HT I

Multiple Integrals

1. (a) For the following integrals sketch the region of integration and so write equivalent integrals with the order of integration reversed. Evaluate the integrals both ways.

$$\int_0^{\sqrt{2}} \mathrm{d}y \int_{y^2}^{2} \mathrm{d}x\, y\ , \qquad \int_0^4 \mathrm{d}x \int_0^{\sqrt{x}} \mathrm{d}y\, y\sqrt{x}\ , \qquad \int_0^1 \mathrm{d}y \int_{-y}^{y^2} \mathrm{d}x\, x\ .$$

(b) Reverse the order of integration and hence evaluate:

$$\int_0^{\pi} \mathrm{d}y \int_y^{\pi} \mathrm{d}x\, \frac{\sin x}{x}\ .$$

2. (a) A mass distribution in the positive x region of the xy-plane and in the shape of a semi-circle of radius a, centred on the origin, has mass per unit area k. Find, using plane polar coordinates,

(i) its mass M, (ii) the coordinates $(\overline{x}, \overline{y})$ of its centre of mass, (iii) its moments of inertia about the x and y axes.

(b) Do as above for a semi-infinite sheet with mass per unit area

$$\sigma = k \exp\left(-\frac{x^2+y^2}{a^2}\right) \quad \text{for} \quad x \geq 0\ , \qquad \sigma = 0 \quad \text{for} \quad x < 0\ .$$

where a is a constant. Comment on the comparisons between the two sets of answers.

Note that

$$\int_0^{\infty} \mathrm{d}u\, \exp\left(-\lambda u^2\right) = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}}\ .$$

(c) Evaluate the following integral:

$$\int_0^{a} \mathrm{d}y \int_0^{\sqrt{a^2-y^2}} \mathrm{d}x\, (x^2+y^2)\, \arctan(y/x)\ .$$

3. The pair of variables (x, y) are each functions of the pair of variables (u, v) and *vice versa.* Consider the matrices

$$A = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}\ .$$

(a) Show using the chain rule that the product AB of these two matrices equals the unit matrix I.

(b) Verify this property explicitly for the case in which (x, y) are Cartesian coordinates and u and v are the polar coordinates (r, θ).

(c) Assuming the result that the determinant of a matrix and the determinant of its inverse are reciprocals, deduce the relation between the Jacobians

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \qquad \text{and} \qquad \frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\ .$$

Vector Calculus

4. Find $\nabla\phi$ in the cases: (a) $\phi = \ln|\boldsymbol{r}|$; (b) $\phi = r^{-1}$, where $r = |\boldsymbol{r}|$.

5. Given that $F = x^2 z + \mathrm{e}^{y/x}$ and $G = 2z^2 y - xy^2$, find $\nabla(F+G)$ and $\nabla(FG)$ at $(1, 0, -2)$.

6. Find the equation for the tangent plane to the surface $2xz^2 - 3xy - 4x = 7$ at $(1, -1, 2)$.

7. A vector field $\boldsymbol{A}(\boldsymbol{r})$ is defined by its components

$$(4x - y^4, -4xy^3 - 3y^2, 4).$$

Evaluate the line integral $\int \boldsymbol{A}.\mathrm{d}\boldsymbol{l}$ between the points with position vectors $(0, 0, 0)$ and $(1, 2, 0)$ along the following paths

(a) the straight line from $(0, 0, 0)$ to $(1, 2, 0)$;

(b) on the path of straight lines joining $(0, 0, 0)$, $(0, 0, 1)$, $(1, 0, 1)$, $(1, 2, 1)$ and $(1, 2, 0)$ in turn.

Show that $\boldsymbol{A}$ is conservative by finding a scalar function $\phi(\boldsymbol{r})$ such that $\boldsymbol{A} = \nabla\phi$.

8. A vector field $\boldsymbol{A}(\boldsymbol{r})$ is defined by its components

$$(3x^2 + 6y, -14yz, 20xz^2).$$

Evaluate the line integral $\int \boldsymbol{A}.\mathrm{d}\boldsymbol{l}$ from $(0, 0, 0)$ to $(1, 1, 1)$ along the following paths

(a) $x = t$, $y = t^2$, $z = t^3$;

(b) on the path of straight lines joining $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, and $(1, 1, 1)$ in turn;

(c) the straight line joining the two points.

Is $\boldsymbol{A}$ conservative?

9. Two circles have equations: (i) $x^2 + y^2 + 2ax + 2by + c = 0$; and (ii) $x^2 + y^2 + 2a'x + 2b'y + c' = 0$. Show that these circles are orthogonal if $2aa' + 2bb' = c + c'$.

HT II

Multiple Integrals

1. (a) Using the change of variable $x+y=u$, $x-y=v$, evaluate the double integral $\iint_R (x^2+y^2)\,\mathrm{d}x\mathrm{d}y$, where R is the region bounded by the straight lines $y=x$, $y=x+2$, $y=-x$ and $y=-x+2$.

(b) Given that $u=xy$ and $v=y/x$, show that $\partial(u,v)/\partial(x,y)=2y/x$. Hence evaluate the integral

$$\iint \exp(-xy)\,\mathrm{d}x\mathrm{d}y$$

over the region $x>0$, $y>0$, $xy<1$, $1/2<y/x<2$.

2. Evaluate

$$\iint \exp[-(x^2+y^2)]\mathrm{d}x\mathrm{d}y$$

over the area of a circle with centre $(0,0)$ and radius a. By letting a tend to ∞, evaluate

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x.$$

3. Spherical polar coordinates are defined in the usual way. Show that

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta\ .$$

Draw a diagram to illustrate the physical significance of this result.

