Merton Maths Problems

First-Year Mathematics (CP3/4)
for Merton College Students

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(with thanks to lecturers and tutors, past and present, in particular to F. I. Parra of Worcester College and M. A. Barnes of University College)

Problems for Michaelmas Term 2018

MT 0

Dimensional Analysis

(problems for the extracurricular lecture)

0.1 G. I. Taylor and the Bomb

In the early autumn of 1940, during some of the most desperate days of the Battle of Britain, a Cambridge Professor of fluid dynamics G. I. Taylor was invited to lunch by an Imperial College Professor and Nobel-prize winner George Thomson, who was then chairman of the MAUD committee (MAUD = "Military Application of Uranium Detonation"). G. I. Taylor was told that it might be possible to produce a bomb in which a very large amount of energy would be released by nuclear fission (this was to become the atomic bomb). The crucial question was what would be the mechanical effect of such an explosion? G. I. Taylor's subsequent solution of this problem may be the most famous example of the application of dimensional analysis of all time. In this problem, you will work through some of his calculation.

Let us simplify the problem by assuming that

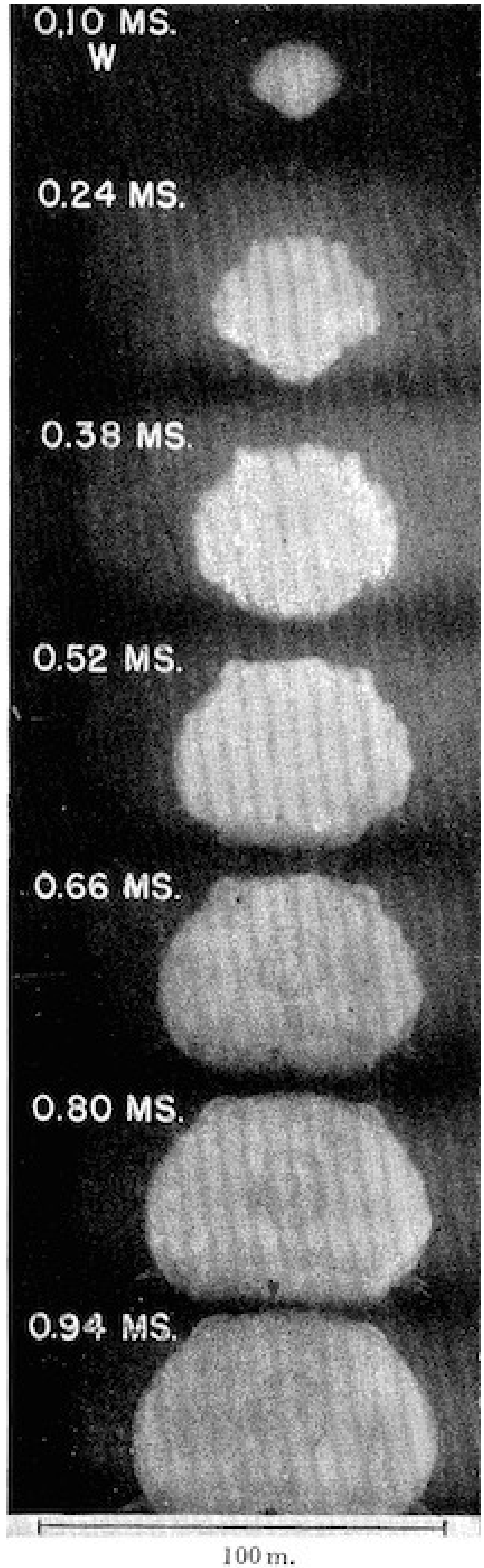
— a finite amount of energy E is released instantaneously at a point (i.e., we will ignore the radius r_0 of the volume where the initial energy release occurs at time $t = 0$, it will not be a relevant parameter);

— there results a *spherically symmetric* shock wave, with its front propagating according to some law $r_f(t)$, where r_f is the radius of the front.

Find $r_f(t)$ as a function of time t. Find also the velocity of the front $u_f(t)$ and the pressure $p_f(t)$ in the surrounding air just outside the front. The density ρ_0 of air before the explosion is given. *If you identify correctly what the governing parameters are (all of them are mentioned above), you should be able to use dimensional analysis to work out r_f, u_f and p_f with only constant dimensionless prefactors left undetermined.*

Based on the result you have obtained, will, in your opinion, making the bomb bigger (say doubling its size) makes much of a difference?

If you did not know the energy of the explosion E (classified!), but had a movie of the fireball, how would you estimate E? When the Americans tested the bomb and released a series of high-speed photographs, G. I. Taylor estimated E and published the result, which caused much embarassment in the American government circles. The high-speed photos are reproduced below: can you come up with your own estimate?


0,10 MS.
W
0.24 MS.
0.38 MS.
0.52 MS.
0.66 MS.
0.80 MS.
0.94 MS.
100 m.

0.2 Poiseuille Flow

This example is also famous, and much more peaceful than the previous one. It was first worked out experimentally by H. Hagen (1839) and J. L. M. Poiseuille (1840) (working independently of each other) and later theoretically explained by G. G. Stokes (1845).

