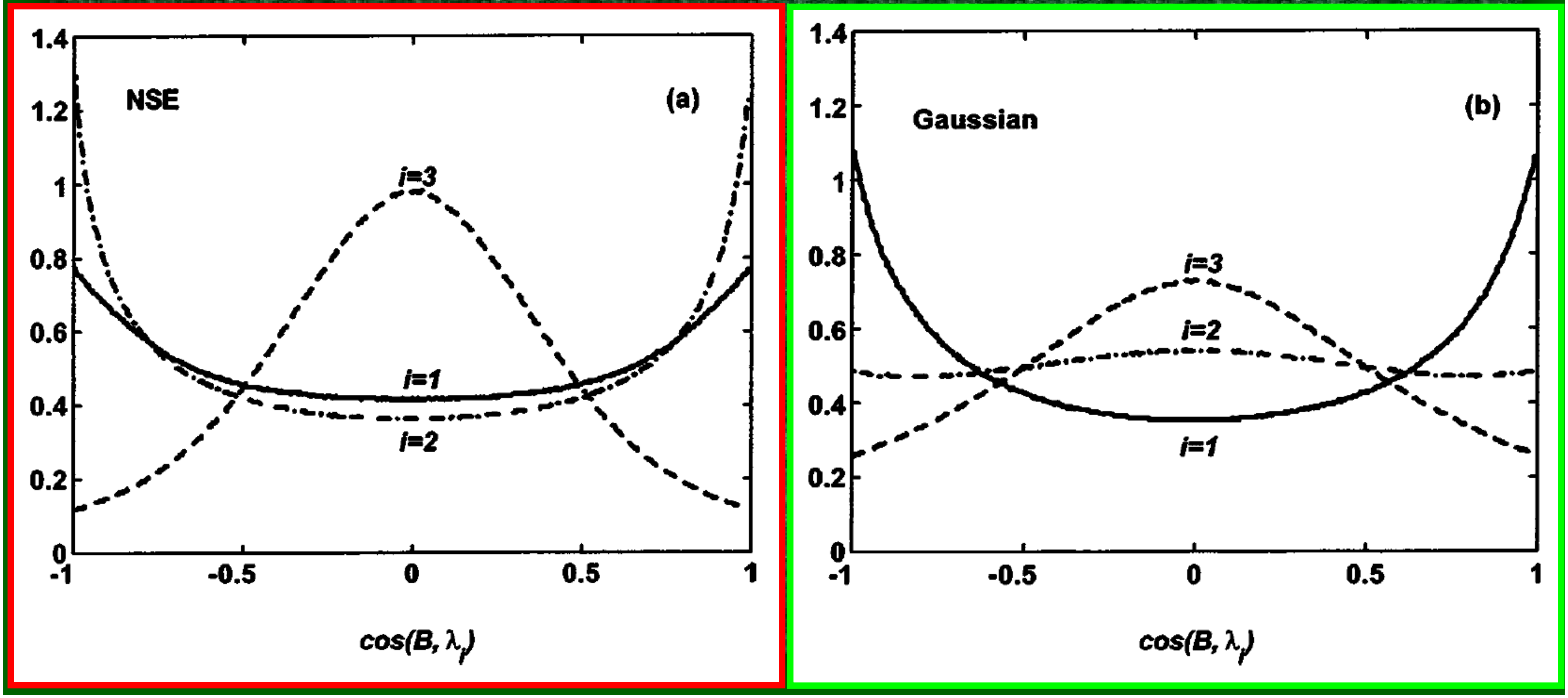
We find that, in general, backwards relative dispersion proceeds at a much faster rate than relative dispersion forwards in time, and the difference between the two is sensitive to the nature of the flow field. The difference vanishes for Gaussian flows and for white-noise in time flows for which relative dispersion can be described by a diffusion equation, suggesting that theories such as two-point closure and kinematic simulation do not differentiate between backwards and forwards dispersion. Backwards relative dispersion is very sensitive to the details of the tails of the probability density function for the Eulerian velocity difference between two points

B. L. Sawford, P. K. Yeung, and M. S. Borgas, 2005 Comparison of backwards and forwards relative dispersion in turbulence, *Phys. Fluids* **17**, 095109.