4. A solid hemisphere of uniform density k occupies the volume $x^2+y^2+z^2\le a^2$, $z\ge 0$. Using symmetry arguments wherever possible, find

(i) its total mass M, (ii) the position $(\overline{x},\overline{y},\overline{z})$ of its centre-of-mass, and (iii) its moments and products of inertia, $I_{xx},\ I_{yy},\ I_{zz},\ I_{xy},\ I_{yz},\ I_{zx}$, where

$$I_{zz}=\int k\,(x^2+y^2)\,\mathrm{d}V,\qquad I_{xy}=\int k\,xy\,\mathrm{d}V,\qquad \text{etc.}$$

5. Show that under certain circumstances the area of a curved surface can be expressed in the form

$$\iint \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,\mathrm{d}x\mathrm{d}y,$$

where the integral is taken over the plane $z=0$. Sketch the surface $z^2=2xy$ and find the area of the part of it which lies inside the hemisphere $x^2+y^2+z^2=1$, $z>0$.

6. Show that the surface area of the curved portion of the hemisphere in Problem 4 is $2\pi a^2$ by

(i) directly integrating the element of area $a^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi$ over the surface of the hemisphere.

(ii) projecting onto an integral taken over the xy plane.

7. (a) Find the area of the plane $x-2y+5z=13$ cut out by the cylinder $x^2+y^2=9$.

(b) A uniform lamina is made of that part of the plane $x+y+z=1$ which lies in the first octant. Find by integration its area and also its centre of mass. Use geometrical arguments to check your result for the area.

Vector Calculus

8. If $\boldsymbol{n}$ is the unit normal to the surface S, evaluate $\int\int \boldsymbol{r}.\boldsymbol{n}\,\mathrm{d}S$ over (a) the unit cube bounded by the coordinate planes and the planes $x = 1$, $y = 1$ and $z = 1$; (b) the surface of a sphere of radius a centred at the origin.

9. Calculate the solid angle of a cone of half-angle α.

10. Evaluate $\int \boldsymbol{A}.\boldsymbol{n}\ \mathrm{d}S$ for the following cases:

(a) $\boldsymbol{A} = (y, 2x, -z)$ and S is the surface of the plane $2x + y = 6$ in the first octant cut off by the plane $z = 4$.

(b) $\boldsymbol{A} = (x + y^2, -2x, 2yz)$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant

(c) $\boldsymbol{A} = (6z, 2x+y, -x)$ and S is the entire surface of the region bounded by the cylinder $x^2 + z^2 = 9$, $x = 0$, $y = 0$, $z = 0$ and $y = 8$.

Oscillations

11. Two identical pendula each of length l and with bobs of mass m are free to oscillate in the same plane. The bobs are joined by a massless spring with a small spring constant k, such that the tension in the spring is k times its extension.

(a) Show that the motion of the two bobs is governed by the equations

$$m\ddot{x} = -\frac{mgx}{l} + k(y - x) \quad \text{and} \quad m\ddot{y} = -\frac{mgy}{l} - k(y - x)$$

(b) By looking for solutions in which x and y vary harmonically at the same angular frequency ω, convert these differential equations into two ordinary simultaneous equations for the amplitudes of oscillation x_0 and y_0.

(c) Why do we not expect these equations to determine the absolute values of x_0 and y_0?

(d) Each of these new equations gives the ratio of x_0/y_0 in terms of ω. Find the values of ω that make these equations consistent.

(e) For each of these values of ω, find the ratio x_0/y_0. Describe the relative motions of the two pendula for each of these normal modes. Is one of the values of ω obvious?

(f) At $t = 0$, both pendula are at rest, with $x = A$ and $y = A$. Describe the subsequent motion of the two pendula.

12. Two coupled simple pendula are of equal length l, but their bobs have different masses m_1 and m_2. Their equations of motion are:

$$\ddot{x} = -\frac{g}{l}x - \frac{k}{m_1}(x - y) \quad \text{and} \quad \ddot{y} = -\frac{g}{l}y + \frac{k}{m_2}(x - y)$$

(a) Use the standard method first to find the frequencies and the relative amplitudes of the bobs for the normal modes of the system.

(b) By taking suitable linear combinations of the two equations of motion, obtain two uncoupled differential equations for linear combinations of x and y. Hence again find the normal mode frequencies and the relative amplitudes.

[Hint: One of these linear combinations is fairly obvious. For the other, it may be helpful to consider the centre of mass of the two bobs.]

HT III

Vector Calculus

1. The vector $\boldsymbol{A}$ is a function of position $\boldsymbol{r} = (x, y, z)$ and has components (xy^2, x^2, yz). Calculate the surface integral $\int \boldsymbol{A}.\mathrm{d}\boldsymbol{S}$ over each face of the triangular prism bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x + y = 1$ and $z = 1$. Show that the integral $\int \boldsymbol{A}.\mathrm{d}\boldsymbol{S}$ taken outwards over the whole surface is not zero. Show that it equals $\int \nabla.\boldsymbol{A}\,\mathrm{d}V$ calculated over the volume of the prism. Why?

2. Show that $\nabla.(\phi\boldsymbol{A}) = \boldsymbol{A}.\nabla\phi + \phi\nabla.\boldsymbol{A}$, where ϕ is any scalar field and $\boldsymbol{A}$ is any vector field.

3. For $\boldsymbol{A} = (3xyz^2, 2xy^3, -x^2yz)$ and $\phi = 3x^2 - yz$, find: (a) $\nabla.\boldsymbol{A}$; (b) $\boldsymbol{A}.\nabla\phi$; (c) $\nabla.(\nabla\phi)$.

