Merton Maths

First-Year Mathematics (CP3&4)
for Merton College Students

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(with thanks to lecturers and tutors, past and present)

Problems for Hilary Term 2026

HT I

Vector Calculus

1.1 Two circles have equations:

(i) $x^2 + y^2 + 2ax + 2by + c = 0$;

(ii) $x^2 + y^2 + 2a'x + 2b'y + c' = 0$.

Show that these circles are orthogonal if $2aa' + 2bb' = c + c'$.

1.2 Find the equation for the tangent plane to the surface $2xz^2 - 3xy - 4x = 7$ at $(1, -1, 2)$.

Multiple Integrals: 2D

1.3 (a) For the following integrals sketch the region of integration and so write equivalent integrals with the order of integration reversed. Evaluate the integrals both ways.

$$\int_0^{\sqrt{2}} \mathrm{d}y \int_{y^2}^{2} \mathrm{d}x\, y\;, \qquad \int_0^4 \mathrm{d}x \int_0^{\sqrt{x}} \mathrm{d}y\, y\sqrt{x}\;, \qquad \int_0^1 \mathrm{d}y \int_{-y}^{y^2} \mathrm{d}x\, x\;.$$

(b) Reverse the order of integration and hence evaluate:

$$\int_0^{\pi} \mathrm{d}y \int_y^{\pi} \mathrm{d}x\, \frac{\sin x}{x}\;.$$

[This is the same as A. Lukas' Problem Set III: Question 1]

1.4 Evaluate

$$\iint \exp[-(x^2 + y^2)]\mathrm{d}x\mathrm{d}y$$

over the area of a circle with centre $(0, 0)$ and radius a. By letting a tend to ∞, evaluate

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x.$$

1.5 (a) A mass distribution in the positive x region of the xy-plane and in the shape of a semi-circle of radius a, centred on the origin, has mass per unit area k. Find, using plane polar coordinates,

(i) its mass M,

(ii) the coordinates $(\overline{x}, \overline{y})$ of its centre of mass,

(iii) its moments of inertia about the x and y axes.

(b) Do as above for a semi-infinite sheet with mass per unit area

$$\sigma = k \exp\left(-\frac{x^2 + y^2}{a^2}\right) \quad \text{for} \quad x \geq 0\;, \qquad \sigma = 0 \quad \text{for} \quad x < 0,$$

where a is a constant. Comment on the comparisons between the two sets of answers.

Note that the result of Q1.4 will be useful in the evaluation of the required integrals.

1.6 Evaluate the following integral:

$$\int_0^a \mathrm{d}y \int_0^{\sqrt{a^2-y^2}} \mathrm{d}x\,(x^2+y^2)\,\arctan\frac{y}{x}\,.$$

[This is the same as A. Lukas' Problem Set III: Question 2(a)]

1.7 The pair of variables (x, y) are each functions of the pair of variables (u, v) and *vice versa*. Consider the matrices

$$A = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}\,.$$

(a) Show using the chain rule that the product AB of these two matrices equals the unit matrix I.

(b) Verify this property explicitly for the case in which (x, y) are Cartesian coordinates and u and v are the polar coordinates (r, θ).

(c) Assuming the result that the determinant of a matrix and the determinant of its inverse are reciprocals, deduce the relation between the Jacobians

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \qquad \text{and} \qquad \frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\,.$$

1.8 (a) Using the change of variable $x + y = u$, $x - y = v$, evaluate the double integral $\iint_R (x^2+y^2)\,\mathrm{d}x\mathrm{d}y$, where R is the region bounded by the straight lines $y = x$, $y = x+2$, $y = -x$ and $y = -x+2$.

[This is similar to A. Lukas' Problem Set III: Question 2(b)]

(b) Given that $u = xy$ and $v = y/x$, show that $\partial(u,v)/\partial(x,y) = 2y/x$. Hence evaluate the integral

$$\iint \exp(-xy)\,\mathrm{d}x\mathrm{d}y$$

over the region $x > 0$, $y > 0$, $xy < 1$, $1/2 < y/x < 2$.

[This is the same as A. Lukas' Problem Set III: Question 2(c)]

Normal Modes

1.9 ODE Problem Set 6: Question 6.1

1.10 ODE Problem Set 6: Question 6.2

HT II

Multivariate Calculus

2.1 A. Lukas' Problem Set I: Question 1 (gradients)

2.2 A. Lukas' Problem Set I: Question 2 (radius function)

2.3 A. Lukas' Problem Set I: Question 4 (cylindrical coordinates)

2.4 A. Lukas' Problem Set I: Question 6 (stationary points)

Multiple Integrals: 3D

2.5 A solid hemisphere of uniform density k occupies the volume $x^2 + y^2 + z^2 \leq a^2$, $z \geq 0$. Using symmetry arguments wherever possible, find

(i) its total mass M,

(ii) the position $(\overline{x}, \overline{y}, \overline{z})$ of its centre of mass,

(iii) its moments and products of inertia, I_{xx}, I_{yy}, I_{zz}, I_{xy}, I_{yz}, I_{zx}, where

$$I_{zz} = \int k\ (x^2 + y^2)\ \mathrm{d}V, \qquad I_{xy} = \int k\ xy\ \mathrm{d}V, \qquad \text{etc.}$$

Probabilities

2.6 If X is a continuous random variable with probability density function (PDF) $f(x) = ce^{-x}$ for $x \geq 0$ and zero otherwise,

(a) find c;

(b) find the cumulative distribution function $F(x)$;

(c) find $P(1 < X < 3)$.

