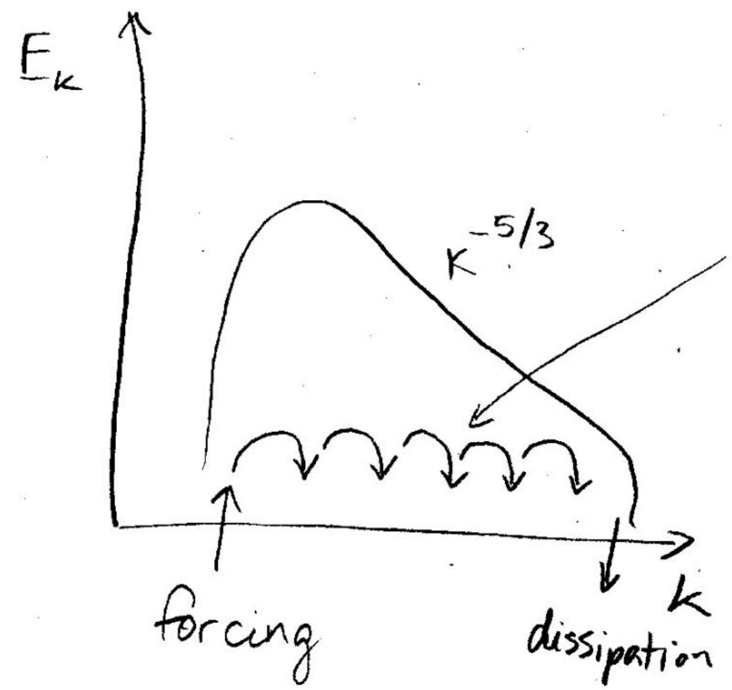
S. Nazarenko

Local vs. nonlocal turbulence

3D local turbulence. Kolmogorov – Obukhov theory K41 (Alex' lecture).



Locality, or step-by-step nature of the energy cascade, is crucial for existence of K41 scalings.

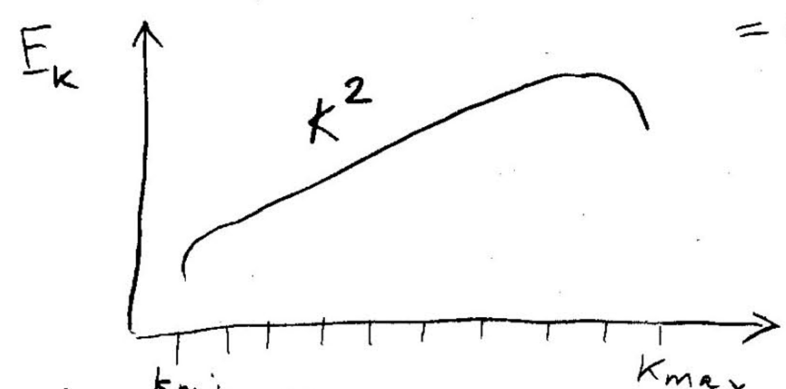
Truncated Euler equation

ideas can be traced back to Burgers'30s.

finite-dimensional non-dissipative system.

Typical for early numerics (70's – 80's – S. Orszag, D. Montgomery...): thermalization

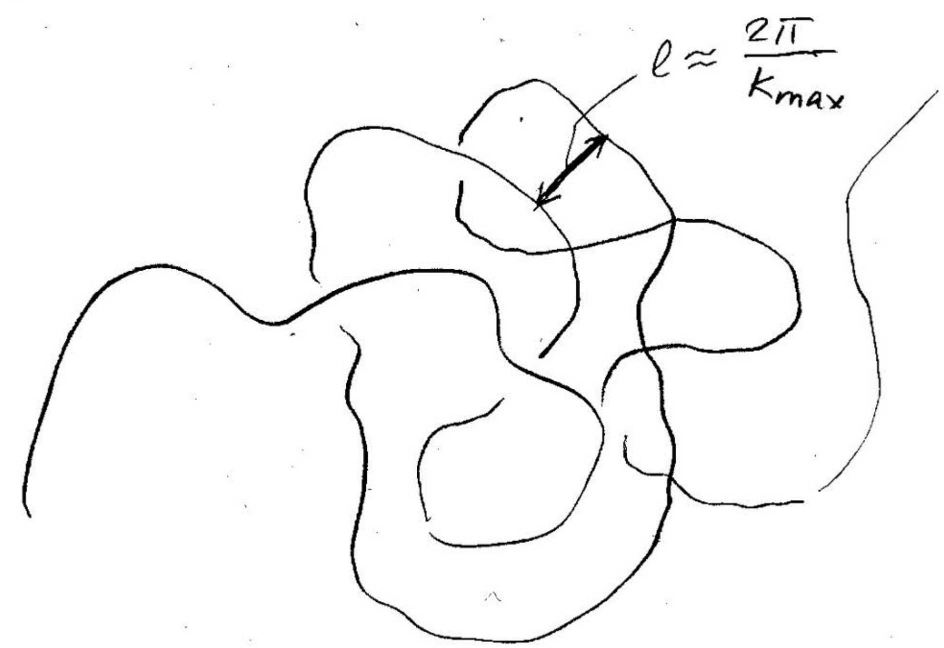
= equipartition of E among k-modes



T. D. Lee, 1952

Recent theoretical advances – Uriel's lectures.

Note. Thermalization is not just a numerical artifact, but can be relevant to a real physical effect, e.g. in superfluid He: Here, truncation appears naturally due to discreteness of quantum vortices. (L'vov et al 2006).



What is the minimal model which captures both K41 and thermalization?

Leith model (C. Leith, 1968).