4. A body expands linearly by a factor $1 + \alpha$ because of a rise in temperature. The expansion shifts the particle originally at $\boldsymbol{r}$ to $\boldsymbol{r} + \boldsymbol{h}$. Calculate $\nabla.\boldsymbol{h}$. By what fraction does the volume increase?

5. The magnetic field $\boldsymbol{B}$ at a distance r from a straight wire carrying a current I has magnitude $\mu_0 I/2\pi r$. The lines of force are circles centred on the wire and in planes perpendicular to it. Show that $\nabla.\boldsymbol{B} = 0$.

Oscillations

6. Consider again the two identical pendula of Problem 11 on the last sheet.

(a) At $t = 0$, both pendula are at rest, with $x = A$ and $y = 0$. They are then released. Describe the subsequent motion of the system. Given that $k/m = 0.105g/l$, show that

$$x = A\cos\Delta t\cos\overline{\omega}t \quad \text{and} \quad y = A\sin\Delta t\sin\overline{\omega}t$$

where $\Delta = 0.05\sqrt{g/l}$ and $\overline{\omega} = 1.05\sqrt{g/l}$.

Sketch x and y as functions of t, and note that the oscillations are transferred from the first pendulum to the second and back. Approximately how many oscillations does the second pendulum have before the first pendulum is oscillating again with its initial amplitude?

(b) State a different set of initial conditions such that the subsequent motion of the pendula corresponds to that of a normal mode.

(c) In the most general initial conditions, each bob has a given initial displacement and a given initial velocity. Explain as fully as you can why the solution in this general case has the form:

$$x = \alpha\cos(\omega_1 t + \phi_1) + \beta\cos(\omega_2 t + \phi_2)$$
$$y = \alpha\cos(\omega_1 t + \phi_1) - \beta\cos(\omega_2 t + \phi_2),$$

where ω_1 and ω_2 are the normal mode angular frequencies and α, β, ϕ_1 and ϕ_2 are arbitrary constants. How are these arbitrary constants determined?

(d) At $t = 0$, both bobs are at their equilibrium positions: the first is stationary but the second is given an initial velocity v_0. Show that subsequently

$$x = \tfrac{1}{2}v_0\left(\frac{1}{\omega_1}\sin\omega_1 t - \frac{1}{\omega_2}\sin\omega_2 t\right) \quad ; \quad y = \tfrac{1}{2}v_0\left(\frac{1}{\omega_1}\sin\omega_1 t + \frac{1}{\omega_2}\sin\omega_2 t\right).$$

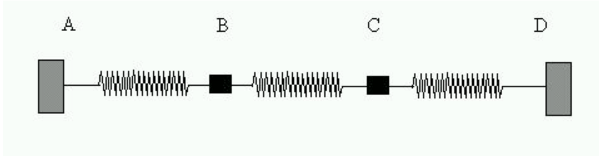
(e) For the initial conditions of part (d), and with $k/m = 0.105g/l$, describe as fully as possible the subsequent velocities of the two bobs.

7. A mass m is suspended from a beam by a massless spring with spring constant k. From this mass is suspended a second identical mass by an identical spring. Considering only motion in the vertical direction, obtain the differential equations for the displacements of the two masses from their equilibrium positions. Show that the angular frequencies of the normal modes are given by

$$\omega^2 = (3 \pm \sqrt{5})\frac{k}{2m}.$$

Find the ratio of the amplitudes of the two masses in each separate mode. Why does the acceleration due to gravity not appear in these answers?

8. AB, BC, and CD are identical springs with negligible mass, and stiffness constant k:



The masses m, fixed to the springs at B and C, are displaced by small distances x_1 and x_2 from their equilibrium positions along the line of the springs, and execute small oscillations. Show that the angular frequencies of the normal modes are $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{3k/m}$. Sketch how the two masses move in each mode. Find x_1 and x_2 at times $t > 0$ given that at $t = 0$ the system is at rest with $x_1 = a$, $x_2 = 0$.

9. * The setup is as for Problem 8, except that in this case the springs AB and CD have stiffness constant k_0, while BC has stiffness constant k_1. C is clamped, B vibrates with frequency $\nu_0 = 1.81\,\mathrm{Hz}$. The frequency of the lower-frequency normal mode is $\nu_1 = 1.14\,\mathrm{Hz}$. Calculate the frequency of the higher-frequency normal mode, and the ratio k_1/k_0. (From French 5-7).

10. * Two particles, each of mass m, are connected by a light spring of stiffness k, and are free to slide along a smooth horizontal track. What are the normal frequencies of this system? Describe the motion in the mode of zero frequency. Why does a zero-frequency mode appear in this problem, but not in Problem 8, for example?

HT IV

Vector Calculus

1. Sketch the vector fields $\boldsymbol{A} = (x, y, 0)$ and $\boldsymbol{B} = (y, -x, 0)$. Calculate the divergence and curl of each vector field and explain the physical significance of the results obtained.

2. Evaluate $\mathrm{curl}\,(\alpha\mathbf{a})$, $\mathrm{div}\,\mathrm{curl}\,\mathbf{a}$, $\mathrm{curl}\,\mathrm{curl}\,\mathbf{a}$.

3. Evaluate $\nabla \times \nabla\phi$, $\nabla(\phi\psi)$, $\nabla \cdot (\mathbf{a} \times \mathbf{b})$, $\nabla \times (\mathbf{a} \times \mathbf{b})$, $\nabla \cdot (\nabla\phi \times \nabla\psi)$.