2.7 Let X and Y be two jointly continuous random variables with a joint PDF $f(x, y) = cx^2y$ for $0 \leq y \leq x \leq 1$ and zero otherwise.

(a) Sketch the region in the (x, y) plane for which the PDF is non-zero.

(b) Find c.

(c) Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.

(d) Find $P(Y \leq X/2)$.

Line Integrals

2.8 A vector field $\mathbf{A}(\mathbf{r})$ is defined by its components $(4x - y^4, -4xy^3 - 3y^2, 4)$. Evaluate the line integral $\int \mathbf{A} \cdot \mathrm{d}\mathbf{l}$ between the points with position vectors $(0,0,0)$ and $(1,2,0)$ along the following paths

(a) the straight line from $(0,0,0)$ to $(1,2,0)$;

(b) on the path of straight lines joining $(0,0,0)$, $(0,0,1)$, $(1,0,1)$, $(1,2,1)$ and $(1,2,0)$ in turn.

Show that $\mathbf{A}$ is conservative by finding a scalar function $\phi(\mathbf{r})$ such that $\mathbf{A} = \nabla\phi$.

2.9 A vector field $\mathbf{A}(\mathbf{r})$ is defined by its components $(3x^2 + 6y, -14yz, 20xz^2)$. Evaluate the line integral $\int \mathbf{A} \cdot \mathrm{d}\mathbf{l}$ from $(0,0,0)$ to $(1,1,1)$ along the following paths

(a) $x = t$, $y = t^2$, $z = t^3$;

(b) on the path of straight lines joining $(0,0,0)$, $(1,0,0)$, $(1,1,0)$, and $(1,1,1)$ in turn;

(c) the straight line joining the two points.

Is $\mathbf{A}$ conservative?

2.10 If $\phi = 2xyz^2$, $\mathbf{F} = (xy, -z, x^2)$ and C is the curve $x = t^2$, $y = 2t$, $z = t^3$ from $t = 0$ to $t = 1$, evaluate the line integrals:

(a) $\int_C \phi \, \mathrm{d}\mathbf{r}$;

(b) $\int_C \mathbf{F} \times \mathrm{d}\mathbf{r}$.

Surface Integrals

2.11 Show that under certain circumstances the area of a curved surface can be expressed in the form

$$\iint \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, \mathrm{d}x\mathrm{d}y,$$

where the integral is taken over the plane $z = 0$.

Sketch the surface $z^2 = 2xy$ and find the area of the part of it that lies inside the hemisphere $x^2 + y^2 + z^2 = 1$, $z > 0$.

2.12 Show that the surface area of the curved portion of the hemisphere in Question 2.5 is $2\pi a^2$ by

(i) directly integrating the element of area $a^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi$ over the surface of the hemisphere.

(ii) projecting onto an integral taken over the xy plane.

2.13 (a) Find the area of the plane $x - 2y + 5z = 13$ cut out by the cylinder $x^2 + y^2 = 9$.

(b) A uniform lamina is made of that part of the plane $x + y + z = 1$ which lies in the first octant. Find by integration its area and also its centre of mass. Use geometrical arguments to check your result for the area.

2.14 Calculate the solid angle of a cone of half-angle α.

2.15 Calculate the area of a unit sphere in n dimensions. Check that your answer is 2π for $n = 2$ and 4π for $n = 3$.

Hint. Calculate the Gaussian integral $\int \cdots \int \mathrm{d}x_1 \ldots \mathrm{d}x_n \exp[-(x_1^2 + \cdots + x_n^2)]$ in Cartesian and in polar coordinates.

Normal Modes

2.16 ODE Problem Set 6: Question 6.3

2.17 ODE Problem Set 6: Question 6.4

HT III

Multivariate Calculus

3.1 A. Lukas' Problem Set II: Question 1 (vector identities)

3.2 A. Lukas' Problem Set II: Question 2 (potential, curl)

3.3 A. Lukas' Problem Set II: Question 3 (vector potential, div)

3.4 (a) Use index notation to find the divergence and curl of $\mathbf{r}\times(\mathbf{a}\times\mathbf{r})$, where $\mathbf{a}$ is a constant vector and $\mathbf{r}$ is the position vector.

(b) Prove that $\nabla[\mathbf{a}\cdot\nabla(r^{-1})] = -\nabla\times[\mathbf{a}\times\nabla(r^{-1})]$, where $r = |\mathbf{r}|$.

3.5 For $\mathbf{A} = (3xyz^2, 2xy^3, -x^2yz)$ and $\phi = 3x^2 - yz$, find: (a) $\nabla\cdot\mathbf{A}$; (b) $\mathbf{A}\cdot\nabla\phi$; (c) $\nabla\cdot(\phi\mathbf{A})$; (d) $\nabla\cdot(\nabla\phi)$.