Belongs to the class of Differential Approximation Models (DAM).

$$\frac{\partial E_k}{\partial t} = \frac{1}{8}\frac{\partial}{\partial k}\left(k^{\frac{11}{2}} E_k^{\frac{1}{2}} \frac{\partial}{\partial k}\left(\frac{E_k}{k^2}\right)\right) + f - \nu k^2 E_k.$$

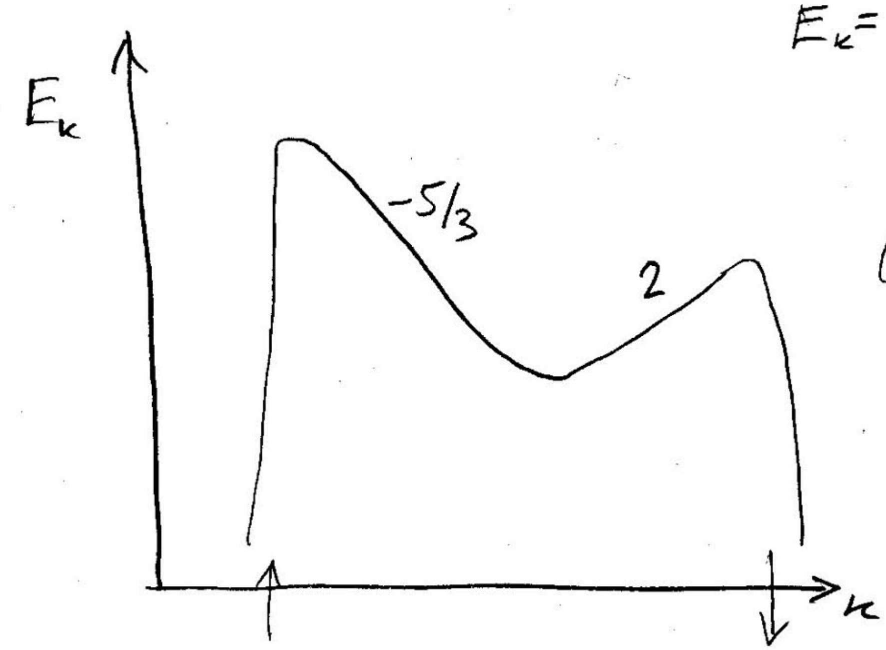
Simplest <u>local</u> model of turbulence.

- Contains both K41 and thermo solutions
- Can model nonstationary turbulence.
- Hybrid K41-thermo solution:

$$E_k = C k^2 (\epsilon k^{-11/2} + T)$$

flux "temperature"

Connaughton & SN'2004.



2D Turbulence

Extra invariant - enstrophy:

$$Z = \int (\nabla \times \vec{u})^2 \, d\vec{x} = \text{const}$$

Indeed, start with 2D Euler in vorticity form:

$$\underbrace{\partial_t \omega + \vec{u} \cdot \nabla \omega}_{D_t \omega} = 0$$

$$\boxed{\begin{array}{l} \omega = (\nabla \times \vec{u})_z \\ \text{Vorticity} \end{array}}$$

Vorticity is "frozen into" fluid particles.

$\Rightarrow$ Any integral of form

$$\int f(\omega) \, d\vec{x} \quad \text{is conserved}$$

(f - is arbitrary).

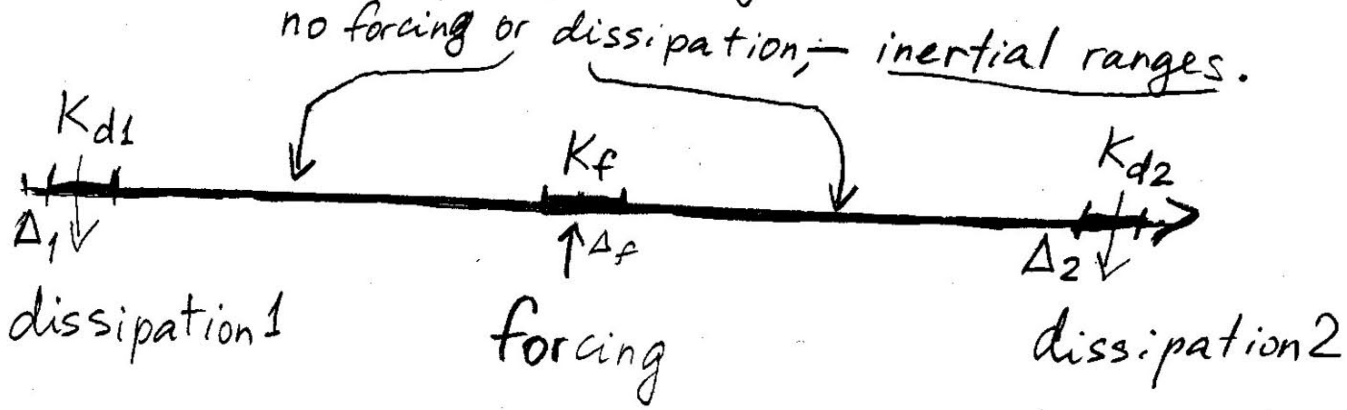
How do E and Z cascade?

Fjørtoft 1953 (No locality assumed!)

In the k-space, the spectral density of Z is related to the energy spectrum E_k like this:

$$Z_k = k^2 E_k .$$

Consider the following setup:



1). Let's suppose that in range Δ_f around k_f we produce energy per unit time: $\boxed{\epsilon}$.

This amounts to respective rate of enstrophy production: $\boxed{\eta = k_f^2 \epsilon}$.

2). Suppose now that a comparable amount of energy, $\sim \epsilon$, is dissipated (per unit time) near k_{d2}. But this would imply the rate of enstrophy dissipation

$$\sim k_{d2}^2 \epsilon \gg k_f^2 \epsilon = \eta$$

This is not possible in steady state (i.e. to dissipate Z at rate $\gg$ rate of production η).

$\Rightarrow$ $\boxed{\text{Most energy is dissipated near } k_{d1}}$.

3) Similarly, assuming dissipation of Z near k_{d1} at rate $\sim \eta$ also leads to contradiction (i.e. diss. of E at rate $k_{d1}^{-2}\eta \gg k_f^{-2}\eta = \epsilon$).