J. Berg, B. Lüthi, J. Mann, and S. Ott, An experimental investigation: backwards and forwards relative dispersion in turbulent flow, *Phys Rev.* **E 74**, 016304

Alignment of passive vector (B) with the eigenfrarme of the rate of strain tensor in genuine (NSE) and gaussian velocity fields

$$D\mathbf{B}/Dt = (\mathbf{B}\cdot\nabla)\mathbf{u} + \eta\Delta\mathbf{B}$$



A COUNTEREXAMPLE

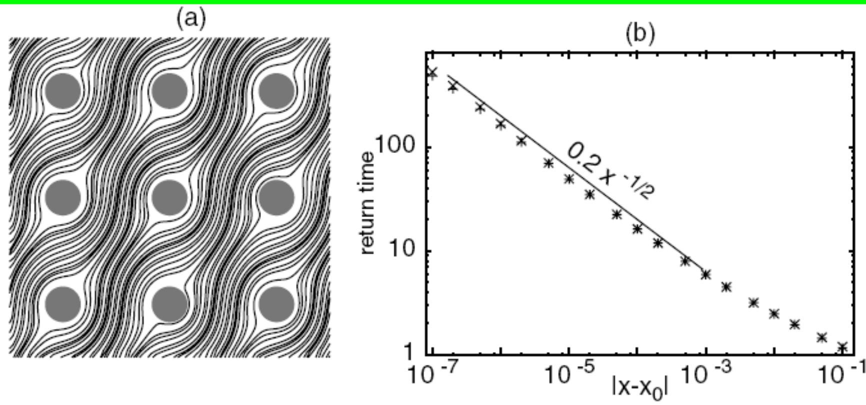


FIG. 1. (a) Flow pattern with $\alpha/\beta = (\sqrt{5}-1)/2$; (b) divergence of passage time for streamlines near the cylinder. Crosses: numerical values; straight line: power law $\sim|x - x_0|^{-1/2}$.

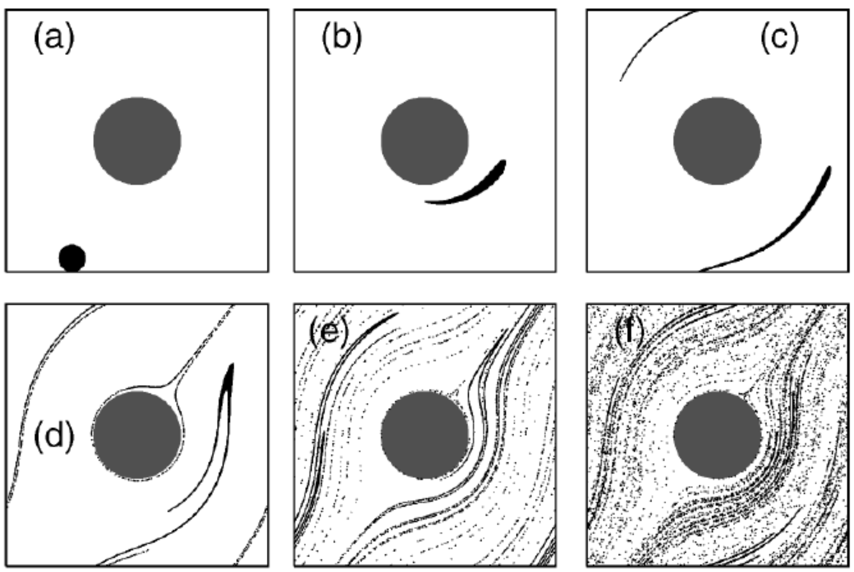


FIG. 3. Cloud of passive tracers carried by the flow. (a) $t = 0$; (b) $t = 0.5$; (c) $t = 2$; (d) $t = 5$; (e) $t = 20$; (f) $t = 100$.