4. For $\mathbf{c}$ a constant vector, evaluate $\mathrm{div}\,\mathbf{r}$, $\mathrm{curl}\,\mathbf{r}$, $\mathrm{div}\,(r^n\mathbf{r})$, $\mathrm{curl}\,(r^n\mathbf{r})$, $\mathrm{div}\,[\mathbf{r} \times (\mathbf{c} \times \mathbf{r})]$, $\mathrm{curl}\,[\mathbf{r} \times (\mathbf{c} \times \mathbf{r})]$.

5. For $\mathbf{c}$ a constant vector, prove that $\nabla[\mathbf{c} \cdot \nabla(r^{-1})] = -\nabla \times [\mathbf{c} \times \nabla(r^{-1})]$.

6. Find a vector field $\mathbf{A}$ such that $\nabla \times \mathbf{A} = (0, 0, B)$, where B is a constant. Is this field unique?

7. The vector $\boldsymbol{A}(\boldsymbol{r}) = (y, -x, z)$. Verify Stokes' theorem for the hemispherical surface $|\boldsymbol{r}| = 1$, $z \geq 0$.

8. $\boldsymbol{A} = (y, -x, 0)$. Find $\int \boldsymbol{A}.\mathrm{d}\boldsymbol{l}$ for a closed loop on the surface of the cylinder $(x - 3)^2 + y^2 = 2$. Consider the cases (i) in which the loop wraps round the cylinder's axis, and (ii) in which the loop does not (so it can be continuously deformed to a point).

9. A bucket of water is rotated slowly with angular velocity ω about its vertical axis. When a steady state has been reached the water rotates with a velocity field $\boldsymbol{v}(\boldsymbol{r})$ as if it were a rigid body. Calculate $\nabla.\boldsymbol{v}$ and interpret the result. Calculate $\nabla \times \boldsymbol{v}$. Can the flow be represented in terms of a velocity potential ϕ such that $\boldsymbol{v} = \nabla\phi$? If so, what is ϕ?

10. If $\phi = 2xyz^2$, $\boldsymbol{F} = (xy, -z, x^2)$ and C is the curve $x = t^2$, $y = 2t$,$z = t^3$ from $t = 0$ to $t = 1$, evaluate the line integrals: (a) $\int_C \phi\,\mathrm{d}\boldsymbol{r}$; (b) $\int_C \boldsymbol{F} \times \mathrm{d}\boldsymbol{r}$.

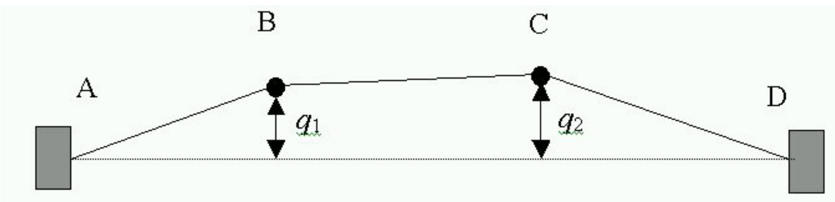
Oscillations

11. * A stretched massless spring has its ends at $x = 0$ and $x = 3l$ fixed, and has equal masses attached at $x = l$ and $x = 2l$. The masses slide on a smooth horizontal table. Convince yourself that, for small oscillations, it is reasonable to neglect the changes in tension caused by the variation in length of the three sections of the spring resulting from the transverse motion of the masses. Show that the equations of the transverse motion of the masses are approximately

$$m\ddot{y}_1 = \frac{T}{l}(y_2 - 2y_1) \quad \text{and} \quad m\ddot{y}_2 = \frac{T}{l}(y_1 - 2y_2),$$

where T is the tension in the spring. Find the frequencies and the ratio of amplitudes of the transverse oscillations for the normal modes of the two masses. Is the relative motion of the higher-frequency mode reasonable?

12. The figure shows two masses m at points B and C of a string fixed at A and D, executing small transverse oscillations. The tensions are assumed to be all equal, and in equilibrium $AB = BC = CD = l$.



If the (small) transverse displacements of the masses are denoted by q_1 and q_2, the equations of motion are

$$m\ddot{q}_1 = -k(2q_1 - q_2) \quad ; \quad m\ddot{q}_2 = -k(2q_2 - q_1), \tag{1}$$

where $k = T/l$, and terms of order q_1^2, q_2^2 and higher have been neglected.

(a) Define the normal coordinates Q_1, Q_2 by

$$Q_1 = \frac{1}{\sqrt{2}}(q_1 + q_2) \quad ; \quad Q_2 = \frac{1}{\sqrt{2}}(q_1 - q_2)$$

Show that $m\ddot{Q}_1 = -kQ_1$, $m\ddot{Q}_2 = -3kQ_2$, and hence that the general solution of (1) is

$$Q_1 = E\cos(\omega_1 t + \phi_1) \quad ; \quad Q_2 = F\cos(\omega_2 t + \phi_2), \tag{2}$$

where $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{3k/m}$ are the normal mode frequencies. Hence find the general solution for q_1 and q_2.

(b) The forces on the RHS of (1) may be interpreted in terms of a potential energy function $V(q_1, q_2)$, as follows. We write the equations as

$$m\ddot{q}_1 = -\frac{\partial V}{\partial q_1} \quad ; \quad m\ddot{q}_2 = -\frac{\partial V}{\partial q_2}$$

generalising "$m\ddot{x} = -\partial V/\partial x$". Show that V may be taken to be

$$V = k(q_1^2 + q_2^2 - q_1 q_2).$$

Derive the same result for V by considering the work done in giving each section of the string its deformation [e.g. for AB the work done is equal to $T(\sqrt{l^2 + q_1^2} - l)$], and expanding in powers of q_1^2/l^2. Show that, when written in terms of the variables Q_1 and Q_2, V becomes

$$V = \tfrac{1}{2}m\omega_1^2 Q_1^2 + \tfrac{1}{2}m\omega_2^2 Q_2^2,$$

where $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{3k/m}$ as before.