Consider a pipe of length l and diameter d. A pressure drop between the ends of the pipe, $p_1 - p_2$, is maintained to pump an incompressible fluid of viscosity μ through the pipe. Find the volumetric flow rate Q, i.e., the volume of the fluid that passes through any cross-section of the pipe per unit time.

If I double the diameter of the pipe, by what factor will Q change? What if I double the pressure contrast? And what if I double viscosity? Does the answer make sense? (Why does viscosity matter?) What if I double viscosity *and* cut the pipe length by half?

Hint. A judicious choice of governing parameters in this problem is d, μ and $(p_1 - p_2)/l$ — the pressure drop per unit length (think about why that is).

Now find the velocity U at which the fluid flows through the pipe.

Why do you think the density of the fluid does not matter here? Under what conditions would you expect it to start being an important parameter? (Think about the discussion in the lectures — what is the dimensionless number that controls this?)

Further Reading

- P. W. Bridgman, *Dimensional Analysis* (there is a very cheap Amazon reprint of this classic 1920 text — a bit dated, but still quite readable)
- G. I. Barenblatt, *Scaling* (this is quite advanced, but you can read the first couple of chapters; this is the book from which I lifted the G. I. Taylor example — but don't look until you have attempted to solve it unaided! It also contains several other intesting examples of dimensional analysis, including one giving important insights into rowing, which you will find fascinating in the unlikely case that you are into that kind of thing, despite being a Merton physicist — and if you are not into it, this will give you the satisfying feeling that you know more of the physics of it than those who are)
- L. Landau and E. Lifshitz, *Fluid Mechanics* (Vol. 6 of their *Course of Theoretical Physics*, all of the ten volumes of which every self-respecting physicist should keep on his/her desk at all times; you will find there an account of some rather complicated issues that arise in the Poiseuille problem at large Reynolds numbers; you can also read there how bubbles rise, bodies move through fluid etc.)

MT I

Calculus: Differentiation

1.1 From the definition of the derivative,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \left\{ \frac{y(x+\delta x) - y(x)}{\delta x} \right\},$$

evaluate $\mathrm{d}(x^2)/\mathrm{d}x$. In the same way evaluate $\mathrm{d}(\sin x)/\mathrm{d}x$.

1.2 Differentiate

$$\begin{aligned}
&\text{(a)} \quad y = \sin x\, \mathrm{e}^{x^3}, \\
&\text{(b)} \quad y = \mathrm{e}^{x^3 \sin x}, \\
&\text{(c)} \quad y = \ln\left[\cosh\left(\frac{1}{x}\right)\right], \\
&\text{(d)} \quad y = x^{\cos x}, \\
&\text{(e)} \quad y = \log_{10}(x^2), \\
&\text{(f)} \quad y = \arccos x, \\
&\text{(g)} \quad y = \operatorname{arctanh}\left(\frac{x}{1+x}\right).
\end{aligned}$$

1.3 Be fearless and differentiate

$$\text{(i)} \quad y = \sqrt{x^2+1} - \ln\left(\frac{1}{x} + \sqrt{1+\frac{1}{x^2}}\right), \quad \text{answer :} \quad y' = \sqrt{1+\frac{1}{x^2}},$$

$$\text{(ii)} \quad y = \frac{\cosh(x^2)}{\sinh^2(x^2)} - \ln\coth\left(\frac{x^2}{2}\right), \quad \text{answer :} \quad y' = -\frac{4x}{\sinh^3(x^2)},$$

$$\text{(iii)} \quad y = \ln\sqrt{\frac{\sqrt{x^4+1} - \sqrt{2}\,x}{\sqrt{x^4+1} + \sqrt{2}\,x}} - \arctan\frac{\sqrt{2}\,x}{\sqrt{x^4+1}}, \quad \text{answer :} \quad y' = 2\sqrt{2}\,\frac{\sqrt{x^4+1}}{x^4-1}.$$

Make sure you get the right answer (and if it does not come out right the first time, don't give up and try again).

1.4 (i) Find $\mathrm{d}y/\mathrm{d}x$ as a function of x and y if

$$y\,\mathrm{e}^{y \ln x} = x^2 + y^2. \tag{1}$$

(ii) Find $\mathrm{d}^2y/\mathrm{d}x^2$ as a function of x and y if

$$\mathrm{e}^{x-y} = x + y. \tag{2}$$

(iii) A particle moves a distance x in time t where $t = ax^2 + bx + c$ with a, b, c constants. Prove that the acceleration is proportional to the cube of the velocity.

1.5 (i) For $y = \sinh\theta$ and $x = \cosh\theta$, find $\mathrm{d}y/\mathrm{d}x$ and $\mathrm{d}^2y/\mathrm{d}x^2$.