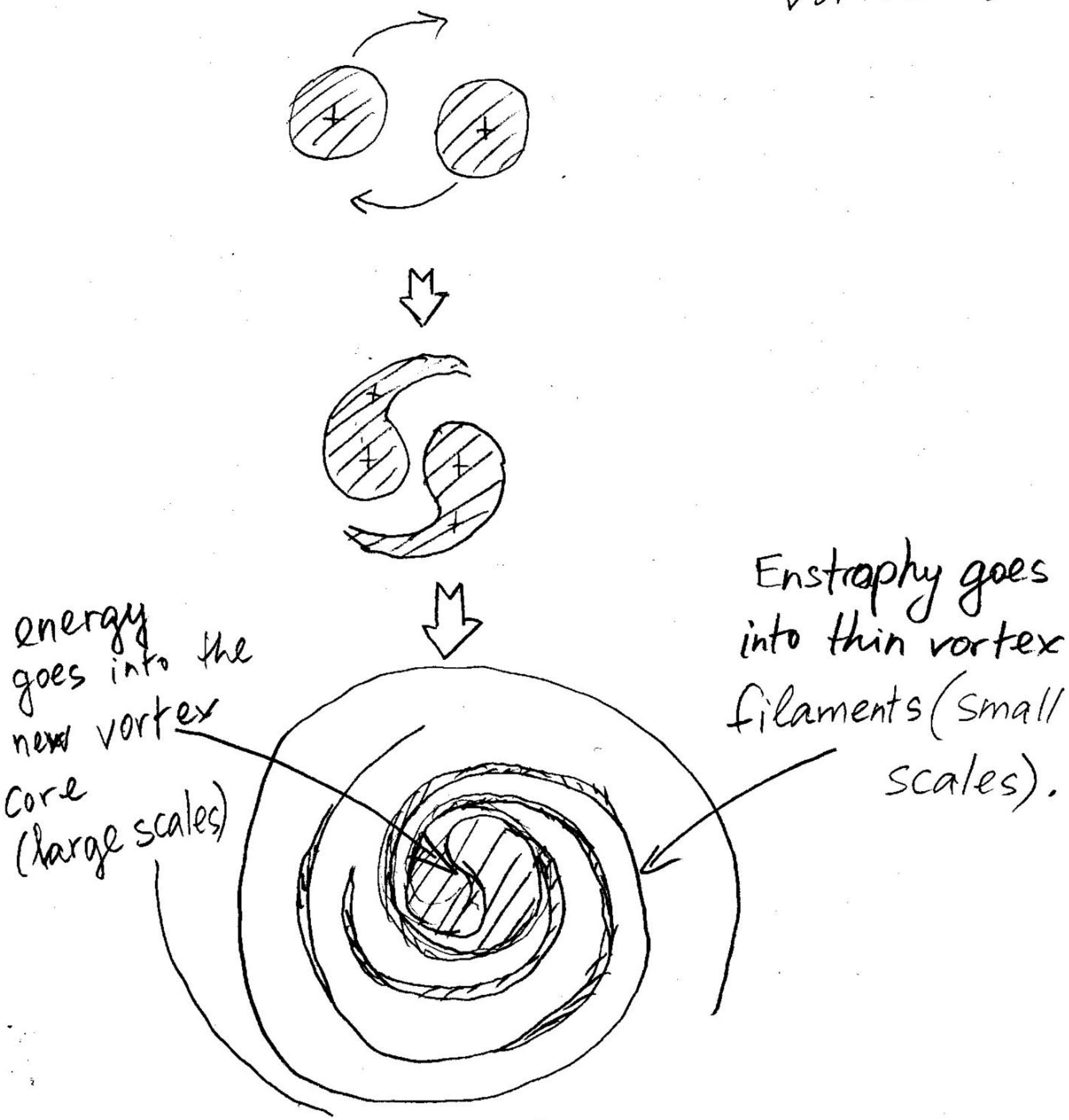
$\Rightarrow$ $\boxed{\text{Most enstrophy is dissipated near } k_{d2}}$.

<u>Again</u>: no locality was used in Fjørtoft.

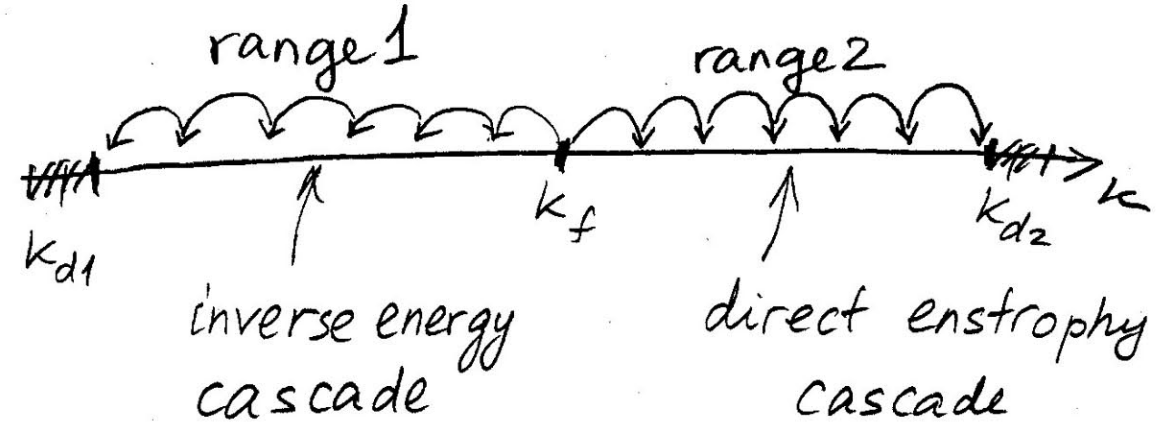
Prototype event for the E&Z transfers:

2 D Vortex merger

Initially: two like-signed round vortices.



Now, assuming locality (Kraichnan '67)



Due to locality, spectrum in range 1 will depend only on ϵ, and in range 2 — only on η.

1). Inverse energy cascade spectrum:

dim. argument identical to K41

$$\Rightarrow \quad E_k = C_1\, \epsilon^{2/3}\, k^{-5/3}$$

Note, this order-one constant is different from the Kolmogorov constant of 3D turbulence.

2) Direct enstrophy cascade:

$$\dim \eta = \frac{[\text{enstrophy density}]}{[\text{time}]} = \frac{[\omega^2]}{[t]} =$$

$$= [t]^{-3}.$$

$$\dim E_k = \frac{[\ell]^3}{[t]^2}.$$

$$\Rightarrow \boxed{E_k = C_2\, \eta^{2/3}\, k^{-3}}$$

the only possible dimensional combination

dim'less constant

Batchelor – Kraichnan spectrum

(First found by Batchelor in 1959 in the context of passive scalar turbulence).