Zaks, M.A. and Straube, A.V. (2002) Steady Stokes flow with long-range correlations, fractal Fourier spectrum and anomalous transport, *Phys. Rev. Lett.*, 89, 244101--1-4.

Concluding Remarks

Visualizations E- versus L-

Flow visualizations used for studying the structure of dynamical fields (velocity, vorticity, etc.) of turbulent flows may be quite misleading, making the question "what do we see?" extremely nontrivial.

Seeing is not necessarily believing.

E- versus L-structure(s),

i.e. structure(s) in E- versus L-settings.

Passive objects have lots of structure(s) in Gaussian velocity fields* which by definition is "structureless".

*and other artificial, random and not random.

Eulerian and Lagrangian settings are different conceptually not just/only technically. Eulerian setting is revealing the pure dynamical chaotic aspects of genuine turbulence as contrasted to "mixing" of kinematical with the dynamical ones in the Lagrangian setting, i.e. in genuine turbulence the latter contains both which seem to be essentially inseparable.

On the mathematical side there is an important aspect associated with the 'more chaotic' nature of the Lagrangian setting. Namely, one is tempted to conjecture that the pure Lagrangian **dynamical** equations (so far intractable for viscous flows)

$$\partial^2 X_i/\partial^2 t = [X_j, X_k, p] + \nu[X_n, X_{n+1}, [X_n, X_{n+1}, \partial X_i/\partial t],$$

$$\mathcal{D}(X_i)/\mathcal{D}(a_j) \equiv [X_1, X_2, X_3] = 1$$

are more rich than their Navier Stokes counterpart

$$Du_i/Dt \equiv \partial U_i/\partial t + U_k \partial U_i/\partial x_k = -\partial p/\partial x_i + \nu \nabla^2 U_i$$

$$\partial U_i/\partial x_i = 0$$

The former being equivalent to the latter plus the equation

$$\frac{\partial \mathbf{X}(\mathbf{a},t)}{\partial t} = \mathbf{U}[\mathbf{X}(\mathbf{a},t);t]$$

Some important consequences

First, Euler setting seems (?!) to be preferable for studying genuine dynamical aspects of (e.g. NSE) genuine turbulence due to impossibility to separate the Lagrangian (kinematic) chaos from the genuinly dynamical (Eulerian/intinsic) stochasticity. **Second**, generally, simple relations (and even not so simple) cannot be expected between Eulerian and Lagrangian statistics. For example, there exist no such a relation for a host of Lagrangian chaotic flows having no Eulerian counterpart at all. So one has to resort to an ad hoc approach for different cases/classes of flows.

Third, studying Lagrangian statistics of a variety of artificial and/or purely E-laminar flows may not provide adequate information of the L-statistics of genuine turbulence as not containing the pure dynamical stochasticity of genuine turbulence.

All the above brings us to the questions posed at the beginning: Is it true that dynamical issues per se can and should be treated satisfactory in Eulerian setting only? Is there any need to use for this purpose the Lagrangian setting too? Are there problems which require such an approach?

A plausible answer is that there are important problems/questions of dynamical nature for which Lagrangian information is of utmost importance (as well), i.e one has to employ both settings. The first example is given by the class of flows where turbulence memory and/or sensitivity to the inflow conditions plays an essential role (e.g. jets, mixing layes, wakes and flows past grids too – the recent example of flows past fractal grids provides especially strong evidence for this).

It has to be mentioned that the issues concerning the Taylor and the Random Taylor hypotheses and a number of questions on accelerations belong to this sort of problems too.