(ii) For $y = t^m + t^{-m}$ and $x = t + t^{-1}$ show that

$$(x^2-4)\left\{\frac{\mathrm{d}y}{\mathrm{d}x}\right\}^2 = m^2(y^2-4), \qquad (x^2-4)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} - m^2y = 0.$$

1.6 If x_i is an approximation to a root of the equation $f(x) = 0$, Newton's method of finding a better approximation x_{i+1} is $x_{i+1} = x_i - f(x_i)/f'(x_i)$, where $f'(x) = df/dx$. Explain this method graphically or otherwise (e.g., using a series expansion) in terms of the linear approximation to $f(x)$ near $x = x_i$.

1.7 Use Taylor's theorem to show that when h is small

(a) $f'(a) = \dfrac{f(a+h) - f(a-h)}{2h}$ with an error $\mathrm{O}(h^2 f'''(a))$.

(b) $f''(a) = \dfrac{f(a+h) - 2f(a) + f(a-h)}{h^2}$ with an error $\mathrm{O}(h^2 f''''(a))$.

Taking $f(x) = \sin x$, $a = \pi/6$, and $h = \pi/180$ find from (a) and (b) the approximate values of $f'(a)$ and $f''(a)$ and compare them to exact values.

(c) These finite-difference formulae are often used to calculate derivatives numerically. How would you construct a more precise finite-difference approximation to $f'(a)$, namely, an approximation with an error of order $\mathrm{O}(h^n)$?

1.8 The function $I(x)$ is defined by $I(x) = \int_a^x f(x')\mathrm{d}x'$. Show graphically that $\mathrm{d}I(x)/\mathrm{d}x = f(x)$.

1.9 (a) Explain why

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{-1}.$$

(b) Given that y is a function of x, show, by putting $\mathrm{d}y/\mathrm{d}x = p$, that

$$\frac{\mathrm{d}^2x}{\mathrm{d}y^2} = -\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\Big/\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3.$$

1.10 Given that

$$y = x + \frac{x^5}{5},$$

find $\mathrm{d}x/\mathrm{d}y$ at $y = 0$ and at $y = 6/5$. Is the solution you have found in each case the only solution?

1.11 (a) In the differential equation

$$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (4x + 3x^2)\frac{\mathrm{d}y}{\mathrm{d}x} + (2 + 6x + 2x^2)y = x$$

replace the dependent variable y by $z = yx^2$ to give

$$\frac{\mathrm{d}^2z}{\mathrm{d}x^2} + 3\frac{\mathrm{d}z}{\mathrm{d}x} + 2z = x.$$

(b) In the differential equation

$$4x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2(1-\sqrt{x})\frac{\mathrm{d}y}{\mathrm{d}x} - 6y = \mathrm{e}^{3\sqrt{x}}$$

replace the independent variable x by $t = \sqrt{x}$ to give

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - \frac{\mathrm{d}y}{\mathrm{d}t} - 6y = \mathrm{e}^{3t}.$$

(These are equations with constant coefficients that you will soon be able to solve.)

1.12 Use Leibnitz's theorem to find the 8th derivative of $x^2 \sin x$.

1.13 By finding their stationary points and examining their general forms, determine the range of values that each of the following functions can take. In each case, make a sketch graph incorporating the features you have identified.

$$\begin{aligned}
&\text{(a)}\quad y(x) = \frac{x-1}{x^2+2x+6},\\
&\text{(b)}\quad y(x) = \frac{1}{4+3x-x^2},\\
&\text{(c)}\quad y(x) = \frac{8\sin x}{15+8\tan^2 x}.
\end{aligned}$$

1.14 Sketch the following functions. Are they (i) continuous, (ii) differentiable, throughout the domain $-1 \le x \le 1$?

(a) $f(x) = 0$ for $x \le 0$, $f(x) = x$ for $x > 0$,

(b) $f(x) = 0$ for $x \le 0$, $f(x) = x^2$ for $x > 0$,

(c) $f(x) = 0$ for $x \le 0$, $f(x) = \cos x$ for $x > 0$,

(d) $f(x) = |x|$.

1.15 Sketch the graph of

$$f(x) = \mathrm{e}^{-x} + 2x, \qquad x \ge 0; \qquad f(x) = \mathrm{e}^{x}, \qquad x < 0$$

and sketch its 1st, 2nd and 3rd derivatives. Show that the third derivative is discontinuous at $x = 0$.

This problem set continues on the next page.

Calculus: Integration

1.16 Evaluate

$$\text{(a)} \quad \int \frac{(x+a)\,\mathrm{d}x}{(1+2ax+x^2)^{3/2}},$$

$$\text{(b)} \quad \int_0^{\pi/2} \cos x\, \mathrm{e}^{\sin x} \mathrm{d}x,$$

$$\text{(c)} \quad \int_0^{\pi/2} \cos^3 x\, \mathrm{d}x,$$

$$\text{(d)} \quad \int_{-2}^{2} |\, x\,|\, \mathrm{d}x,$$

$$\text{(e)} \quad \int \frac{\mathrm{d}x}{(3+2x-x^2)^{1/2}} \quad \text{(complete square first)},$$

$$\text{(f)} \quad \int_0^{\pi} \frac{\mathrm{d}\theta}{5+3\cos\theta} \quad \left(\text{use } t = \tan\frac{\theta}{2}\right),$$