2D turbulence context – Kraichnan 1967.

What is minimal model for local 2D turbulence?

Leith-type DAM (L'vov & Nazarenko'2006)

2-cascades + 2-thermo solutions (E & Z equipartitions) $\Rightarrow$ 4-th order equation:

$$\frac{\partial E}{\partial t} = C\sqrt{x}\,\frac{\partial^2}{\partial x^2}\left[E^{\frac{5}{2}}\, x^{\frac{15}{2}}\,\frac{\partial^2}{\partial x^2}\left(\frac{\sqrt{x}}{E}\right)\right] - \nu x E$$

where $x = k^2$.

Similar uses as Leith model:

1). Unsteady turbulence, T. with friction, "thermalization", double cascade (D. Lilly'87 – Naström-Gage spectrum)

2) Accurate prediction for Kolmogorov–Kraichan constants: $\frac{C_1}{C_2} = \left(\frac{2}{9}\right)^{2/3} \simeq 0.367$ | L'vov & Nazarenko 2006

More complex turbulence models

Direct interaction approximation - DIA

(Kraichnan 1958, Kadomtsev 1964).

Diff. eqn $\longrightarrow$ Integral eqn.

i.e. locality is more relaxed and is not guaranteed (even though assumed in derivation).

Locality = convergence of integral modeling the interaction term.

(Similarly to locality in Wave Turbulence).

DIA's children – LHDIA, EDQNM...

In spirit – all these models share K41 scaling, i.e. assume locality during their derivations.

All these models, as well are "mean field" type.

For the role of fluctuations and intermittency — see Alex', Antonio's and Berengere's talks.

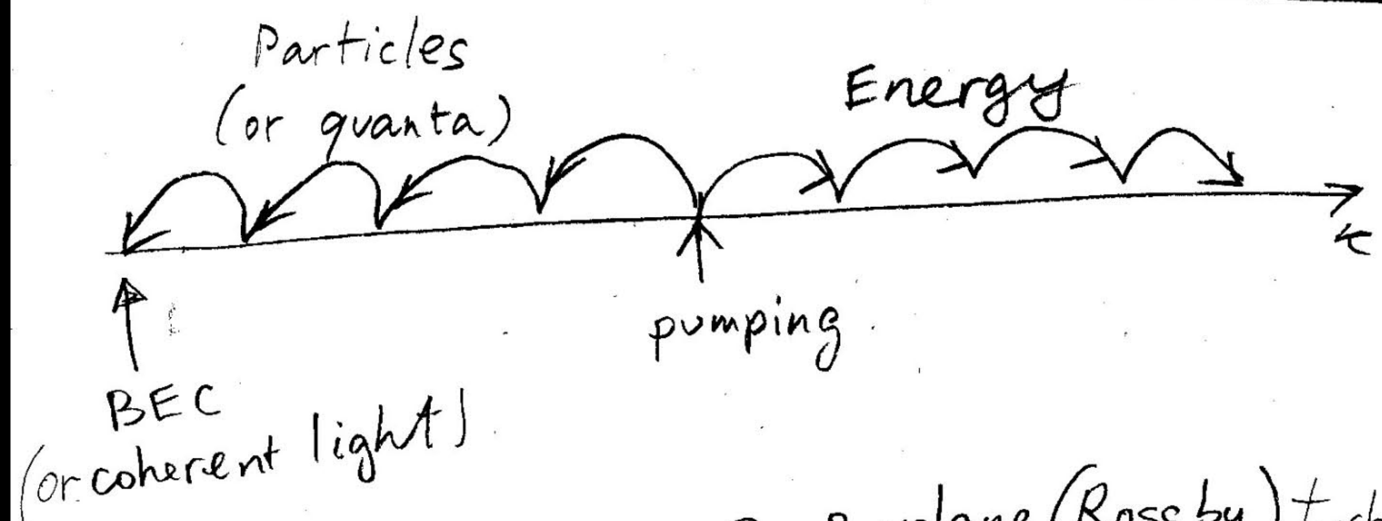
Antonio will also describe other interesting objects in 2D turbulence not covered in my lectures, e.g. fractality and conformal invariance of vorticity level sets.

Other Dual-cascade systems.

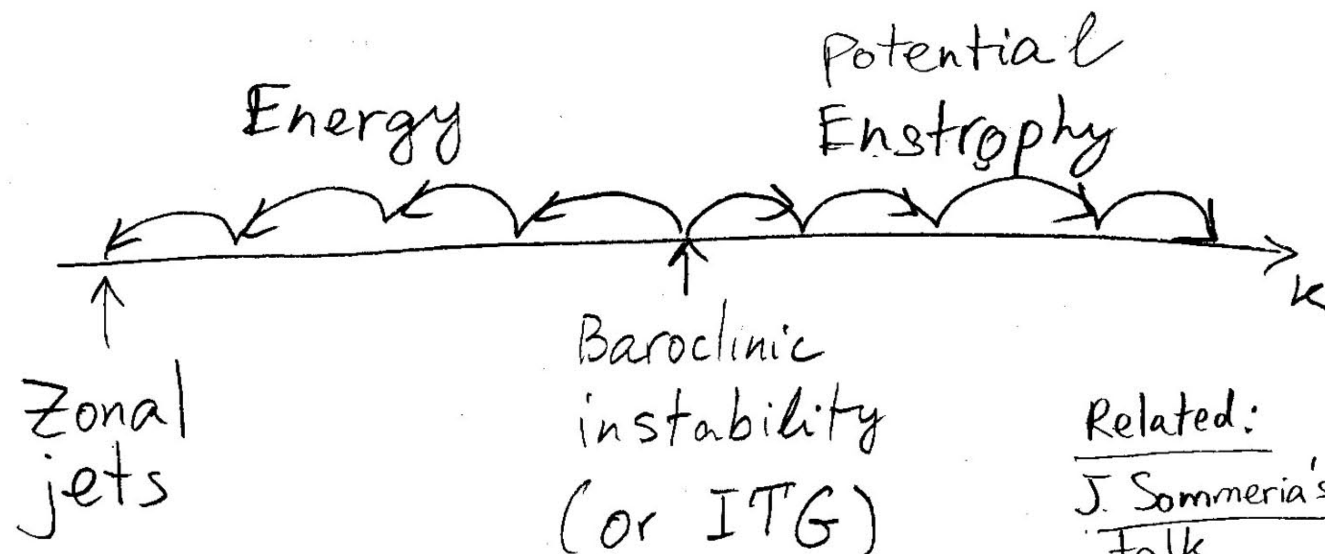
1) Optical Turbulence, BEC.

$$i\frac{\partial\psi}{\partial t} + \Delta\psi \pm |\psi|^2\psi = 0$$ - NLS eqn.

