Oxford Physics

MERTON COLLEGE OXFORD

Merton Maths

First-Year Mathematics (CP3&4)
for Merton College Students

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(with thanks to lecturers and tutors, past and present)

Problems for Hilary Term 2019

HT I

Vector Calculus

1.1 Two circles have equations:

(i) $x^2 + y^2 + 2ax + 2by + c = 0$;

(ii) $x^2 + y^2 + 2a'x + 2b'y + c' = 0$.

Show that these circles are orthogonal if $2aa' + 2bb' = c + c'$.

1.2 Find the equation for the tangent plane to the surface $2xz^2 - 3xy - 4x = 7$ at $(1, -1, 2)$.

Multiple Integrals: 2D

1.3 (a) For the following integrals sketch the region of integration and so write equivalent integrals with the order of integration reversed. Evaluate the integrals both ways.

$$\int_0^{\sqrt{2}} \mathrm{d}y \int_{y^2}^{2} \mathrm{d}x\, y\,, \qquad \int_0^4 \mathrm{d}x \int_0^{\sqrt{x}} \mathrm{d}y\, y\sqrt{x}\,, \qquad \int_0^1 \mathrm{d}y \int_{-y}^{y^2} \mathrm{d}x\, x\,.$$

(b) Reverse the order of integration and hence evaluate:

$$\int_0^{\pi} \mathrm{d}y \int_y^{\pi} \mathrm{d}x\, \frac{\sin x}{x}\,.$$

1.4 Evaluate

$$\iint \exp[-(x^2 + y^2)]\mathrm{d}x\mathrm{d}y$$

over the area of a circle with centre $(0, 0)$ and radius a. By letting a tend to ∞, evaluate

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\mathrm{d}x.$$

1.5 (a) A mass distribution in the positive x region of the xy-plane and in the shape of a semi-circle of radius a, centred on the origin, has mass per unit area k. Find, using plane polar coordinates,

(i) its mass M,

(ii) the coordinates $(\overline{x}, \overline{y})$ of its centre of mass,

(iii) its moments of inertia about the x and y axes.

(b) Do as above for a semi-infinite sheet with mass per unit area

$$\sigma = k \exp\left(-\frac{x^2 + y^2}{a^2}\right) \quad \text{for} \quad x \geq 0\,, \qquad \sigma = 0 \quad \text{for} \quad x < 0,$$

where a is a constant. Comment on the comparisons between the two sets of answers.

Note that the result of Q1.4 will be useful in the evaluation of the required integrals.

1.6 Evaluate the following integral:

$$\int_0^a \mathrm{d}y \int_0^{\sqrt{a^2-y^2}} \mathrm{d}x\,(x^2+y^2)\,\arctan\frac{y}{x}\,.$$

1.7 The pair of variables (x, y) are each functions of the pair of variables (u, v) and *vice versa*. Consider the matrices

$$A = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}\,.$$

(a) Show using the chain rule that the product AB of these two matrices equals the unit matrix I.

(b) Verify this property explicitly for the case in which (x, y) are Cartesian coordinates and u and v are the polar coordinates (r, θ).

(c) Assuming the result that the determinant of a matrix and the determinant of its inverse are reciprocals, deduce the relation between the Jacobians

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \qquad \text{and} \qquad \frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\,.$$

1.8 (a) Using the change of variable $x + y = u$, $x - y = v$, evaluate the double integral $\iint_R (x^2 + y^2)\,\mathrm{d}x\mathrm{d}y$, where R is the region bounded by the straight lines $y = x$, $y = x+2$, $y = -x$ and $y = -x + 2$.

(b) Given that $u = xy$ and $v = y/x$, show that $\partial(u, v)/\partial(x, y) = 2y/x$. Hence evaluate the integral

$$\iint \exp(-xy)\,\mathrm{d}x\mathrm{d}y$$

over the region $x > 0$, $y > 0$, $xy < 1$, $1/2 < y/x < 2$.

HT II

Multiple Integrals: 3D

2.1 Spherical polar coordinates are defined in the usual way. Show that

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2 \sin\theta \ .$$

Draw a diagram to illustrate the physical significance of this result.

2.2 A solid hemisphere of uniform density k occupies the volume $x^2 + y^2 + z^2 \leq a^2$, $z \geq 0$. Using symmetry arguments wherever possible, find

(i) its total mass M,

(ii) the position $(\overline{x}, \overline{y}, \overline{z})$ of its centre of mass,

(iii) its moments and products of inertia, I_{xx}, I_{yy}, I_{zz}, I_{xy}, I_{yz}, I_{zx}, where

$$I_{zz} = \int k\,(x^2 + y^2)\,\mathrm{d}V, \qquad I_{xy} = \int k\,xy\,\mathrm{d}V, \qquad \text{etc.}$$

Probabilities

2.3 If X is a continuous random variable with probability density function (PDF) $f(x) = ce^{-x}$ for $x \geq 0$ and zero otherwise,

(a) find c;

(b) find the cumulative distribution function $F(x)$;

(c) find $P(1 < X < 3)$.

2.4 Let X and Y be two jointly continuous random variables with a joint PDF $f(x,y) = cx^2y$ for $0 \leq y \leq x \leq 1$ and zero otherwise.