$$\text{(g)} \quad \int \frac{\mathrm{d}x}{x(1+x^2)},$$

$$\text{(h)} \quad \int x \sin x\, \mathrm{d}x,$$

$$\text{(i)} \quad \int \ln x\, \mathrm{d}x \quad \text{(by parts)},$$

$$\text{(j)} \quad \int (\cos^5 x - \cos^3 x)\, \mathrm{d}x,$$

$$\text{(k)} \quad \int \sin^5 x\, \cos^4 x\, \mathrm{d}x,$$

$$\text{(l)} \quad \int \sin^2 x\, \cos^4 x\, \mathrm{d}x,$$

$$\text{(m)} \quad \int \frac{\sqrt{x^2-9}}{x}\, \mathrm{d}x,$$

$$\text{(n)} \quad \int \frac{\mathrm{d}x}{x^2\sqrt{16-x^2}}.$$

1.17 (a) Derive general formulae for the integrals

$$I_n = \int_0^\infty \mathrm{d}x\, x^n e^{-x^2},$$

where n is a positive integer. You will find it opportune to consider odd and even n separately and to know that

$$\int_{-\infty}^{\infty} \mathrm{d}x\, e^{-x^2} = \sqrt{\pi}.$$

Notice that if we define

$$f(\lambda) \equiv \int_0^\infty \mathrm{d}x\, e^{-\lambda x} = \frac{1}{\lambda},$$

then, for example,

$$\int_0^\infty \mathrm{d}x\, x e^{-\lambda x} = -f'(1) = 1$$

and you can similarly find integrals of this sort with higher powers of x in the integrand. Generalising this method to the integrals I_n above is not hard.

(b) By the method analogous to that employed in the derivation of Stirling's formula, or otherwise, show that, when $n \gg 1$,

$$I_n \approx \sqrt{\frac{\pi}{2}} \left(\frac{n}{2}\right)^{n/2} e^{-n/2}.$$

(c*) Can you prove that, slightly more precisely,

$$I_n \approx \sqrt{\frac{\pi}{2}} \left(\frac{n}{2}\right)^{n/2} e^{-n/2} \left(1 - \frac{1}{12n}\right).$$

1.18 By sketching the integrand of the following integrals, determine which integrals vanish:

$$\text{(i)} \int_{-\infty}^{\infty} x\mathrm{e}^{-x^2} \mathrm{d}x, \qquad \text{(ii)} \int_{-\pi}^{\pi} x \sin x\, \mathrm{d}x, \qquad \text{(iii)} \int_{-\pi}^{\pi} x^2 \sin x\, \mathrm{d}x.$$

1.19 Prove that if $f(x)$ is an odd function of x, then $\displaystyle\int_{-a}^{a} f(x)\mathrm{d}x = 0$.

1.20 Given that $\ln x$ is defined by $\int_1^x t^{-1}\mathrm{d}t$, show that $\ln x + \ln y = \ln xy$.

Vectors & Matrices

Problem Set 1: Questions 1–7 (vectors, vector spaces).

MT II

Complex Numbers

2.1 For a) $z_1 = 1 + \mathrm{i}$, $z_2 = -3 + 2\mathrm{i}$ and b) $z_1 = 2\mathrm{e}^{\mathrm{i}\pi/4}$, $z_2 = \mathrm{e}^{-3\mathrm{i}\pi/4}$ find

(i) $z_1 + z_2$, (ii) $z_1 - z_2$, (iii) $z_1 z_2$, (iv) z_1/z_2, (v) $|z_1|$, (vi) z_1^*.

2.2 For $z = x + \mathrm{i}y$ find the real and imaginary parts of

(i) $2 + z$; (ii) z^2; (iii) z^*; (iv) $1/z$; (v) $|z|$; (vi) i^{-5}; (vii) $(1 + \mathrm{i})^2$; (viii) $(2 + 3\mathrm{i})/(1 + 6\mathrm{i})$; (ix) $\mathrm{e}^{\mathrm{i}\pi/6} - \mathrm{e}^{-\mathrm{i}\pi/6}$.

2.3 Find the modulus and argument of each of (i) $R + \mathrm{i}\omega L$ (ii) $R + \mathrm{i}\omega L + 1/\mathrm{i}\omega C$ where R, L, C are all real.

Hence find the modulus and argument of each of (iii) $\frac{V_0\mathrm{e}^{\mathrm{i}\omega t}}{R+\mathrm{i}\omega L}$ (iv) $\frac{V_0\mathrm{e}^{\mathrm{i}\omega t}}{R+\mathrm{i}\omega L+1/\mathrm{i}\omega C}$ where V_0 is also real. Find also the real and imaginary parts of of (iii) and (iv). (These manipulations are important in a.c. circuit theory, where ω is the angular frequency and $Z = E + \mathrm{i}\omega L + 1/\mathrm{i}\omega C$ is the complex impedance of a resistance R, inductance L and capicitance C in series.)