DAM: Newell et al '91

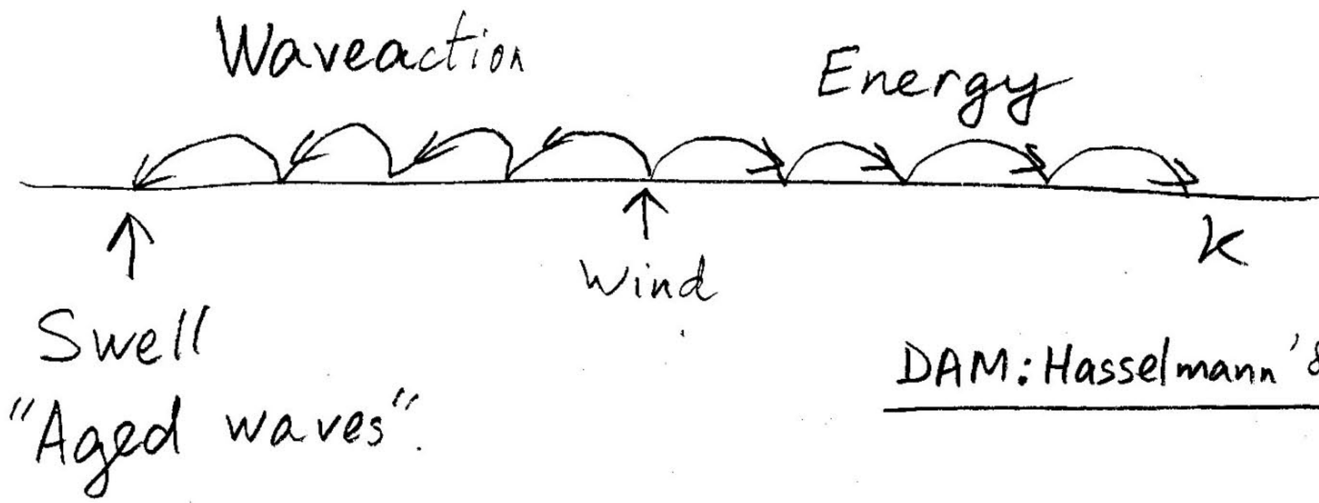


2) Drift turbulence & β-plane (Rossby) turbulence



Related:
J. Sommeria's
Talk

3) Water wave turbulence



DAM: Hasselmann '84

BUT: often there is not efficient dissipation at low k (large scales).

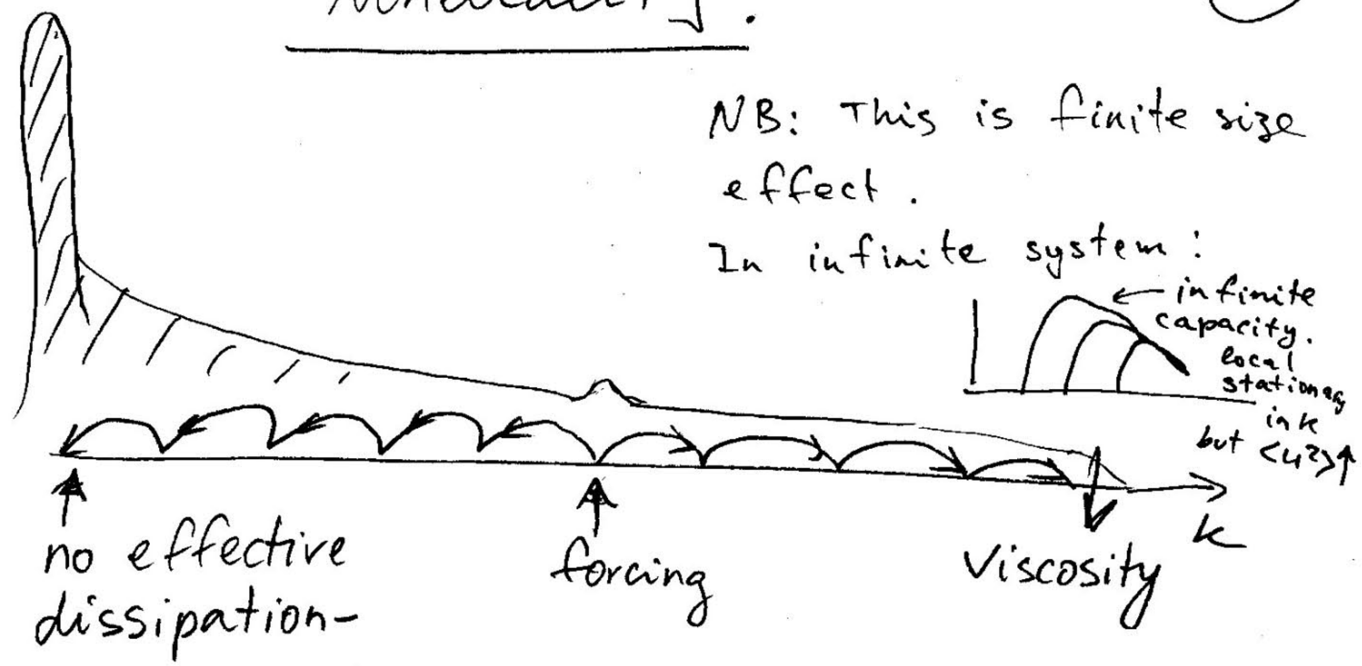
What happens then?