3.6 A body expands linearly by a factor $1+\alpha$ because of a rise in temperature. The expansion shifts the particle originally at $\mathbf{r}$ to $\mathbf{r}+\mathbf{h}$. Calculate $\nabla\cdot\mathbf{h}$. By what fraction does the volume increase?

3.7 The magnetic field $\mathbf{B}$ at a distance r from a straight wire carrying a current I has magnitude $\mu_0 I/2\pi r$. The lines of force are circles centred on the wire and in planes perpendicular to it. Show that $\nabla\cdot\mathbf{B} = 0$.

3.8 Sketch the vector fields $\mathbf{A} = (x, y, 0)$ and $\mathbf{B} = (y, -x, 0)$. Calculate the divergence and curl of each vector field and explain the physical significance of the results obtained.

3.9 A bucket of water is rotated slowly with angular velocity ω about its vertical axis. When a steady state has been reached the water rotates with a velocity field $\mathbf{v}(\mathbf{r})$ as if it were a rigid body. Calculate $\nabla\cdot\mathbf{v}$ and interpret the result. Calculate $\nabla\times\mathbf{v}$. Can the flow be represented in terms of a velocity potential ϕ such that $\mathbf{v} = \nabla\phi$? If so, what is ϕ?

3.10 Find a vector field $\mathbf{A}$ such that $\nabla\times\mathbf{A} = (0, 0, B)$, where B is a constant. Is this field unique?

Surface Integrals: Fluxes

3.11 If $\mathbf{n}$ is the unit normal to the surface S, evaluate $\int\int \mathbf{r}\cdot\mathbf{n}\,\mathrm{d}S$ over

(a) the unit cube bounded by the coordinate planes and the planes $x = 1$, $y = 1$ and $z = 1$;

(b) the surface of a sphere of radius a centred at the origin.

3.12 Evaluate $\int \mathbf{A} \cdot \mathbf{n} \, \mathrm{d}S$ for the following cases:

(a) $\mathbf{A} = (y, 2x, -z)$ and S is the surface of the plane $2x + y = 6$ in the first octant cut off by the plane $z = 4$.

(b) $\mathbf{A} = (x + y^2, -2x, 2yz)$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant

(c) $\mathbf{A} = (6z, 2x+y, -x)$ and S is the entire surface of the region bounded by the cylinder $x^2 + z^2 = 9$, $x = 0$, $y = 0$, $z = 0$ and $y = 8$.

[This is the same as A. Lukas' Problem Set IV: Question 2(a,b,c)]

3.13 Air is flowing with a speed 0.4 m s^{-1} in the direction of the vector $(-1, -1, 1)$. Calculate the volume of air flowing per second through the loop which consists of straight lines joining, in turn, the following: $(1, 1, 0)$, $(1, 0, 0)$, $(0, 0, 0)$, $(0, 1, 1)$, $(1, 1, 1)$ and $(1, 1, 0)$.

Wave Equation

3.14 F. Parra's Waves Problem Set I: Question 1.1

3.15 F. Parra's Waves Problem Set I: Question 1.2

HT IV

Multivariate Calculus

4.1 A. Lukas' Problem Set II: Question 4 (Laplacian)

4.2 A. Lukas' Problem Set II: Question 5 (uncurling)

Gauss and Stokes Theorems

4.3 The vector $\mathbf{A}$ is a function of position $\mathbf{r} = (x, y, z)$ and has components (xy^2, x^2, yz). Calculate the surface integral $\int \mathbf{A} \cdot \mathrm{d}\mathbf{S}$ over each face of the triangular prism bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x+y = 1$ and $z = 1$. Show that the integral $\int \mathbf{A} \cdot \mathrm{d}\mathbf{S}$ taken outwards over the whole surface is not zero. Show that it equals $\int \nabla \cdot \mathbf{A} \, \mathrm{d}V$ calculated over the volume of the prism. Why?

4.4 Solve Questions 3.11(a,b) and 3.12(c) using Gauss's Theorem.

4.5 The vector $\mathbf{A}(\mathbf{r}) = (y, -x, z)$. Verify Stokes' Theorem for the hemispherical surface $|\mathbf{r}| = 1$, $z \geq 0$.

[This is the same as A. Lukas' Problem Set IV: Question 4(a)]

4.6 Let $\mathbf{A} = (y, -x, 0)$. Find $\int \mathbf{A} \cdot \mathrm{d}\mathbf{l}$ for a closed loop on the surface of the cylinder $(x - 3)^2 + y^2 = 2$. Consider the cases

(i) in which the loop wraps round the cylinder's axis;

(ii) in which the loop does not (so it can be continuously deformed to a point).

Waves: d'Alembert's Solution

4.7 At time $t = 0$, the displacement of an infinitely long string is:

$$y(x,t) = \begin{cases} \sin(\pi x/a) & \text{for } -a \leq x \leq a, \\ 0 & \text{otherwise.} \end{cases}$$

The string is initially at rest. Using d'Alembert's solution with phase speed c, sketch the displacement of the string at $t = 0$, $t = a/2c$, and $t = a/c$.