2.4 Change to polar form ($z = r\mathrm{e}^{\mathrm{i}\theta}$)

(i) $-\mathrm{i}$, (ii) $\frac{1}{2} - \frac{\sqrt{3}\mathrm{i}}{2}$, (iii) $-3 - 4\mathrm{i}$, (iv) $1 + \mathrm{i}$, (v) $1 - \mathrm{i}$, (vi) $(1 + \mathrm{i})/(1 - \mathrm{i})$.

2.5 Draw in the complex plane

(i) $3 - 2\mathrm{i}$, (ii) $4\mathrm{e}^{-\mathrm{i}\pi/6}$, (iii) $|z - 1| = 1$, (iv) $\mathrm{Re}(z^2) = 4$, (v) $z - z^* = 5\mathrm{i}$,

(vi) $z = t\mathrm{e}^{\mathrm{i}t}$ (for real values of the parameter t),

(vii) $\arg(z + 3\mathrm{i}) = \pi/4$, (viii) $|z + 1| + |z - 1| = 8$.

2.6 Find (i) $(1 + 2\mathrm{i})^7$, (ii) $(1 - 2\mathrm{i})^7/(1 + 2\mathrm{i})^7$.

2.7 Solve for all possible values of the real numbers x and y

(i) $2\mathrm{i}x + 3 = y - \mathrm{i}$, (ii) $(x + 2y + 3) + \mathrm{i}(3x - y - 1) = 0$, (iii) $z^2 = z^{*2}$ $\quad (z = x + \mathrm{i}y)$,

(iv) $|2x - 1 + \mathrm{i}y| = x^2 + \mathrm{i}y$.

2.8 Write the following in the form $a + \mathrm{i}b$, where a and b are real:

(i) e^{i}, (ii) $\sqrt{\mathrm{i}}$, (iii) $\ln \mathrm{i}$, (iv) $\cos \mathrm{i}$, (v) $\sin \mathrm{i}$, (vi) $\ln(-\mathrm{e})$,

(vii) $\ln \frac{1}{2}(\sqrt{3} + \mathrm{i})$, (viii) $(1 + \mathrm{i})^{\mathrm{i}y}$, (ix) $e^{3\ln 2 - \mathrm{i}\pi}$, (x) $\cos(\pi - 2\mathrm{i}\ln 3)$, (xi) $\arctan(\sqrt{3}\mathrm{i})$,

(xii) $\sinh(x + \mathrm{i}y)$, (xiii) $\tanh(x + \mathrm{i}y)$.

(Note $\mathrm{i} = \mathrm{e}^{\mathrm{i}\pi/2}$)

2.9 Sketch the curves C_1 and C_2 in the Argand diagram for z defined respectively by $\arg[(z-4)/(z - 1)] = \pi/2$ and $\arg[(z - 4)/(z - 1)] = 3\pi/2$.

2.10 Show that

$$\sum_{n=0}^{\infty} 2^{-n} \cos n\theta = \frac{1 - \frac{1}{2}\cos\theta}{\frac{5}{4} - \cos\theta}.$$

2.11 Prove that

$$\sum_{r=1}^{n} \begin{pmatrix} n \\ r \end{pmatrix} \sin 2r\theta = 2^n \sin n\theta \cos^n \theta \quad \text{where} \quad \begin{pmatrix} n \\ r \end{pmatrix} \equiv \frac{n!}{(n-r)!r!}.$$

[Hint: express the left side as $\mathrm{Im}\left(\sum \begin{pmatrix} n \\ r \end{pmatrix} \mathrm{e}^{\mathrm{i}2r\theta}\right)$.]

2.12 (a) Obtain and sketch the locus in the complex plane defined by $\mathrm{Re}(z^{-1}) = 1$. On the same picture, sketch the locus defined by $\mathrm{Im}(z^{-1}) = 1$. At what angle do these loci intersect one another? Show that the unit circle touches both loci but crosses neither of them.

(b) Make a sketch of the complex plane showing a typical pair of complex numbers z_1 and z_2 that satisfy the equations

$$\begin{aligned} z_2 - z_1 &= (z_1 - a)\mathrm{e}^{2\pi\mathrm{i}/3} \\ a - z_2 &= (z_2 - z_1)\mathrm{e}^{2\pi\mathrm{i}/3}, \end{aligned}$$

where a is a real positive constant. Describe the geometrical figures whose vertices are z_1, z_2 and a.

Complex Numbers: Roots

2.13 Find all the values of the following roots

(i) $\sqrt[4]{\frac{-1-\sqrt{3}\mathrm{i}}{2}}$,

(ii) $(-8\mathrm{i})^{2/3}$,

(iii) $\sqrt[8]{16}$.

2.14 Solve the equation $z^4 = -4\mathrm{i}$.

2.15 Find the 5th roots of unity and plot them on an Argand diagram. What is the sum of the roots? What is the sum of n roots of any complex number?