(c) Show that the kinetic energy of the masses is

$$K = \tfrac{1}{2}m(\dot{Q}_1^2 + \dot{Q}_2^2)$$

and hence that the total energy, in terms of Q_1 and Q_2, is

$$H = K + V == \tfrac{1}{2}m\left(\dot{Q}_1^2 + \omega_1^2 Q_1^2\right) + \tfrac{1}{2}m\left(\dot{Q}_2^2 + \omega_2^2 Q_2^2\right) = E_1 + E_2$$

where E_1 is the total energy of 'oscillator' Q_1 with frequency ω_1, and similarly for E_2. What is the expression for the total energy when written in terms of q_1, q_2, $\dot{q}_1$, and $\dot{q}_2$? Discuss the similarities and differences.

(d) Find the equations of motion for Q_1 and Q_2 from Newton's law in the form

$$m\ddot{Q}_1 = -\frac{\partial H}{\partial Q_1} \quad ; \quad m\ddot{Q}_1 = -\frac{\partial H}{\partial Q_2},$$

and hence re-derive the solution (2).

HT V

Oscillations

1. (a) What is the difference between a travelling wave and a standing wave?

(b) Convince yourself that

$$y_1 = A\sin(kx - \omega t)$$

corresponds to a travelling wave. Which way does it move? What are the amplitude, wavelength, frequency, period and velocity of the wave?

(c) Show that y_1 satisfies the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

provided that ω and k satisfy a *dispersion relation*.

(d) Write down a wave y_2 of equal amplitude travelling in the opposite direction. Show that $y_1 + y_2$ can be written in the form

$$y_1 + y_2 = f(x)g(t),$$

where $f(x)$ is a function of x only, and $g(t)$ is a function just of t. Convince yourself that the combination of two travelling waves is a standing wave. By determining $f(x)$ and $g(t)$ explicitly, determine the wavelength and frequency of $y_1 + y_2$. Comment on the velocity of the waves.

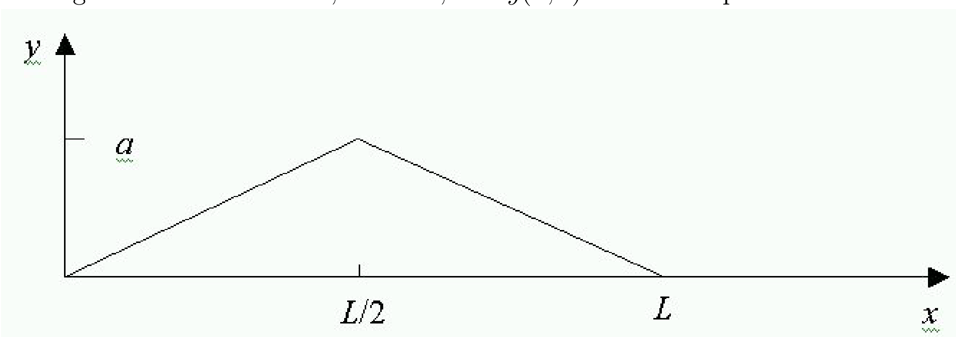
2. (a) A string of uniform linear density ρ is stretched to a tension T, its ends being fixed at $x = 0$ and $x = L$. If $y(x,t)$ is the transverse displacement of the string at position x and time t, show that $c^2\partial^2 y/\partial x^2 = \partial^2 y/\partial t^2$ where $c^2 = T/\rho$. What is meant by the statement that this equation is 'linear'?

Verify that

$$y(x,t) = A_r \sin\left(\frac{r\pi x}{L}\right)\sin\left(\frac{r\pi ct}{L}\right) \quad \text{and} \quad y(x,t) = B_r \sin\left(\frac{r\pi x}{L}\right)\cos\left(\frac{r\pi ct}{L}\right),$$

where r is any integer, are both solutions of this equation, obeying the boundary conditions $y(0,t) = y(L,t) = 0$. Explain why sums of such solutions are also solutions.

(b) The string is such that at $t = 0$, for all x, and $y(x,0)$ has the shape



i.e. the mid-point is drawn aside a small distance a. Explain why the solution after the mid-point is released has the form

$$y(x,t) = \sum_{r=1}^{\infty} B_r \sin\left(\frac{r\pi x}{L}\right)\cos\left(\frac{r\pi ct}{L}\right)$$

[N.B. There are infinitely many constants $(B_1, B_2, \ldots)$ in this expression. They can be determined from the initial displacement of the string by the technique of Fourier Analysis:

$$y(x,0) = \sum_{r=1}^{\infty} B_r \sin\left(\frac{r\pi x}{L}\right) \quad \Rightarrow \quad B_r = \frac{2}{L}\int_0^L \mathrm{d}x\, y(x,0)\sin\left(\frac{r\pi x}{L}\right).$$

Fourier Analysis is now on the 2nd year course.]

3. (a) Standing waves $y = f(x)g(t)$ exist on a string of length L, as in question 1. Given that the x dependence is

$$f(x) = A\sin(kx),$$

what is $g(t)$? [This involves 2 arbitrary constants.]