4.8 F. Parra's Waves Problem Set I: Question 1.3

4.9 F. Parra's Waves Problem Set I: Question 1.4

4.10 F. Parra's Waves Problem Set I: Question 1.5

Nonlinear Oscillations: Limit Cycles

4.11 ODE Problem Set 5: Question 5.3

4.12 ODE Problem Set 5: Question 5.4

HT V

Multivariate Calculus

5.1 Prove that

$$\text{(i)} \qquad \int_{\partial V} p\,\mathrm{d}\mathbf{S} = \int_V \nabla p\,\mathrm{d}^3\mathbf{r}$$

$$\text{(ii)} \qquad \int_{\partial V} \mathrm{d}\mathbf{S}\times\mathbf{a} = \int_V \nabla\times\mathbf{a}\,\mathrm{d}^3\mathbf{r},$$

where ∂V is the closed surface bounding the volume V.

5.2 If $\mathbf{G} = (\mathbf{a}\cdot\mathbf{r})\mathbf{a}$, where $\mathbf{a}$ is a constant vector, and S is a closed surface, show that

$$\int_S \mathbf{G}\times\mathrm{d}\mathbf{S} = 0,$$

where S is any closed surface.

Verify this result for the special case in which $\mathbf{a} = (0,0,1)$ and S is the bounding surface of the hemisphere $r \le 1$, $z \ge 0$.

5.3 Prove that

$$\oint_{\partial S} \phi\mathrm{d}\mathbf{l} = \int_S \mathrm{d}\mathbf{S}\times\nabla\phi,$$

where ∂S is the closed curve bounding the surface S.

Verify this relation for the function $\phi = x^3$, with ∂S the circle $x^2+y^2=a^2$, $z=0$, and S the part of the plane $z=0$ enclosed by ∂S.

5.4 $\mathbf{A}(\mathbf{r})$ is defined by

$$\mathbf{A}(\mathbf{r}) = \int_V \nabla_{\mathbf{r}'} f(|\mathbf{r}-\mathbf{r}'|)\mathrm{d}^3\mathbf{r}',$$

where $\nabla_{\mathbf{r}'}$ denotes the gradient with respect to $\mathbf{r}'$, $\mathbf{r}$ remaining fixed, f is a well-behaved function of a single variable and the volume V is fixed. Show that $\nabla\times\mathbf{A} = 0$. [Hint: Find a function of which $\mathbf{A}$ is the gradient.]

Hence find $\mathbf{A}$ when $f(x) = x^4$ and V is the sphere $r=1$.

5.5 The region V is bounded by a simple, closed surface ∂V. Prove that

$$\int_{\partial V} \psi\nabla\phi\cdot\mathrm{d}\mathbf{S} = \int_V \psi\nabla^2\phi\mathrm{d}^3\mathbf{r} + \int_V (\nabla\psi\cdot\nabla\phi)\mathrm{d}^3\mathbf{r}.$$

Let $\nabla^2\phi = 0$ in V and $\phi(\mathbf{r}) = g(\mathbf{r})$ on ∂V. A function $f(\mathbf{r})$ is chosen so that it also satisfies $f(\mathbf{r}) = g(\mathbf{r})$ on ∂V. By writing $\psi = f-\phi$ show that

$$\int_V |\nabla\phi|^2\mathrm{d}^3\mathbf{r} \le \int_V |\nabla f|^2\mathrm{d}^3\mathbf{r}.$$

If ∂V is the sphere $x^2 + y^2 + z^2 = a^2$ and in polar coordinates (r, θ, ϕ), $g = a\cos\theta$, show that

$$\int_V |\nabla\phi|^2 \mathrm{d}^3\mathbf{r} \leq V = \frac{4}{3}\pi a^3.$$

5.6 If S is a surface bounded by the closed curve ∂S, prove that

$$\oint_{\partial S} \mathbf{r}(\mathbf{r}\cdot \mathrm{d}\mathbf{l}) = \int_S \mathbf{r} \times \mathrm{d}\mathbf{S}.$$

Verify this formula when ∂S is

(i) the intersection of the cylinder $x^2 - x + y^2 = 2$ with the plane $z = 0$,

(ii) the intersection of the same cylinder with the sphere $x^2 + y^2 + z^2 = 9$.

What is the value of the right-hand integral for the portion of the cylinder cut off by the plane and the sphere?

Waves: Energetics

5.7 F. Parra's Waves Problem Set I: Question 1.6

5.8 F. Parra's Waves Problem Set I: Question 1.7

Waves in Confined Spaces

5.9 (a) What is the difference between a travelling wave and a standing wave?

(b) Convince yourself that

$$y_1 = A\sin(kx - \omega t)$$

corresponds to a travelling wave. Which way does it move? What are the amplitude, wavelength, frequency, period and velocity of the wave?

(c) Show that y_1 satisfies the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

provided that ω and k are suitably related (this is called the "dispersion relation").