2.16 By considering the roots of $z^{2n+1} + 1 = 0$, with n a positive integer, show that

$$\sum_{k=-n}^{n} \cos\left(\frac{2k+1}{2n+1}\pi\right) = 0.$$

2.17 Show that the equation with the four roots $z = \frac{1}{2}(\pm\sqrt{3} \pm \mathrm{i})$ is $z^4 - z^2 + 1 = 0$.

2.18 Show that the equation $(z + \mathrm{i})^n - (z - \mathrm{i})^n = 0$ has roots $z = \cot(r\pi/n)$, where $r = 1, 2, \ldots, n-1$.

2.19 Find the roots of the equation $(z-1)^n + (z+1)^n = 0$. Hence solve the equation $x^3 + 15x^2 + 15x + 1 = 0$.

2.20 Prove that the sum and product of the roots, x_i, of the polynomial $a_n x^n + \cdots + a_0$ satisfy $\sum x_i = -a_{n-1}/a_n$ and $\prod x_i = (-1)^n a_0/a_n$. Hence find the sum and the product of the roots of $P = x^3 - 6x^2 + 11x - 6$. Show that $x = 1$ is a root and by writing $P = (x-1)Q$, where Q is a quadratic, find the other two roots. Verify that the roots have the expected sum and product.

2.21 The polynomial $f(z)$ is defined by

$$f(z) = z^5 - 6z^4 + 15z^3 - 34z^2 + 36z - 48.$$

Show that the equation $f(z) = 0$ has two purely imaginary roots. Hence, or otherwise, factorize $f(z)$, and find all of its roots. Check that the sum and product of the roots take the expected values.

2.22 Show that the equation

$$(z+1)^n - \mathrm{e}^{2in\theta}(z-1)^n = 0$$

has roots $z = -\mathrm{i}\cot(\theta + r\pi/n)$, with $r = 0, \ldots, n-1$. Hence show that

$$\prod_{r=1}^{n} \cot\left(\theta + \frac{r\pi}{n}\right) = \begin{cases} (-1)^{n/2}, & \text{for } n \text{ even,} \\ (-1)^{(n-1)/2}\cot n\theta, & \text{for } n \text{ odd.} \end{cases}$$

2.23 Find all the roots, real and complex, of the equation $z^3 - 1 = 0$. If ω is one of the complex roots, prove that $1 + \omega + \omega^2 = 0$. Find the sums of the following series:

$$S_1 = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \ldots, \quad S_2 = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \ldots, \quad S_3 = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \ldots$$

MT III

Calculus: Curves, Line Integrals

3.1 (a) Find the arc length of the curve $y = \cosh x$ between $x = 0$ and $x = 1$.

(b) Find the arc length of the curve $x = \cos t,\ \ y = \sin t$ for $0 < t < \pi/2$.

(c) Find the surface area and volume of a sphere of radius R by treating it as obtained by rotating the curve $y = \sqrt{R^2 - x^2}$ about the x-axis.

For (b) and (c) do you get the answers you expect?

3.2 Evaluate the following line integrals:

(a) $\int_C (x^2 + 2y)\mathrm{d}x$ from (0,1) to (2,3) where C is the line $y = x + 1$.

(b) $\int_C xy\ \mathrm{d}x$ from (0,4) to (4,0) where C is the circle $x^2 + y^2 = 16$

(c) $\int_C (y^2\mathrm{d}x + xy\mathrm{d}y + zx\mathrm{d}z)$ from $A = (0, 0, 0)$ to $B = (1, 1, 1)$ where (i) C is the straight line from A to B; (ii) C is the broken line from A to $(0, 0, 1)$ and then from $(0, 1, 1)$ to B.

Calculus: Series, Expansions, Limits

3.3 (a) Find a_n and b_n for $\frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots = \sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} b_n$. Sum the series.

(b) Write out the first few terms of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$

What is this series equal to exactly?

(c) Squares and products of whole series can also occur, for example
$(a_1 + a_2 + a_3 + \cdots)^2$ and $(a_1 + a_2 + a_3 + \cdots) \times (b_1 + b_2 + b_3 + \cdots)$.
How would you write these in $\sum$ notation?

3.4 (a) Find by differentiation the expansion of each of the following functions in powers of x up to and including terms in x^3:

$$\text{(i) } \mathrm{e}^x, \qquad \text{(ii)} \sqrt{(1+x)}, \qquad \text{(iii)} \tan^{-1} x.$$

(b) Obtain the value of $\sin 31^{\mathrm{o}}$ by expanding $\sin x$ to four terms about the point $x = \pi/6$. How accurate is your answer?

3.5 (a) From the series for $\sin x$ and $\cos x$ show that

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

(b) Using the power series for e^y and $\ln(1+x)$, find the first four terms in the series for $\exp\{(\ln(1+x)\}$, and comment on the result.

3.6 Write down the power series expansion for $x^{-1}\sin x$. Hence evaluate, to four significant figures, the integral $\displaystyle\int_0^1 x^{-1}\sin x\,\mathrm{d}x$.