(b) At $t = 0$, the displacement is

$$y(x,0) = \sin\left(\frac{\pi x}{L}\right) + 2\sin\left(\frac{2\pi x}{L}\right)$$

and the string is instantaneously standing. Find the displacement at subsequent times.

Make rough sketches of $y(x,t)$ at the following times: 0, $L/4c$, $L/2c$, $3L/4c$, L/c.

3. What is meant by (a) a dispersive medium, and (b) the phase velocity v? Explain the relevance of group velocity g for the transmission of signals in a dispersive medium. Justify the equation

$$g = \frac{\mathrm{d}\omega}{\mathrm{d}k}. \qquad (1)$$

Show that for electromagnetic waves alternative expressions for g are

$$g = v + k\frac{\mathrm{d}v}{\mathrm{d}k} \quad ; \quad g = v - \lambda\frac{\mathrm{d}v}{\mathrm{d}\lambda} \quad ; \quad g = \frac{c}{\mu}\left(1 + \frac{\mathrm{d}\ln\mu}{\mathrm{d}\ln\lambda}\right),$$

where μ is the refractive index for waves of wavelength λ and wavenumber k (in the medium). Show that

$$g = v\left(1 - \frac{1}{1 + \mathrm{d}\ln\lambda'/\mathrm{d}\ln v}\right),$$

where λ' is the wavelength in vacuum.

4. In quantum mechanics, a particle of momentum p and energy E has associated with it a wave of wavelength λ and frequency ν given by

$$\lambda = h/p \quad \text{and} \quad \nu = E/h,$$

where h is Planck's constant. Find the phase and group velocities of these waves when the particle (a) is non-relativistic, given that

$$p = m_0 v \quad \text{and} \quad E = \tfrac{1}{2}m_0 v^2.$$

and (b) is relativistic, in which case

$$p = \frac{m_0 v}{\sqrt{1 - v^2/c^2}} \quad \text{and} \quad E = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}},$$

where m_0 is the particle's rest mass. Comment on your answers.

5. * Explain the terms *group velocity* and *phase velocity*, illustrating your answers by reference to the propagation of a quantity

$$\phi = a\left[\mathrm{e}^{\mathrm{i}(\omega t - kx)} + \mathrm{e}^{\mathrm{i}(\omega' t - k' x)}\right],$$

where the differences $\Delta\omega \equiv \omega' - \omega$ and $\Delta k \equiv k' - k$ are small compared with ω and k.

In a certain dispersive medium a disturbance ϕ is propagating according to the equation

$$\tau\frac{\partial}{\partial t}\left(\frac{\partial^2\phi}{\partial t^2} - c_1^2\frac{\partial^2\phi}{\partial x^2}\right) + \frac{\partial^2\phi}{\partial t^2} - c_0^2\frac{\partial^2\phi}{\partial x^2} = 0.$$

Show that a disturbance with frequency $\omega \ll 1/\tau$ travels with phase velocity c_0, and that its amplitude decreases by a factor

$$\simeq \exp\left[-\pi\omega\tau\left(\frac{c_1^2}{c_0^2} - 1\right)\right]$$

in each wavelength. [Assume $c_1 > c_0$.]

Vector Calculus

6. Prove that

$$\text{(i)} \qquad \int_S \phi \, \mathrm{d}^2\mathbf{S} = \int_V \nabla\phi \, \mathrm{d}^3 r$$

$$\text{(ii)} \qquad \int_S \mathrm{d}^2\mathbf{S} \times \mathbf{a} = \int_V \nabla \times \mathbf{a} \, \mathrm{d}^3 r,$$

where S is the closed surface bounding the volume V.

7. If $\mathbf{G} = (\mathbf{a} \cdot \mathbf{r})\mathbf{a}$, where $\mathbf{a}$ is a constant vector, and S is a closed surface, show that

$$\int_S \mathbf{G} \times \mathrm{d}^2\mathbf{S} = 0,$$

where S is any closed surface. Verify this result for the special case in which $\mathbf{a} = (0, 0, 1)$ and S is the bounding surface of the hemisphere $r \leq 1$, $z \geq 0$.

8. Prove that

$$\oint_C \phi \mathrm{d}\mathbf{l} = \int_S \mathrm{d}^2\mathbf{S} \times \nabla\phi,$$

where C is the closed curve bounding the surface S. Verify this relation for the function $\phi = x^3$, with C the circle $x^2 + y^2 = a^2$, $z = 0$, and S the part of the plane $z = 0$ enclosed by C.

9. * $\mathbf{A}(\mathbf{r})$ is defined by

$$\mathbf{A}(\mathbf{r}) = \int_V \nabla_{\mathbf{r}'} f(|\mathbf{r} - \mathbf{r}'|) \mathrm{d}^3 r',$$

where $\nabla_{\mathbf{r}'}$ denotes the gradient with respect to $\mathbf{r}'$, $\mathbf{r}$ remaining fixed, f is a well-behaved function of a single variable and the volume V is fixed. Show that $\nabla \times \mathbf{A} = 0$. [Hint: Find a function of which $\mathbf{A}$ is the gradient.] Hence find $\mathbf{A}$ when $f(x) = x^4$ and V is the sphere $r = 1$.