(d) Write down a wave y_2 of equal amplitude travelling in the opposite direction. Show that $y_1 + y_2$ can be written in the form

$$y_1 + y_2 = f(x)g(t),$$

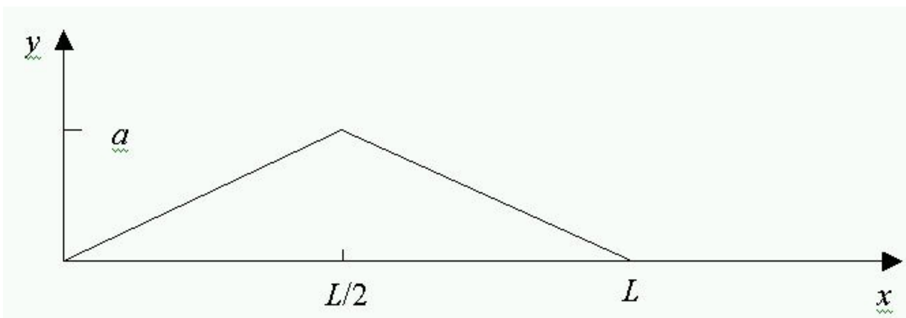
where $f(x)$ is a function of x only, and $g(t)$ is a function just of t. Convince yourself that the combination of two travelling waves is a standing wave. By determining $f(x)$ and $g(t)$ explicitly, find the wavelength and frequency of $y_1 + y_2$. Comment on the velocity of the waves.

5.10 (a) Consider a string of uniform linear density ρ, stretched to a tension T, with its ends fixed at $x = 0$ and $x = L$. Show that

$$y(x,t) = A_p \sin\left(\frac{p\pi x}{L}\right) \sin\left(\frac{p\pi ct}{L}\right) \quad \text{and} \quad y(x,t) = B_p \sin\left(\frac{p\pi x}{L}\right) \cos\left(\frac{p\pi ct}{L}\right),$$

where p is any integer and $c = \sqrt{T/\rho}$, are both solutions describing posible evolution of the transverse displacement of the string at position x vs. time t. Explain why sums of such solutions are also solutions.

(b) Imagine that at $t = 0$, the string is plucked at its midpoint to a small distance a and held still:



Show that, after the midpoint is released, the transverse displacement of the string will evolve according to

$$y(x,t) = \sum_{p=1}^{\infty} B_p \sin\left(\frac{p\pi x}{L}\right) \cos\left(\frac{p\pi ct}{L}\right),$$

prove that the coefficients are

$$B_p = \frac{2}{L} \int_0^L \mathrm{d}x\, y(x,0) \sin\left(\frac{p\pi x}{L}\right),$$

and calculate them all.

In dealing with this problem, you will find it illuminating to consider a vector space of functions that vanish at $x = 0$ and $x = L$, come up with an orthogonal basis in this vector space, then turn the wave equation into an infinite set of ODEs, each corresponding to one of the basis vectors, and solve them by the usual method of projecting the initial condition onto the basis vectors, solving for the subsequent time evolution of each projection, and then reassambling these solutions to predict the complete evolution of the initial displacement.

(c) Ponder and discuss the connexion between what you have done here and Question 6.6(c,d) in the ODE Problem Set 6.

(d) Consider the same problem as in (b), but instead of plucking the string, assume that, at $t = 0$, its transverse displacement is

$$y(x,0) = \sin\left(\frac{\pi x}{L}\right) + 2\sin\left(\frac{2\pi x}{L}\right)$$

and the string is instantaneously stationary. Find the displacement at subsequent times and make rough sketches of $y(x,t)$ at the following times: 0, $L/4c$, $L/2c$, $3L/4c$, L/c.

(e) Again consider the same problem as in (b), except assuming that the string is initially in its equilibrium position, $y(x,t=0)=0$, but is imparted an initial velocity profile $[\partial y/\partial t](x,t=0)=v(x)$. Find the subsequent motion of the string, $y(x,t)$.

Multivariate Calculus and Waves: Electromagnetism

5.11 Microscopic *Maxwell's equations* in integral form are, if written in Gauss units (I leave it to you to figure out the conversion to the abominable SI system):

$$\begin{aligned}
\int_{\partial V} \mathbf{E}\cdot \mathrm{d}\mathbf{S} &= 4\pi \int_V \mathrm{d}^3\mathbf{r}\,\rho \quad \text{(Gauss's law)}, && (1)\\
\int_{\partial V} \mathbf{B}\cdot \mathrm{d}\mathbf{S} &= 0, && (2)\\
\oint_{\partial S} \mathbf{E}\cdot \mathrm{d}\mathbf{l} &= -\frac{1}{c}\frac{\partial}{\partial t}\int_S \mathbf{B}\cdot \mathrm{d}\mathbf{S} \quad \text{(Faraday's law)}, && (3)\\
\oint_{\partial S} \mathbf{B}\cdot \mathrm{d}\mathbf{l} &= \frac{1}{c}\int_S \left(4\pi\mathbf{j} + \frac{\partial \mathbf{E}}{\partial t}\right)\cdot \mathrm{d}\mathbf{S} \quad \text{(Ampère-Maxwell law)}, && (4)
\end{aligned}$$

where $\mathbf{E}$ is electric field, $\mathbf{B}$ magnetic field, ρ charge density, $\mathbf{j}$ current density and c the speed of light. The integrals are over an arbitrary volume V (whose bounding surface is ∂V) or arbitrary surface S (whose bounding loop is ∂S).