3.7 Find the following limits using power series expansion and asymptotic forms of functions in various limts; convince yourself that this is the same as using L'Hôpital's rule.

$$\begin{array}{ll}
\text{(a)} & \displaystyle\lim_{x\to 0}\frac{\sin x}{x}, \quad \lim_{x\to\infty}\frac{\sin x}{x}, \\
\text{(b)} & \displaystyle\lim_{x\to 0}\frac{1-\cos^2 x}{x^2}, \quad \lim_{x\to\infty}\frac{1-\cos^2 x}{x^2}, \\
\text{(c)} & \displaystyle\lim_{x\to 0}\frac{\sin x - x}{e^{-x}-1+x}, \quad \lim_{x\to\infty}\frac{\sin x - x}{e^{-x}-1+x}, \\
\text{(d)} & \displaystyle\lim_{x\to -1}\frac{\sin \pi x}{1+x}, \\
\text{(e)} & \displaystyle\lim_{x\to\infty}\frac{2x\cos x}{1+x}, \\
\text{(f)} & \displaystyle\lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x}, \\
\text{(g)} & \displaystyle\lim_{x\to 0}\frac{\sec x - \cos x}{\sin x}, \\
\text{(h)} & \displaystyle\lim_{x\to\pi/2}(\sin x)^{\tan x}.
\end{array}$$

3.8 Expand $[\ln(1+x)]^2$ in powers of x as far as x^4. Hence determine:

(a) whether $\cos 2x + [\ln(1+x)]^2$ has a maximum, minimum or point of inflection at $x = 0$,

(b) whether

$$\frac{[\ln(1+x)]^2}{x(1-\cos x)}$$

has a finite limit as $x \to 0$ and, if so, its value.

3.9 For the function

$$y = \cos(a\cos^{-1} x)$$

show that

$$(1-x^2)y'' - xy' + a^2 y = 0 \qquad (1)$$

where a is a constant.

Use Leibnitz' theorem to differentiate (1) n times and then put $x = 0$ to show that for $n \geq 0$

$$y^{(n+2)}(0) = (n^2 - a^2)y^{(n)}(0)$$

where $y^{(n)}(0)$ is the n^{th} derivative of $y(x)$ evaluated at $x = 0$.

Use this result to obtain a terminating power series expansion for $y = \cos(3\cos^{-1} x)$ in terms of x. Verify that your solution solves (1).

Vectors & Matrices

Problem Set 1: Questions 8–16 (geometry).

MT IV

Multivariate Calculus

4.1 (a) Sketch (in three dimensions) and (b) draw a contour map of the surfaces

(i) $z = (4 - x^2 - y^2)^{1/2}$,

(ii) $z = 1 - 2(x^2 + y^2)$,

(iii) $z = xy$,

(iv) $z = x^2 - y^2$.

4.2 (a) Find $\partial f/\partial x$ for

$$\text{(i) } f = (x^2 + y^2)^{1/2}, \quad \text{(ii) } f = \arctan\left(\frac{y}{x}\right), \quad \text{(iii) } f = y^x.$$

(b) Verify that $f_{xy} = f_{yx}$ for

$$\text{(i) } f = (x^2 + y^2)\sin(x + y), \quad \text{(ii) } f = x^m y^n.$$

(c) The function $f(x, y)$ is such that $f_{xy} = 0$. Find the most general forms for f_x and f_y and deduce that f has the form $f(x, y) = F(x) + G(y)$, where the functions F and G are arbitrary.

(d) If $V = f(x - ct) + g(x + ct)$, where c is a constant, prove that

$$V_{xx} - \frac{1}{c^2}V_{tt} = 0.$$

4.3 The acceleration of gravity can be found from the length l and period T of a pendulum; the formula is $g = 4\pi^2 l/T^2$. Using the linear approximation, find the relative error in g (i.e. $\Delta g/g$) in the worst case if the relative error in l is 5 % and the relative error in T is 2%.

4.4 (a) Find $\mathrm{d}u/\mathrm{d}t$ in two ways given that $u = x^n y^n$ and $x = \cos at$, $\; y = \sin at$, where a, n are constants.

(b) Find $\mathrm{d}u/\mathrm{d}x$ in two ways given that $u = x^2 y + y^{-1}$ and $y = \ln x$.

4.5 Given that $w = \exp(-x^2 - y^2)$, $\; x = r\cos\theta$, $\; y = r\sin\theta$, find $\dfrac{\partial w}{\partial r}$ and $\dfrac{\partial w}{\partial \theta}$ in two ways.

4.6 (a) The perfect gas law $PV = RT$, R is a constant, may be regarded as expressing any one of the quantities pressure P, volume V or temperature T of a perfect gas as a function of the other two. Verify explicitly that

$$\left(\frac{\partial P}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_P \left(\frac{\partial T}{\partial P}\right)_V = -1 \quad \text{and} \quad \left(\frac{\partial P}{\partial V}\right)_T = 1\Big/\left(\frac{\partial V}{\partial P}\right)_T.$$

(b) Show that these relations hold whatever the relation $f(P, V, T) = 0$ between P, V and T.

4.7 A variable z may be expressed either as a function of (u, v) or of (x, y), where $u = x^2+y^2$, and $v = 2xy$.

(a) Find $\left(\dfrac{\partial z}{\partial x}\right)_y$ in terms of $\left(\dfrac{\partial z}{\partial u}\right)_v$ and $\left(\dfrac{\partial z}{\partial v}\right)_u$.