10. * The region V is bounded by a simple, closed surface S. Prove that

$$\int_S \psi \nabla\phi \cdot \mathrm{d}^2\mathbf{S} = \int_V \psi \nabla^2 \phi \mathrm{d}^3 r + \int_V (\nabla\psi \cdot \nabla\phi) \mathrm{d}^3 r.$$

Let $\nabla^2\phi = 0$ in V and $\phi(\mathbf{r}) = g(\mathbf{r})$ on S. A function $f(\mathbf{r})$ is chosen so that it also satisfies $f(\mathbf{r}) = g(\mathbf{r})$ on S. By writing $\psi = f - \phi$ show that

$$\int_V |\nabla\phi|^2 \mathrm{d}^3 r \leq \int_V |\nabla f|^2 \mathrm{d}^3 r.$$

If S is the sphere $x^2 + y^2 + z^2 = a^2$ and in polar coordinates (r, θ, ϕ), $g = a\cos\theta$, show that

$$\int_V |\nabla\phi|^2 \mathrm{d}^3 r \leq V = \tfrac{4}{3}\pi a^3.$$

11. * If S is a surface bounded by the closed curve C, prove that

$$\oint_C \mathbf{r}(\mathbf{r} \cdot \mathrm{d}\mathbf{l}) = \int_S \mathbf{r} \times \mathrm{d}^2\mathbf{S}.$$

Verify this formula when C is (i) the intersection of the cylinder $x^2 - x + y^2 = 2$ with the plane $z = 0$, (ii) the intersection of the same cylinder with the sphere $x^2 + y^2 + z^2 = 9$.

What is the value of the right-hand integral for the portion of the cylinder cut off by the plane and the sphere?

HT VI

Oscillations

1. By changing to the variables $u = x - ct$, $v = x + ct$, show that the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

may be reduced to the form

$$\frac{\partial^2 y}{\partial u \partial v} = 0.$$

Hence show that the wave equation is satisfied by d'Alembert's solution

$$y = f(x - ct) + g(x + ct),$$

where f and g are arbitrary functions.

2. At time $t = 0$, the displacement of an infinitely long string is:

$$y(x,t) = \begin{cases} \sin(\pi x/a) & \text{for } -a \le x \le a \\ 0 & \text{otherwise.} \end{cases}$$

The string is initially at rest. Using d'Alembert's solution for speed c, sketch the displacement of the string at $t = 0$, $t = a/2c$, and $t = a/c$.

3. Calculate the rate of working of a device which launches small amplitude waves $y(x,t) = A\cos(kx - \omega t)$ into the end of a semi-infinite string by forcing $y(0,t) = A\cos(\omega t)$.

4. Show that the kinetic energy U and the potential energy V for a length $\lambda = 2\pi/k$ of a transverse wave on a string of linear density ρ and at tension T are given by

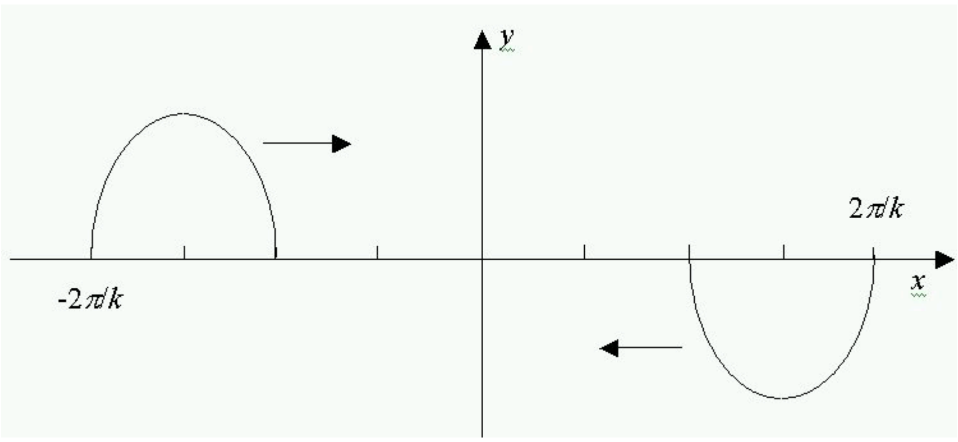
$$U = \int_0^\lambda \mathrm{d}x\, \tfrac{1}{2}\rho \left(\frac{\partial y}{\partial t}\right)^2 \quad \text{and} \quad V = \int_0^\lambda \mathrm{d}x\, \tfrac{1}{2}T \left(\frac{\partial y}{\partial x}\right)^2.$$

Evaluate these for the wave

$$y = A\cos(kx + \omega t + \phi)$$

and show that $U = V$.

5. Two transverse waves are on the same piece of string. The first has displacement y non-zero only for $kx + \omega t$ between π and 2π, when it is equal to $A\sin(kx + \omega t)$. The second has $y = A\sin(kx - \omega t)$ for $kx - \omega t$ between -2π and $-\pi$, and is zero otherwise. When $t = 0$, the displacement is as shown in the figure.



Calculate the energy of the two waves.

What is the displacement of the string at $t = 3\pi/2\omega$? Calculate the energy at this time.

6. Two long strings lie along the x-axis under tension T. They are joined at $x = 0$ so that for $x < 0$ the line density $\rho = \rho_1$, and for $x > 0$, $\rho = \rho_2$. Small transverse oscillations propagate along these strings from $x = -\infty$. Show that at the join

$$\left.\frac{\partial y}{\partial x}\right|_{x=0-} = \left.\frac{\partial y}{\partial x}\right|_{x=0+}.$$

If a train of waves $y(x,t) = \mathrm{e}^{\mathrm{i}(kx-\omega t)}$ is launched into the combined string from $x = -\infty$, find the amplitude and phase of the trains that are (a) reflected from the join, and (b) transmitted through the join.

Using the result of Problem 3, verify that energy is conserved.