(a) Use Gauss's and Stokes' theorems and the fact that V and S are arbitrary and can be made infinitesimal to derive Maxwell's equations in differential form (do make sure you get the right result: look it up!).

(b) Using Gauss's and Ampère-Maxwell laws in their differential form, show that

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\mathbf{j} = 0.$$

Integrate this over some volume V, use Gauss's theorem and use the result to argue that this equation expresses *conservation of charge*.

(c) The energy density of the electromagnetic field is

$$\varepsilon = \frac{E^2+B^2}{8\pi}.$$

Use Maxwell's equations to show that

$$\frac{\partial \varepsilon}{\partial t} + \nabla\cdot\mathbf{P} = -\mathbf{E}\cdot\mathbf{j},$$

where $\mathbf{P} = c\mathbf{E}\times\mathbf{B}/4\pi$ is called the *Poynting vector*. Again integrate over some volume V and argue that the above equation expresses *conservation of energy*. Give physical interpretation of $\mathbf{P}$ and explain what the right-hand side of the above equation represents.

5.12 Consider electromagnetic field in vacuo: $\rho = 0$, $\mathbf{j} = 0$.

(a) Use Maxwell's equations to show that $\mathbf{E}$ and $\mathbf{B}$ satisfy the wave equation:

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = c^2 \nabla^2 \mathbf{E}$$

and similarly for $\mathbf{B}$. These are *electromagnetic waves* (light).

(b) Let $\mathbf{E}$ and $\mathbf{B}$ both be $\propto \exp(-i\omega t + i\mathbf{k}\cdot\mathbf{r})$.

Show that in order for this to be a solution, we must have $\omega = \pm ck$.

Using Maxwell's equations, show that $\mathbf{E}$ and $\mathbf{B}$ are perpendicular to each other and to the direction of the propagation of the wave.

(c) Show that the Poynting vector for light waves is

$$\mathbf{P} = \pm c \frac{\mathbf{k}}{k} \varepsilon$$

and explain what this result means physically.

Multivariate Calculus and Waves: Hydrodynamics

5.13 Consider a fluid or gaseous medium with density $\rho(t, \mathbf{r})$ and velocity $\mathbf{u}(t, \mathbf{r})$.

(a) Taking an arbitrary volume V within the fluid, we can express the *conservation of mass* as follows

$$\frac{\partial}{\partial t} \int_V \mathrm{d}^3\mathbf{r}\, \rho = -\int_{\partial V} \mathbf{F}_{\mathrm{mass}} \cdot \mathrm{d}\mathbf{S},$$

where the flux of mass is $\mathbf{F}_{\mathrm{mass}} = \rho\mathbf{u}$ (density ρ flowing with velocity $\mathbf{u}$). Use Gauss's Theorem and the fact that V is entirely arbitrary and can be taken to be as small as we like to derive the *continuity equation*:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \tag{5}$$

Note that this can be written as

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right) \rho = -\rho \nabla \cdot \mathbf{u}.$$

The left-hand side is the so-called *convective time derivative* of ρ — the rate of change of density in a fluid element moving with velocity $\mathbf{u}$. The above equation then means that negative divergence of the fluid flow $\nabla \cdot \mathbf{u} < 0$ implies local compression and positive divergence $\nabla \cdot \mathbf{u} > 0$ local rarefaction.

(b) Now apply similar logic to the *conservation of momentum*. The momentum density is $\rho\mathbf{u}$. The rate of change of momentum in a volume V is

$$\frac{\partial}{\partial t} \int_V \mathrm{d}^3\mathbf{r}\, \rho\mathbf{u} = -\int_{\partial V} \rho\mathbf{u}\mathbf{u} \cdot \mathrm{d}\mathbf{S} - \int_{\partial V} p\mathrm{d}\mathbf{S}.$$

The first term on the right-hand side is the flux of momentum (which is a tensor quantity: vector flux of each component ρu_i is $\rho u_i \mathbf{u}$). The second term is the force on the boundary of the volume V due to pressure p (pressure times area; minus because $\mathrm{d}\mathbf{S}$ points outward). Use the above equation and Eq. (5) to derive *Euler's Equation* for the velocity of the fluid:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p. \tag{6}$$

(c) Finally we deal with the *conservation of energy.* The energy density of the fluid is

$$\varepsilon = \frac{\rho u^2}{2} + \frac{p}{\gamma - 1},$$

where the first term is kinetic-energy density and the second term is internal-energy density (γ is a constant and depends on the nature of the medium; e.g., for monatomic ideal gases, it is 5/3). Express the rate of change of energy in a volume V as

$$\frac{\partial}{\partial t}\int_V \mathrm{d}^3\mathbf{r}\,\varepsilon = \text{flux of energy through } \partial V + \text{work done by pressure on } \partial V \text{ per unit time.}$$

(the first term is analogous to the other flux terms you have encountered above; the second term you can work out by considering that if the fluid is flowing through the boundary of V at velocity $\mathbf{u}$, it must be pushing against pressure p and so doing work against force $p\mathrm{d}\mathbf{S}$). From the resulting equation, prove, using also Eqs. (5) and (6), that p satisfies

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\frac{p}{\rho^\gamma} = 0. \tag{7}$$

Congratulations, you have derived the *equations of compressible hydrodynamics* — Eqs. (5–7) for ρ, $\mathbf{u}$ and p, a closed system.