Breakdown of local (K41-like) scalings and transition to nonlocal turbulence regime.

This is the subject of the next lecture.

Nonlocality.

NB: This is finite size effect.
In infinite system:

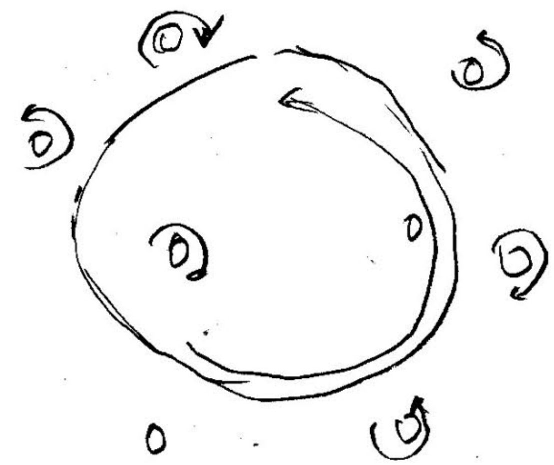


-energy pileup/condensation
Until interactions with these (largest) scales will start dominating the local (Richardson – Kolmogorov type) interactions.

→ Breakdown of K41 – Kraichnan scalings. What happens next? Will the new (nonlocal) scalings be universal?

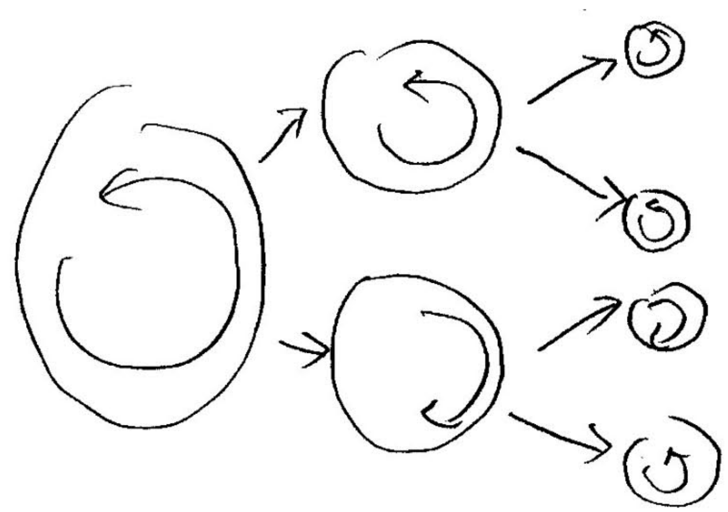
Nonlocal interaction of scales in turbulence:

dominance of interaction among eddies with very different sizes:



Nonlocal

and not:



Local

Remark 1:

Dominance of nonlocality occurs more often in 2D (for the reasons to be discussed) and not so much in 3D (except for cases with external mean flows, magnetic field etc.)

However: even in 3D nonlocal interactions may be important for:

1) Intermittency (Laval, Dubrulle, SN 2001)

2) Numerical modelling of subgrid scales (Berengere's talk).

Remark 2: Direct enstrophy cascade is marginally nonlocal $\to$ log correction. (Kraichnan '71).

We omit this case (for lack of time) and will only concentrate on strong, dominant, nonlocality.

Description of nonlocal interaction

Two components: large-scale, $\vec{u}_L$, ω_L, and small-scale, $\vec{u}_s$, ω_s.

2D Euler equation in vorticity form:

$$\partial_t \omega + \vec{u}\cdot\nabla\omega = 0 \qquad (*)$$

$u = \vec{u}_L + \vec{u}_s$

$\omega = \omega_L + \omega_s$

average over small scales:

$$\partial_t \omega_L + \vec{u}_L\cdot\nabla\omega_L = \underbrace{-\nabla\cdot\langle\vec{u}_s\cdot\omega_s\rangle}_{} \qquad (**)$$

forcing of LS by SS (describes nonlocal energy transfer from SS to LS).

Now, $(*)-(**)=$

$$\partial_t \omega_s + \vec{u}_L\cdot\nabla\omega_s = \underbrace{-u_s\cdot\nabla\omega_L}_{} + \underbrace{\nabla\cdot\left(\langle\vec{u}_s\omega_s\rangle - \vec{u}_s\omega_s\right)}_{}$$

ignore this because $\dfrac{u_s\cdot\nabla\omega_L}{u_L\cdot\nabla\omega_s} \sim \dfrac{k_L^2}{k_s^2} \ll 1$

ignore: these are local interactions of SS among themselves

$$\Rightarrow \quad \partial_t \omega_s + \vec{u}_L \cdot \nabla \omega_s = 0 \qquad (***)$$

formally, it is equivalent to advection of passive scalar by external velocity field $\vec{u}_L$.

Scale of ω_s is $\ll$ scale of $\vec{u}_L$

$\rightarrow$ so called Batchelor's regime of passive scalar turbulence.

To consider cascades we need Fourier space. In this case Fourier has to be performed over a finite window, i.e. $SS \ll$ window $\ll LS$.

(This can be done in several ways, e.g. using Wigner's function or using Gabor transform).

$$\hat{\omega}(\vec{k},\vec{x}) = \int f(|\vec{x}-\vec{x}'|) e^{i\vec{k}\cdot\vec{x}'} \omega_s(\vec{x}') d\vec{x}'$$

distribution of vorticity wavepackets in (k,x) space

Equation for $\hat{\omega}$:

(****) $D_t |\hat{\omega}|^2 = 0$ ← Note: linear equation!

$D_t = \partial_t + \dot{\vec{x}} \cdot \nabla + \dot{\vec{k}} \frac{\partial}{\partial \vec{k}}$ ← derivative along paths in $(\vec{k}, \vec{x})$ space.

$$\left.\begin{aligned} \dot{\vec{x}} &= \vec{u}_L \\ \dot{\vec{k}} &= -(\vec{k} \cdot \nabla)\vec{u}_L \end{aligned}\right\}$$

→ transfer in $\vec{k}$ $\propto$ LS strain tensor.

→ passive advection in $\vec{x}$.