(b) Find $\left(\dfrac{\partial z}{\partial u}\right)_v$ in terms of $\left(\dfrac{\partial z}{\partial x}\right)_y$ and $\left(\dfrac{\partial z}{\partial y}\right)_x$.

(c) Express $\left(\dfrac{\partial z}{\partial u}\right)_v - \left(\dfrac{\partial z}{\partial v}\right)_u$ in terms of $\left(\dfrac{\partial z}{\partial x}\right)_y$ and $\left(\dfrac{\partial z}{\partial y}\right)_x$.

(d) Verify your expression explicitly in the case $z = u + v$.

4.8 Spherical polar coordinates (r, θ, ϕ) are defined in terms of Cartesian coordinates (x, y, z) by

$$x = r \sin\theta \cos\phi, \qquad y = r \sin\theta \sin\phi, \qquad z = r\cos\theta.$$

(a) Find $(\partial x/\partial r)$, treating x as a function of the spherical polar coordinates, and $(\partial r/\partial x)$ treating r as a function of the Cartesian coordinates.

(b) Given that f is a function of r only, independent of θ and ϕ, show that

$$\frac{\partial f}{\partial x} = \frac{x}{r}\frac{\mathrm{d}f}{\mathrm{d}r}, \tag{3}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{r}\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{x^2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}f}{\mathrm{d}r}\right), \tag{4}$$

and hence deduce that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}f}{\mathrm{d}r}\right).$$

4.9 Expand $f(x, y) = \mathrm{e}^{xy}$ to second order around the point $x = 2, y = 3$.

4.10 Find the position and nature of the stationary points of the following functions and sketch rough contour graphs in each case.

$$\text{(i) } f(x, y) = x^2 + y^2, \qquad \text{(ii) } f(x, y) = x^3 + y^3 - 2(x^2 + y^2) + 3xy,$$

$$\text{(iii) } f(x, y) = \sin x \sin y \sin(x + y), \quad 0 < x < \pi/2; \quad 0 < y < \pi/2.$$

4.11 (a) Which of the following are exact differentials? For those that are exact, find f.

$$\text{(i) } \mathrm{d}f = x\,\mathrm{d}y + y\,\mathrm{d}x, \quad \text{(ii) } \mathrm{d}f = x\,\mathrm{d}y - y\,\mathrm{d}x, \quad \text{(iii) } \mathrm{d}f = x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z.$$

(b) What is the value of $\oint x\,\mathrm{d}y + y\,\mathrm{d}x$ around the curve $x^4 + y^4 = 1$?

Vectors & Matrices

Problem Set 2: Questions 1–7 (matrices, linear maps), 8–12 (linear equations).

MT V

Vectors & Matrices

Problem Set 3: Questions 1–5 (determinants), 6–10 (scalar products)

In Question 6, the three polynomials that you are asked to deal with are the first three of an infinite set called Hermite polynomials. The m-th order Hermite polynomial can be generated by the following formula:

$$H_m(x) = (-1)^m e^{x^2} \frac{\mathrm{d}^m}{\mathrm{d}x^m} e^{-x^2}.$$

(a) Convince yourself that these are indeed m-th order polynomials.

(b) Figure out how to prove that they form an orthogonal set of functions, namely that their scalar product is

$$\langle H_m, H_n \rangle = \int_{-\infty}^{+\infty} \mathrm{d}x \, e^{-x^2} H_m(x) H_n(x) = 2^m m! \sqrt{\pi}\, \delta_{m,n}.$$

NB: If you are unable to figure this out entirely on your own, consider this to be an exercise in finding things out, using the broad range of resources that are available to you.

(c) With the above result in hand, prove that the Hermite polynomials form a *complete* set, i.e., for any real, continuous function $f(x) : \mathbb{R} \to \mathbb{R}$, find an infinite set of coefficients f_m such that

$$f(x) = \sum_{m=0}^{\infty} f_m H_m(x).$$

MT VI

Vectors & Matrices

Problem Set 4: Questions 1–9 (eigenvectors & eigenvalues).

ODEs: First Order

Problem Set 1: solve all questions, including “Supplementary”, starred and “Extracurricular”.

MT VII

ODEs: Second Order

Problem Set 2: solve all questions, including “Supplementary”, starred and “Extracurricular”.

MT VIII (Vacation Work)

ODEs: Systems

Problem Set 3: solve all questions, including “Supplementary”, starred and “Extracurricular”.

Some Optional Revision

We have used the set of V&M questions from Andre Lukas’s course, which started in 2013. Download Merton Maths problem sets from before that time and solve all the old V&M questions. This way you will know that you can still do all those things that previous generations learned to do. We will not mark this part of the Vacation Work, but please bring to the tutes in HT all the questions/issues you have identified with your understanding of the subject (or contact graduate mentors).

The questions are (using the numbering scheme from the old problem sets):
MT-III 7-12, MT-IV 8-10, MT-V 5-8, MT-VI 1-9, MT-VII 1-17, MT-VIII 3-7.