5.14 The equations derived in the previous problem have a simple solution: $\rho = \rho_0$, $p = p_0$ and $\mathbf{u} = 0$, where p_0 and ρ_0 are constants independent of t or $\mathbf{r}$. Now consider small disturbances of this static homogeneous state:

$$\rho = \rho_0 + \delta\rho, \quad p = p_0 + \delta p, \quad \mathbf{u} = \delta\mathbf{u},$$

where $\delta\rho$, δp and $\delta\mathbf{u}$ are all infinitesimally small. Since they are infinitesimally small, we can substitute the above expressions for ρ, p, $\mathbf{u}$ into Eqs. (5-7) and neglect all terms where these small perturbations enter quadratically.

(a) Show therefore that $\delta\rho$, δp and $\delta\mathbf{u}$ satisfy

$$\frac{\partial\delta\rho}{\partial t} + \rho_0\nabla\cdot\delta\mathbf{u} = 0, \quad \rho_0\frac{\partial\delta\mathbf{u}}{\partial t} = -\nabla\delta p, \quad \frac{\partial}{\partial t}\left(\delta p - \gamma\frac{p_0}{\rho_0}\delta\rho\right) = 0. \tag{8}$$

(b) Hence show that the perturbations satisfy the wave equation

$$\frac{\partial^2\delta\rho}{\partial t^2} = c_s^2\nabla^2\delta\rho$$

and determine c_s. These are *sound waves*.

(c) Let $\delta\rho$, δp and $\delta\mathbf{u}$ all be $\propto \exp(-i\omega t + i\mathbf{k}\cdot\mathbf{r})$.

What is the relationship between ω and $\mathbf{k}$?

Using Eqs. (8), show that

$$\delta\mathbf{u} = \pm c_s \frac{\mathbf{k}}{k}\frac{\delta\rho}{\rho_0} \quad \text{and} \quad \frac{\delta p}{p_0} = \gamma\,\frac{\delta\rho}{\rho_0}.$$

If you have time and want more hydro action, solve Question 7.25 now.

HT VI

Waves: Travelling, Standing, Dispersive

F. Parra's Waves Problem Set II: do all questions

HT VII

(revision problems—vacation work)

Multivariate Calculus

7.1 A. Lukas' Problem Set I: Question 3 (fixed points of ODEs)

7.2 A. Lukas' Problem Set I: Question 5 (homogeneous functions)

7.3 A. Lukas' Problem Set I: Question 7 (potentials from gradients)

7.4 A. Lukas' Problem Set I: Question 8 (invariant functions)

7.5 A. Lukas' Problem Set II: Question 6 (curve integrals)

7.6 A. Lukas' Problem Set II: Question 7 (commutators)

7.7 A. Lukas' Problem Set III: Question 2(d) (changing variables in 2D integrals)

7.8 A. Lukas' Problem Set III: Question 3 (ellipsoids)

7.9 A. Lukas' Problem Set III: Question 4 (moments of inertia)

7.10 A. Lukas' Problem Set III: Question 5 (Gaussian probabilities)

7.11 A. Lukas' Problem Set IV: Question 1 (minimal distances)

7.12 A. Lukas' Problem Set IV: Question 2(d) (Gauss theorem)

7.13 A. Lukas' Problem Set IV: Question 3 (Gauss theorem on cone)

7.14 A. Lukas' Problem Set IV: Question 4(b) (Stokes theorem)

7.15 A. Lukas' Problem Set IV: Question 5 (areas and volumes)

Normal Modes

7.16 ODE Problem Set 6: Question 6.5

7.17 ODE Problem Set 6: Question 6.6

Travelling Waves

7.18 A semi-infinite string of density ρ per unit length is under tension T. At its free end is a mass m which slides on a smooth horizontal rod that lies perpendicular to the string. Determine the amplitude reflection coefficient for transverse waves incident on the mass. What is the phase difference between the incident and reflected waves?

7.19 An infinite string lies along the x-axis, and is under tension T. It consists of a section at $0 < x < a$, of linear density ρ_1, and two semi-infinite pieces of density ρ_2. A wave of amplitude A travels along the string at $x > a$, towards the short section.

How many types of waves are there in the various sections of the string? How many boundary conditions need to be satisfied?

Show that, if $a = n\lambda_1$, (where λ_1 is the wavelength on the short section, and n is an integer), the amplitude of the wave that emerges at $x < 0$ is A. What is the amplitude of the wave in the short section?