7. The apparatus of the last problem is modified by attaching to the join a particle of mass m which is connected to a fixed support by a light spring of stiffness p. This spring exerts a transverse force on the mass when the latter is displaced from $y = 0$. Show that at the join

$$T\left(\left.\frac{\partial y}{\partial x}\right|_{x=0+} - \left.\frac{\partial y}{\partial x}\right|_{x=0-}\right) = m\frac{\partial^2 y}{\partial t^2} + py$$

is satisfied.

A train of harmonic waves of frequency ω is transmitted from $x = -\infty$. Show that the phase of the transmitted wave lags behind that of the incident wave by an angle

$$\arctan\left(\frac{s_1 s_2(m\omega^2 - p)}{\omega T(s_1 + s_2)}\right),$$

where s_1 and s_2 are the speeds of the waves for $x < 0$ and $x > 0$, respectively.

Vector Calculus

8. * To what scalar or vector quantities do the following expressions in suffix notation correspond (sum where possible): $a_i b_j c_i$; $a_i b_j c_j d_i$; $\delta_{ij} a_i a_j$; $\delta_{ij}\delta_{ij}$; $\epsilon_{ijk} a_i b_k$; and $\epsilon_{ijk}\delta_{ij}$.

9. * Use the summation convention to find the grad of the following scalar functions of position $\boldsymbol{r} = (x, y, z)$: (a) $|r|^n$, (b) $\boldsymbol{a}.\boldsymbol{r}$.

10. * Use the summation convention to find the div and curl of the following vector functions of position $\boldsymbol{r} = (x, y, z)$: (a) r; (b) $|\boldsymbol{r}|^n \boldsymbol{r}$; (c) $(\boldsymbol{a}.\boldsymbol{r})\boldsymbol{b}$; and (d) $\boldsymbol{a} \times \boldsymbol{r}$. Here, $\boldsymbol{a}$ and $\boldsymbol{b}$ are fixed vectors.

11. * Use the summation convention to prove: (a) $\nabla \times (\nabla\phi) = 0$; and (b) $\nabla.(\nabla \times \boldsymbol{A}) = 0$.

HT VII

1. A semi-infinite string of density ρ per unit length is under tension T. At its free end is a mass m which slides on a smooth horizontal rod that lies perpendicular to the string. Determine the amplitude reflection coefficient for transverse waves incident on the mass. What is the phase difference between the incident and reflected waves?

2. * An infinite string lies along the x-axis, and is under tension T. It consists of a section at $0 < x < a$, of linear density ρ_1, and two semi-infinite pieces of density ρ_2. A wave travels along the string at $x > a$, towards the short section.

How many types of waves are there in the various sections of the string? How many boundary conditions need to be satisfied?

Show that, if $a = n\lambda_1$, (where λ_1 is the wavelength on the short section, and n is an integer), the amplitude of the wave that emerges at $x < 0$ is A. What is the amplitude of the wave in the short section?

3. * A bar of uniform cross section A, density ρ and Young's modulus Y transmits longitudinal elastic waves. If a small element at position x is displaced a distance ξ, derive the wave equation for ξ and find the wavelength of a harmonic wave of frequency ω.

4. * Waves of frequency ω travelling in the bar of the last problem are reflected at an end which has a mass M rigidly attached to it. Find the phase change on reflection and discuss the cases $M = 0$ and $M \to \infty$.

5. * For the infinite electrical circuit shown in the figure on page W35 of the lecture notes, show that the voltage V obeys the wave equation and determine the speed of the waves. Find the characteristic impedance Z (i.e. the ratio of voltage to current) for waves travelling in both the positive and negative x-direction. Why is the characteristic impedance positive for waves travelling to the right, but negative for waves travelling to the left? Isn't that paradoxical, since the circuit is the same for left- and right-travelling waves?

6. * A semi-infinite transmission line, of capacitance and inductance C and L per unit length, is terminated by an impedance Z_T (page W36). Find the ratio of the amplitude and the phase difference for the reflected and incident waves if (a) $Z_T = \sqrt{L/c}$, (b) $Z_T = 2\sqrt{L/c}$ or (c) Z_T is a capacitor of capacitance C. In (a) and (b) what type of impedance is required?

7. * A uniform string of length l and density ρ has its end points fixed so that its equilibrium tension is T. A mass M is attached to its mid-point. Show that the angular frequency ω of small vibrations is given by

$$z \tan z = \frac{\rho l}{M}, \quad \text{where} \quad z \equiv \frac{\omega l}{2c} \quad \text{and} \quad c^2 \equiv T/\rho.$$

8. * Two uniform wires of densities ρ_1 and ρ_2 and of equal lengths are fastened together and the two free ends are attached to two fixed points a distance $2l$ apart, so that the equilibrium tension is T. Show that the angular frequency ω of small vibrations satisfies

$$c_1 \tan(\omega l/c_1) = -c_2 \tan(\omega l/c_2),$$

where $c_{1,2}^2 \equiv T/\rho_{1,2}$.

9. * An elastic string of length a consists of two portions, $0 < x < \frac{1}{2}a$ of density ρ_1 and $\frac{1}{2}a < x < a$ of density ρ_2. It is stretched to tension T and the end at $x = a$ is fixed. The end at $x = 0$ is then shaken transversely at frequency ω. Show that throughout the motion, the ratio of the displacement at $x = \frac{1}{2}a$ to that at $x = 0$ is given by

$$\frac{s_2 \csc(\omega a/2s_1)}{s_2 \cot(\omega a/2s_1) + s_1 \cot(\omega a/2s_2)},$$

where $s_{1,2}^2 \equiv T/\rho_{1,2}$.