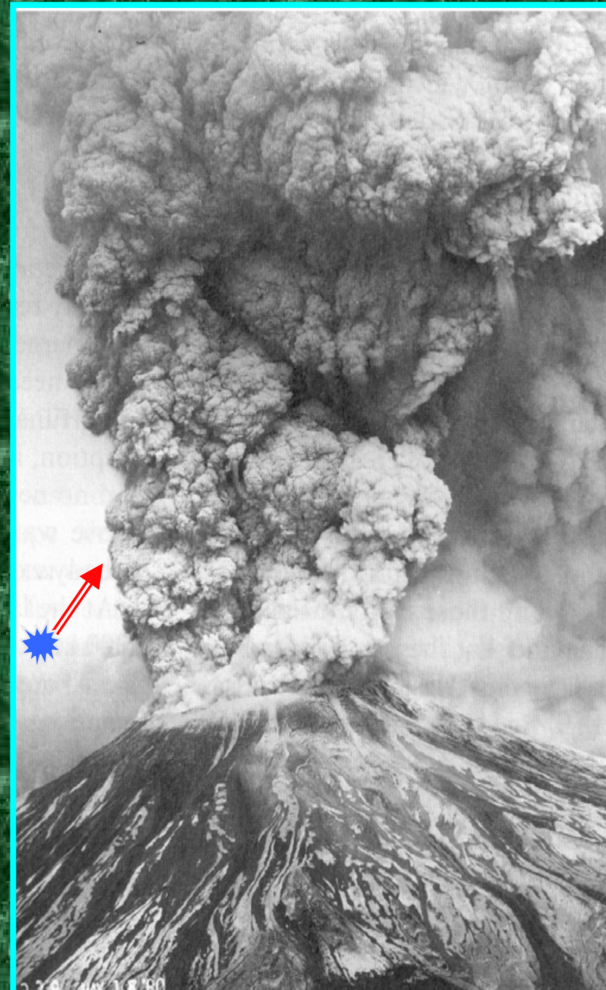
Most of flows mentioned above belong to the kind of the so called partly turbulent flows. The main special features of these flows are the coexistence of regions with laminar and turbulent states of flow and continuous transition of fluid particles (purely Lagrangian objects!) from laminar state into turbulent one via the entrainment process through the boundary between the two. Hence the necessity of Lagrangian approach in studying of this transition process in the proximity of the laminar-turbulent interface. This issue will be addressed in a separate lecture

PTF - ENTRAINMENT

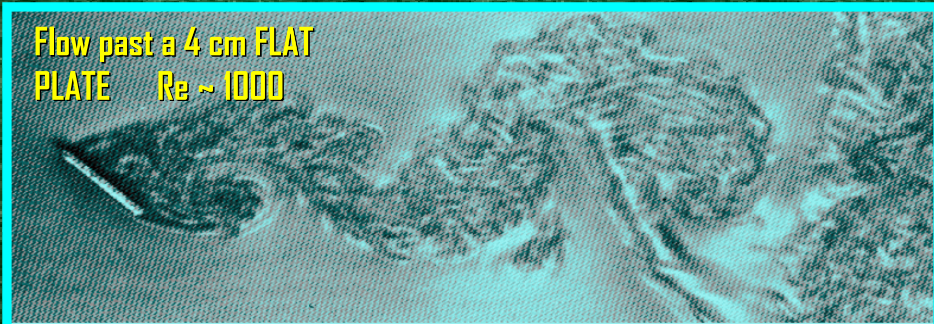
A turbulent jet from testing a Lockheed rocket engine in the Los Angeles hills

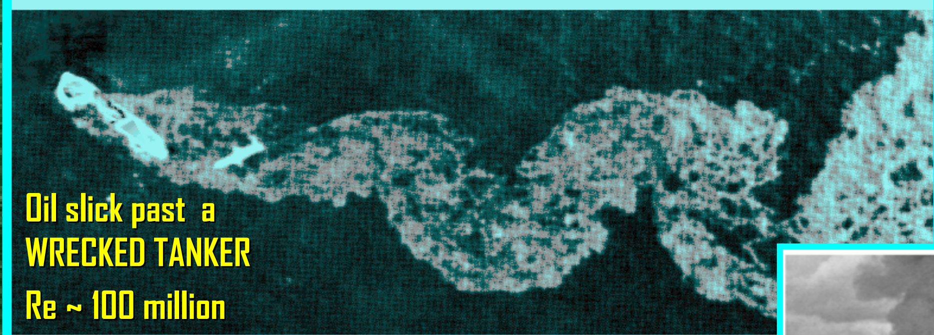
The laminar-turbulent "interface" is sharp so that fluid particles (note the Lagrangian aspect !) "are found" abruptly in a turbulent environment