7.20 A uniform string of length l and density ρ has its end points fixed so that its equilibrium tension is T. A mass M is attached to its mid-point. Show that the angular frequency ω of small vibrations is given by

$$z \tan z = \frac{\rho l}{M}, \quad \text{where} \quad z \equiv \frac{\omega l}{2c} \quad \text{and} \quad c^2 \equiv T/\rho.$$

7.21 Two uniform wires of densities ρ_1 and ρ_2 and of equal lengths are fastened together and the two free ends are attached to two fixed points a distance $2l$ apart, so that the equilibrium tension is T. Show that the angular frequency ω of small vibrations satisfies

$$c_1 \tan(\omega l/c_1) = -c_2 \tan(\omega l/c_2),$$

where $c_{1,2}^2 \equiv T/\rho_{1,2}$.

7.22 An elastic string of length a consists of two portions, $0 < x < a/2$ of density ρ_1 and $a/2 < x < a$ of density ρ_2. It is stretched to tension T and the end at $x = a$ is fixed. The end at $x = 0$ is then shaken transversely at frequency ω. Show that throughout the motion, the ratio of the displacement at $x = a/2$ to that at $x = 0$ is given by

$$\frac{c_2 \csc(\omega a/2c_1)}{c_2 \cot(\omega a/2c_1) + c_1 \cot(\omega a/2c_2)},$$

where $c_{1,2}^2 \equiv T/\rho_{1,2}$.

Waves: Dispersion

7.23 In quantum mechanics, a particle of momentum p and energy E has associated with it a wave of wavelength λ and frequency ν given by

$$\lambda = h/p \quad \text{and} \quad \nu = E/h,$$

where h is Planck's constant. Find the phase and group velocities of these waves when the particle is

(a) non-relativistic, given that

$$p = m_0 v \quad \text{and} \quad E = \frac{1}{2} m_0 v^2;$$

(b) relativistic, in which case

$$p = \frac{m_0 v}{\sqrt{1 - v^2/c^2}} \quad \text{and} \quad E = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}},$$

where m_0 is the particle's rest mass.

Comment on your answers.

7.24 In a certain dispersive medium a disturbance ϕ obeys the equation

$$\tau \frac{\partial}{\partial t}\left(\frac{\partial^2 \phi}{\partial t^2} - c_1^2 \frac{\partial^2 \phi}{\partial x^2}\right) + \frac{\partial^2 \phi}{\partial t^2} - c_0^2 \frac{\partial^2 \phi}{\partial x^2} = 0.$$

Show that a disturbance with frequency $\omega \ll 1/\tau$ travels with phase velocity c_0, and that its amplitude decreases by a factor

$$\simeq \exp\left[-\pi\omega\tau\left(\frac{c_1^2}{c_0^2} - 1\right)\right]$$

in each wavelength. [Assume $c_1 > c_0$.]

Multivariate Calculus: Hydrodynamics

7.25 *Vorticity* of a fluid is defined $\omega = \nabla \times \mathbf{u}$ and tells you how the fluid circulates (locally), as we are about to see.

(a) Let us assume that the fluid is *barotropic*: $p = p(\rho)$, i.e., pressure depends only on density and has no other variation except via ρ, so $\nabla p = p'(\rho)\nabla\rho$ (this would be the case, for example, if $p = \text{const}\rho^\gamma$, which is clearly a solution of Eq. (7) of Problem 5.12).

Use Eq. (6) of Problem 5.12 and vector calculus to prove that

$$\frac{\partial \omega}{\partial t} = \nabla \times (\mathbf{u} \times \omega). \tag{9}$$

(b) *Circulation* is defined $\Gamma = \oint_C \mathbf{u} \cdot \mathrm{d}\mathbf{l}$, where C is a loop. Show that circulation over a loop is the flux of vorticity through a surface (so a vorticity "line" is a fluid swirl, or vortex).

(c) Let $C(t)$ be a "material" loop that moves with the fluid (i.e., each point on the loop moves at the local instantaneous velocity $\mathbf{u}(t, \mathbf{r})$). Prove *Kelvin's Circulation Theorem:*

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = 0, \quad \text{where} \quad \Gamma(t) = \oint_{C(t)} \mathbf{u} \cdot \mathrm{d}\mathbf{l}$$

(circulation through a loop moving with the fluid is conserved).

Strategy: Work out $\Gamma(t + \mathrm{d}t)$ and $\Gamma(t)$ to calculate the time derivative. Express these circulations as fluxes of vorticity through surfaces $S(t)$ and $S(t + \mathrm{d}t)$ for which $C(t) = \partial S(t)$ and $C(t + \mathrm{d}t) = \partial S(t + \mathrm{d}t)$. As the surface $S(t + \mathrm{d}t)$, it is covenient to choose the surface $S(t)+$ the ribbon traced by the loop $C(t)$ as it moved to become $C(t + \mathrm{d}t)$ (i.e., each of its points moved a distance $\mathbf{u}\mathrm{d}t$ in the direction of the local velocity — this should allow you to calculate the surface element $\mathrm{d}\mathbf{S}$ on the ribbon in terms of $\mathbf{u}$, $\mathrm{d}t$ and the line element $\mathrm{d}\mathbf{l}$ of the loop $C(t)$). Judicious application of Stokes' Theorem and Eq. (9) will lead to the desired result.

(d) Convince yourself that this result means that the field lines of ω "move with the fluid."