In homogeneous turbulence

$Z_k \equiv \langle |\hat{\omega}|^2 \rangle$ is independent of $\vec{x}$.

in this case, LS only enters to SS evolution via strain tensor:

$$S_{ij} = \nabla_i u_{L_j}$$

Spectra of nonlocal 2D Turbulence

Dimensional analysis.

1). Linear SS equations (see (***) or (****)) $\Rightarrow$

$E_k \propto \epsilon$ - for energy cascade

$E_k \propto \eta$ - for the enstrophy range.

2) The Only other dim. quantity relevant to SS dynamics is the LS strain (see previous page).

$$\dim S = \frac{1}{[t]}.$$

(NLE1).

$$\Rightarrow \boxed{E_k = C_1 \, \epsilon \, S^{-1} \, K^{-1}}$$

$$\frac{L^3}{T^2} \qquad \frac{L^2}{T^3} \quad T \quad L$$

For the (inverse) energy range.

Obtained by Kraichnan' 1974 in passive scalars' context (but wrongly interpreted as thermo equilibrium).. Corrected in Laval & SN' 1999.

And $\boxed{E_k = C_2 \eta S^{-1} K^{-3}}$ $\leftarrow$ (NLE2)

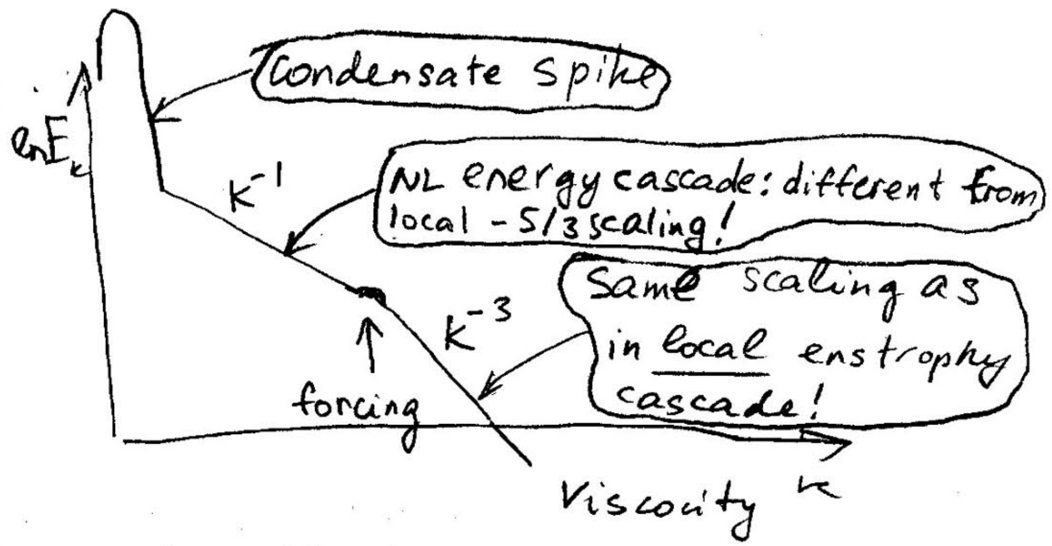
More wellknown spectrum Batchelor 1959 (passive scalar context).

Putting these together,

we get the following scalings of nonlocal

turbulence:

(Laval & SN'99)



Remark 1: Conservation of Z is trivial (eqn $(***)$ is still a Lagrangian conservation of vorticity).
But: Conservation of E is nontrivial in NL case (in principle E can be directly transferred to/from condensate in NL way). However, for isotropic turbulence such NL energy transfer was shown $=0$ (Laval & SN'99) $\Rightarrow$ E is conserved within SS component separately.
This justifies $\epsilon = \text{const}$ assumption $\Rightarrow$ k^{-1} scaling.

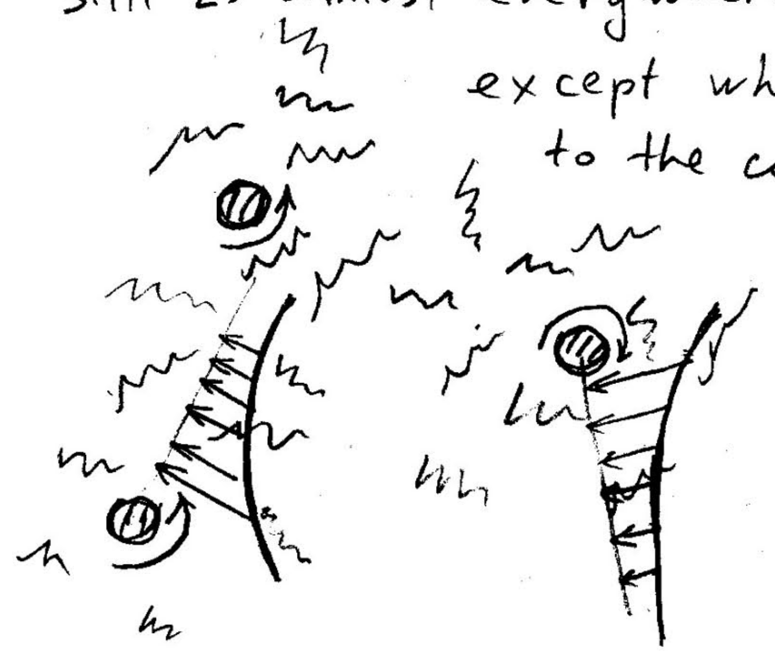
Remark 2: Condensation tends to occur into vortices with small vorticity cores (this naturally maximizes energy for a fixed enstrophy (fixed by k^{-3} part))
$\Rightarrow$ Condensate spectrum is not a spike but is a broader k-distribution (k^{-3} in numerics of Connaughton et al'07)

As a result, the condensate and the energy cascade components overlap in the k-space $\Rightarrow$ k^{-1} is not seen.

But: If one separates the condensate (by wavelets) $\rightarrow$ the remaining part has clear k^{-1} spectrum (Connaughton et al' 2007).