Mount St. Helen volcano on 18 May 1980

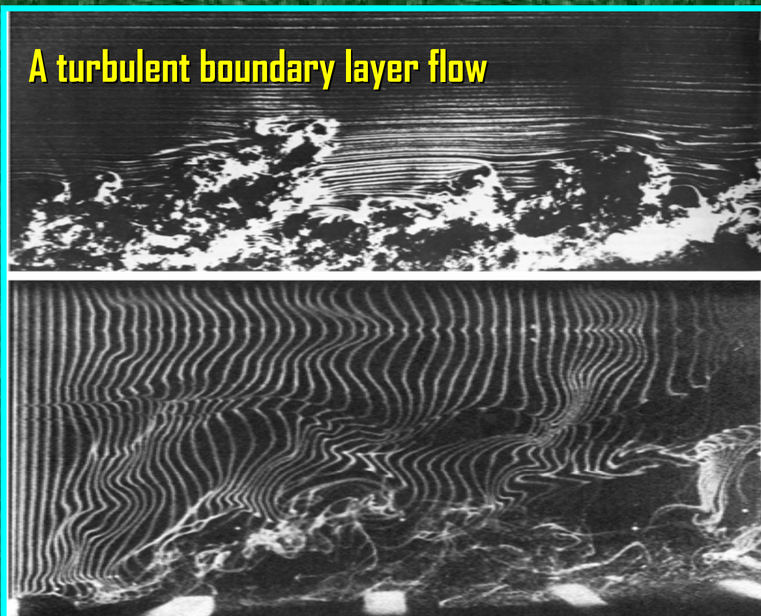


Partly Turbulent Flows


Flow past a 4 cm FLAT PLATE Re ~ 1000


Oil slick past a WRECKED TANKER
Re ~ 100 million


Flow past a bluff body


A turbulent boundary layer flow

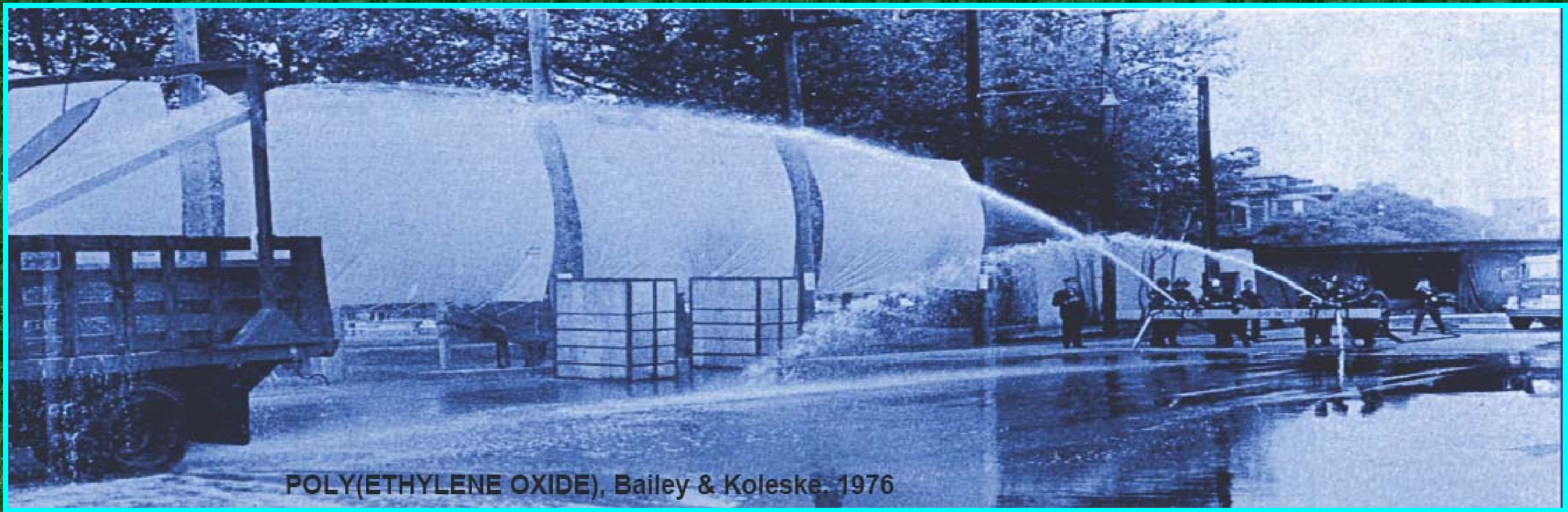

A turbulent jet from testing a Lockheed rocket engine in the Los Angeles hills


Mount St. Helen volcano on 18 May 1980, US Geological Survey

Flows with **polymer solutions** provide another important example where Lagrangian approach is unavoidable at least for two additional reasons 1) since the material elements (again purely Lagrangian objects!) in such flows are not passive and 2) there are no equations (as NSE for Newtonian fluids) reliably describing flows of polymer solutions. So one needs Lagrangian **experimentation** with such turbulent flows in the first place.

A similar statement is true of flows with any other active additives.

DILUTE POLYMER SOLUTIONS



BLOD FLOW ??

Finally, there is a more general consideration. The Lagrangian description of fluid flows is physically more natural than the Eulerian one, since it is related most directly to the motion of fluid elements.

Further insight into the basic physics of turbulent flows requires information on **time evolution** and associated Lagrangian statistics of such quantities like vorticity, strain, accelerations, etc. as relating the spatial structure (the most popular time snapshots) and the time dimension.

A Bit of History

Lagrange versus Euler or vice versa

