Chaos, Turbulence Etc.

Third year physics course (Paper B-I, Part 2)

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Christmas Questions

PROBLEM SET 5: Christmas Questions (extracurricular)

Here are two rather open-ended questions designed to keep you entertained over the Christmas Vacation (sorry, Interterm Time). They may well have more than one solution: 5.1 certainly does, while what I think the solution is for 5.2 may well be incorrect (there is an active discussion going on in the research world about that — this doesn't mean it's mathematically complicated, just that new conceptual thoughts are being thought). These questions will hopefully help you develop a deeper understanding of two extremely fundamental (and fascinating) topics that we only just barely touched on: Brownian Motion and Turbulence. This work is extracurricular, is unlikely to make any difference to your exam performance, and you are under no obligation whatsoever (not even moral) to do anything about it. If those of you who do do something about it want to discuss the fruit of their labours, I am happy to meet some time in HT or TT-2014. You are absolutely encouraged to discuss these questions between yourselves. Merry Interterm Time and Happy New Year!

Brownian Motion

5.1 Consider a large number of small but macroscopic (compared to the molecular size) Brownian particles of diameter a and mass m moving around in a medium of viscosity μ and temperature T. Assume that the motion of these particles can be modelled with the usual Lagevin equation (frictional drag plus white-noise random forcing).

(a) Show that the concentration $n(t, \mathbf{r})$ of these particles will satisfy the diffusion equation and calculate the diffusion coefficient.

(b) How should this equation be modified if the ambient medium is flowing with velocity $\mathbf{u}(t, \mathbf{r})$?

(c***) If you are feeling adventurous and have an appetite for some considerable maths, investigate how to deal with this equation if the velocity $\mathbf{u}$ itself is a random field correlated in time as a white noise (assume incompressibility, $\nabla \cdot \mathbf{u} = 0$). The interesting question is then, e.g., how to find the equal-time spatial correlation function of the concentration field $\langle n(t, \mathbf{r})n(t, \mathbf{r}')\rangle = C(\mathbf{r} - \mathbf{r}')$ (this is called "random advection (or random mixing) of a passive scalar field" and you'd have to identify and study the relevant literature; if you need access to journals, you'll get it if you VPN into your Oxford account).

Rotating Turbulence

5.2 Consider homogeneous turbulence in an incompressible fluid system rotating with a uniform angular velocity Ω (say, in the z direction). The equation for such a fluid is the Navier-Stokes equation with a Coriolis force:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + 2\Omega \hat{\mathbf{z}} \times \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u},$$

where p is fixed by the requirement $\nabla \cdot \mathbf{u} = 0$.

Suppose energy is injected into the system at some "outer" scale L and so turbulence is stirred up with a typical outer-scale velocity of motions U. Consider the case of strong rotation, viz., the Rossby number is

$$\mathrm{Ro} = \frac{U}{L\Omega} \ll 1,$$

while the Reynolds number is

$$\mathrm{Re} = \frac{UL}{\nu} \gg 1.$$

(Would rotation have mattered if I had stipulated $\mathrm{Ro} \gg 1$ instead?)

(a) Such a turbulence has a tendency to be 2D, i.e., instead of 3D "eddies" you'll see 2D "columns" (correlation length along z is infinity). So, consider such a 2D turbulence and work out the scalings of two-point velocity differences $\delta u(\ell)$ vs. point separation (scale) ℓ.[1] Remember that in a purely 2D system, there will be two conserved quadratic invariants, energy $\int d^3\mathbf{r} u^2/2$ and enstrophy $\int d^3\mathbf{r}|\nabla \times \mathbf{u}|^2/2$; they can't both cascade to small scales ("Fjørtoft argument"), so one of them (enstrophy) will cascade to small ($\ell \ll L$), the other (energy) to large ($\ell \gg L$) scales. So you'll find two different scalings in these two regimes, based on Kolmogorov-style constant-flux arguments.

(b) In fact, infinite correlations along z are impossible because information travels at finite speed. In this case, information is carried by "inertial waves." Work out the relevant speed after showing that the dispersion relation for waves in the above equation is

$$\omega^2 = \Omega^2 \frac{k_z^2}{k^2},$$

where $k = |\mathbf{k}|$. If the "columns" get taller than the distance that an inertial wave can travel in one turnover time, their opposite ends will decorrelate and the columns will break up into long-aspect-ratio but 3D cigar-like eddies. Work out, therefore, how tall the rotating box has to get in order for 2D turbulence to break down in this way.

(c) Now let the box be so tall that we have this 3D "cigar-eddy" turbulence. Enstrophy is no longer conserved, but energy is (check both of these statements). Assuming the cascade is again forward (to small scales), follow a Kolmogorov-scale dimensional argument to work out $\delta u(\ell_\perp)$, where $\ell_\perp$ is the "pependicular" (i.e., xy-plane) separation of points ("perpendicular scale"). What is the scaling of the energy spectrum with perpendicular wavenumber ($k_\perp = \sqrt{k_x^2 + k_y^2}$)?

(d) Using the causality principle discussed above, work out the vertical size $\ell_\parallel$ (parallel decorrelation length) of the "cigar eddies" of perpendicular scale $\ell_\perp$. What is the scaling of $\delta u(\ell_\parallel)$ with $\ell_\parallel$?

(e) At what scale $\ell_{\perp,c}$ (and what corresponding parallel scale $\ell_{\parallel,c}$) does rotation become irrelevant and the turbulence turns into the usual isotropic Kolmogorov cascade? [this is called the Zeman scale] Does it make sense to you that this is what must happen?

[1] Note that we now assume that the forcing is also 2D, i.e., the outer scale L is in the xy-plane (the forcing does not impose three-dimensionality).