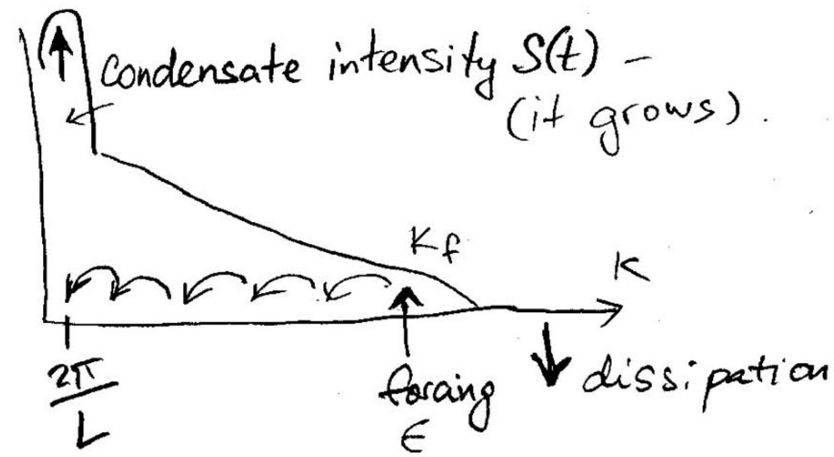
Q: why the scale separation argument (leading to k^{-1}) works even if there is not separation in the k-space?

A: Velocity field of condensate is still LS almost everywhere — except when too close to the condensate vortex cores (shaded).



How is condensate coupled with turbulence?

(feedback loop: SN's PhD thesis, papers with Balk, Zakharov, Dyachenko, Manin, Dubrulle, Laval, 1989 – 2002).



Consider a system with instability forcing:

(not typical for most turbulence theories)

$$\partial_t \omega + \vec{u}\cdot\nabla\omega = \underbrace{\hat{\gamma}\,\omega}_{\text{instability}} - \underbrace{\nu\nabla^2\omega}_{\text{viscosity}}$$

instability: linear operator on ω which becomes algebraic multiplication in the k-space: $\gamma(k)\,\hat{\omega}(k)$.

So $\dot{E} = \int \gamma_k E_k \, dk = \epsilon$

(1) $\epsilon(t) \sim \gamma(k_f)\, E(k_f)\, k_f$

(2) All energy produced at k_f eventually ends up in the condensate. $\Rightarrow$ $E_c = \frac{V_c^2}{2} \sim \frac{S^2}{L^2} \sim \int_0^t \epsilon\, dt'$.

(3) (NLE1) $\rightarrow$ $E_{k_f} \sim \epsilon S^{-1} k_f^{-1}$.

Substitute (3) $\to$ (1):

$$E_{k_f} = \gamma_{k_f} E_{k_f} \cancel{k_f}\, S^{-1} \cancel{k_f^{-1}}$$

Two solutions:

(i) $S = \gamma_{k_f}$ (saturation of condensate).

(ii) $E_{k_f} = 0$ (suppression of small-scale turbulence).

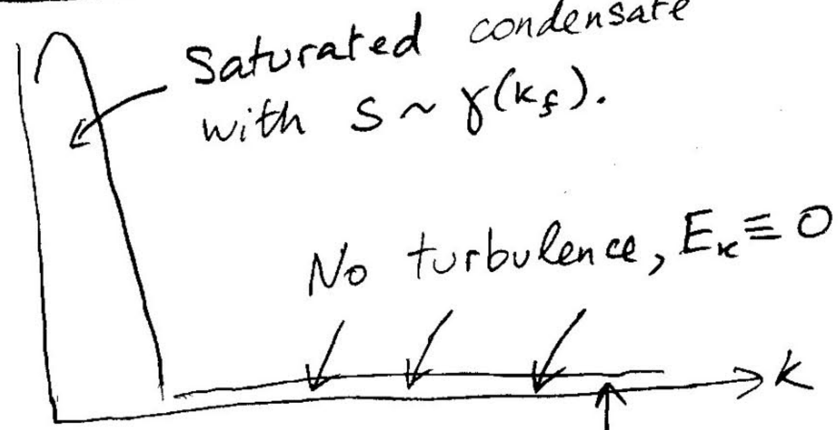
Which one do we chose?

Both (i) and (ii) get realized

because (i) implies (ii) via (2):

If $E_{k_f} \neq 0 \to \epsilon \neq 0 \to S$ would keep growing and not saturate.

So, final state:



Turned-off forcing: $\epsilon \sim \gamma_{k_f} E_{k_f} k_f = 0$ (since $E_{k_f} = 0$).

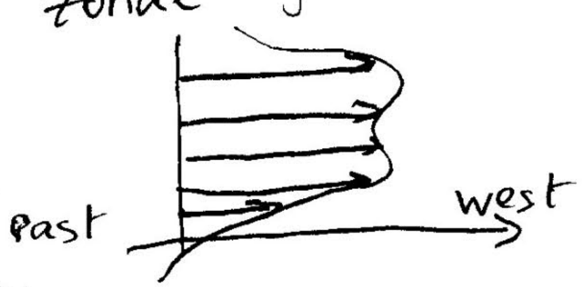
We have obtained the following <u>feedback loop in turbulence</u> (FLT):

(1). Turbulence creates condensate via an inverse cascade.

(2). The condensate <u>kills turbulence</u>.

<u>NB</u>: Historically FLT was first obtained for a slightly more complicated fluid model which includes β-effect - so called Charney-Hassegawa-Mima equation. It describes drift turbulence in plasmas and Rossby waves/vortices in atmospheres and oceans.

In this case, condensate takes form of zonal jets rather than round(ish) vortices:



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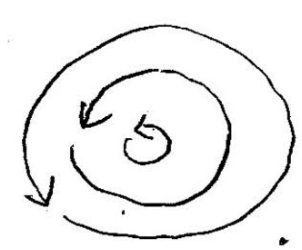

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In fact this is in fusion plasmas where FLT has found its most important application. Namely, FLT is generally accepted among the fusion scientists as a mechanism explaining experimentale transitions from Low to High confinement regimes experimentaly observed in tokamaks (LH transition). Indeed, suppression of turbulence means suppression of anomalous losses of energy and particles from the plasma core. FLT mechanism is the reason why we can hope that ITER tokamak will work.

END