H.LAMB 1932, *Hydrodynamics*, Cambridge Univ. Press, pp 2-3

3. The equations of motion of a fluid have been obtained in two different forms, corresponding to the two ways in which the problem of determining the motion of a fluid mass, acted on by given forces and subject to given conditions, may be viewed. We may either regard as the object of our investigations a knowledge of the velocity, the pressure, and the density, at all points of space occupied by the fluid, for all instants; or we may seek to determine the history of every particle. The equations obtained on these two plans are conveniently designated, as by German mathematicians, the 'Eulerian' and the 'Lagrangian' forms of the hydrokinetic equations, although both forms are in reality due to Euler†.

† "Principes généraux du mouvement des fluides," *Hist. de l'Acad. de Berlin*, 1755. "De principiis motus fluidorum," *Novi Comm. Acad. Petrop.* xiv. 1 (1759).

P. FRANK 1935, *Die differential- und integral Gleichungen der Mechanik und Physik*, 2nd ed., Part 2 Vieweg; L.D.LANDAU AND E.M.LIFSHITS (1959) *Fluid Mechanics*, Pergamon and many others.
A detailed account on the 'misnomer' by which the 'Lagrangian' equations are ascribed to Lagrange is found in C. TRUESDELL 1954, *The Kinematics of Vorticity*, Indiana University Press, pp. 30-31 and references therein (see two next slides)

H.LAMB 1932, *Hydrodynamics*, Cambridge Univ. Press, pp 12-13

13, 14. 'Lagrangian' form of the equations of motion and of the equation of continuity 12

Note **'Lagrangian'** instead of Lagrangian

$$\left(\frac{\partial^2 x}{\partial t^2} - X\right)\frac{\partial x}{\partial a} + \left(\frac{\partial^2 y}{\partial t^2} - Y\right)\frac{\partial y}{\partial a} + \left(\frac{\partial^2 z}{\partial t^2} - Z\right)\frac{\partial z}{\partial a} + \frac{1}{\rho}\frac{\partial p}{\partial a} = 0,$$

$$\left(\frac{\partial^2 x}{\partial t^2} - X\right)\frac{\partial x}{\partial b} + \left(\frac{\partial^2 y}{\partial t^2} - Y\right)\frac{\partial y}{\partial b} + \left(\frac{\partial^2 z}{\partial t^2} - Z\right)\frac{\partial z}{\partial b} + \frac{1}{\rho}\frac{\partial p}{\partial b} = 0,$$

$$\left(\frac{\partial^2 x}{\partial t^2} - X\right)\frac{\partial x}{\partial c} + \left(\frac{\partial^2 y}{\partial t^2} - Y\right)\frac{\partial y}{\partial c} + \left(\frac{\partial^2 z}{\partial t^2} - Z\right)\frac{\partial z}{\partial c} + \frac{1}{\rho}\frac{\partial p}{\partial c} = 0.$$

$$\varrho\,\frac{\partial(x, y, z)}{\partial(a, b, c)} = \varrho_0,$$

For incompressible flows

$$\frac{\partial(x, y, z)}{\partial(a, b, c)} = 1.$$

Note that these equations (given in Lamb) are equivalent to those quoted above with $\nu=0$.

INDIANA UNIVERSITY PUBLICATIONS
SCIENCE SERIES NO. 19

The Kinematics of Vorticity

by

C. TRUESDELL

See pp. 30-32

Indiana University Press
BLOOMINGTON 1954
$6.00

[2] In this work we eschew the general misnomer by which X, Y, Z are called "Lagrangian" co-ordinates, while x, y, z are called "Eulerian" co-ordinates. The origin of this incorrect usage is as follows.

By the middle nineteenth century the history of fluid dynamics in the eighteenth century had apparently sunk into obscurity. Euler's papers were not often read, of his results which were not forgotten several were attributed to more recent authors who had appropriated them without acknowledgement or discovered them afresh, and indeed his supreme achievements in mathematics, mechanics, and mathematical physics were undervalued then, though not so much as now. The erroneous terminology still current was introduced in the posthumous memoir of Dirichlet [1860, 1, Introd.], edited by Dedekind, where [1757, 2] was quoted as the source of the "Eulerian" method, while it was stated that Lagrange in the *Méchanique Analitique* [1788, 1, Part II, Sect. II, ¶¶4–7] had introduced the "Lagrangian" method, but had immediately converted the resulting equations to "Eulerian" form. Although in the next year Hankel [1861, 1, §1] stated that his teacher Riemann had told him that Euler had introduced the "Lagrangian" method in [1770, 1], one year's priority has been sufficient to perpetuate the error.

Riemann's attribution is correct, but the references quoted are not the earliest, either for Euler or for Lagrange. Subsequent writers on hydrodynamics have followed Hankel in adopting the printer's error on the title page by which [1770, 1] is dated 1759, while the correct date is 1769; Lagrange's first exposition of the "Lagrangian" description is not in the *Méchanique Analitique* but actually in [1762, 3, Chs. XL, XLIV, XLVIII, LII]. The whole matter is easily clarified, however. In a letter [1862, 2], written to Lagrange under the date 27 October 1759, Euler after expressing his admiration for Lagrange's first memoir on the propagation of sound stated that one had reason to doubt that propagation in two or three dimensions would follow the same law as in the one dimensional case, since he had already found the fundamental equations to be of different form. The equations he gives are the linearized equations of plane flow of a perfect fluid expressed in terms of the variables X, Y. (That the date of Euler's discovery of the material description is 1759 or earlier is shown also by [1766, 1, §§4–13, 31–40], a memoir dated 1759. In [1767, 1], written in 1750–1751, Euler for plane motions had used a description partly spatial and partly material.)