(a) Sketch the region in the (x,y) plane for which the PDF is non-zero.

(b) Find c.

(c) Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.

(d) Find $P(Y \leq X/2)$.

Vector Calculus

2.5 Find $\nabla\phi$ in the cases:

(a) $\phi = \ln r$;

(b) $\phi = r^{-1}$,

where $r = |\mathbf{r}|$.

2.6 Given that $F = x^2z + e^{y/x}$ and $G = 2z^2y - xy^2$, find $\nabla(F+G)$ and $\nabla(FG)$ at $(1, 0, -2)$.

Line Integrals

2.7 A vector field $\mathbf{A}(\mathbf{r})$ is defined by its components $(4x - y^4, -4xy^3 - 3y^2, 4)$. Evaluate the line integral $\int \mathbf{A} \cdot d\mathbf{l}$ between the points with position vectors $(0, 0, 0)$ and $(1, 2, 0)$ along the following paths

(a) the straight line from $(0, 0, 0)$ to $(1, 2, 0)$;

(b) on the path of straight lines joining $(0, 0, 0)$, $(0, 0, 1)$, $(1, 0, 1)$, $(1, 2, 1)$ and $(1, 2, 0)$ in turn.

Show that $\mathbf{A}$ is conservative by finding a scalar function $\phi(\mathbf{r})$ such that $\mathbf{A} = \nabla\phi$.

2.8 A vector field $\mathbf{A}(\mathbf{r})$ is defined by its components $(3x^2 + 6y, -14yz, 20xz^2)$. Evaluate the line integral $\int \mathbf{A} \cdot d\mathbf{l}$ from $(0, 0, 0)$ to $(1, 1, 1)$ along the following paths

(a) $x = t$, $y = t^2$, $z = t^3$;

(b) on the path of straight lines joining $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, and $(1, 1, 1)$ in turn;

(c) the straight line joining the two points.

Is $\mathbf{A}$ conservative?

2.9 If $\phi = 2xyz^2$, $\mathbf{F} = (xy, -z, x^2)$ and C is the curve $x = t^2$, $y = 2t$, $z = t^3$ from $t = 0$ to $t = 1$, evaluate the line integrals:

(a) $\int_C \phi \, d\mathbf{r}$;

(b) $\int_C \mathbf{F} \times d\mathbf{r}$.

Surface Integrals

2.10 Show that under certain circumstances the area of a curved surface can be expressed in the form

$$\iint \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dx dy,$$

where the integral is taken over the plane $z = 0$.

Sketch the surface $z^2 = 2xy$ and find the area of the part of it that lies inside the hemisphere $x^2 + y^2 + z^2 = 1$, $z > 0$.

2.11 Show that the surface area of the curved portion of the hemisphere in Problem 2.2 is $2\pi a^2$ by

(i) directly integrating the element of area $a^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi$ over the surface of the hemisphere.

(ii) projecting onto an integral taken over the xy plane.

2.12 (a) Find the area of the plane $x - 2y + 5z = 13$ cut out by the cylinder $x^2 + y^2 = 9$.

(b) A uniform lamina is made of that part of the plane $x + y + z = 1$ which lies in the first octant. Find by integration its area and also its centre of mass. Use geometrical arguments to check your result for the area.

2.13 Calculate the solid angle of a cone of half-angle α.

2.14 Calculate the area of a unit sphere in n dimensions. Check that your answer is 2π for $n = 2$ and 4π for $n = 3$.

Hint. Calculate the Gaussian integral $\int \cdots \int \mathrm{d}x_1 \ldots \mathrm{d}x_n \exp[-(x_1^2 + \cdots + x_n^2)]$ in Cartesian and in polar coordinates.

Normal Modes

Do Questions 1.1 to 1.5 from Christopher Palmer's Problem Set on Normal Modes

HT III

Surface Integrals: Fluxes

3.1 If $\mathbf{n}$ is the unit normal to the surface S, evaluate $\int\int \mathbf{r}\cdot\mathbf{n}\,\mathrm{d}S$ over

(a) the unit cube bounded by the coordinate planes and the planes $x = 1$, $y = 1$ and $z = 1$;

(b) the surface of a sphere of radius a centred at the origin.

3.2 Evaluate $\int \mathbf{A}\cdot\mathbf{n}\ \mathrm{d}S$ for the following cases:

(a) $\mathbf{A} = (y, 2x, -z)$ and S is the surface of the plane $2x + y = 6$ in the first octant cut off by the plane $z = 4$.

(b) $\mathbf{A} = (x + y^2, -2x, 2yz)$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant

(c) $\mathbf{A} = (6z, 2x+y, -x)$ and S is the entire surface of the region bounded by the cylinder $x^2 + z^2 = 9$, $x = 0$, $y = 0$, $z = 0$ and $y = 8$.

3.3 Air is flowing with a speed 0.4 m s^{-1} in the direction of the vector $(-1, -1, 1)$. Calculate the volume of air flowing per second through the loop which consists of straight lines joining, in turn, the following: $(1, 1, 0)$, $(1, 0, 0)$, $(0, 0, 0)$, $(0, 1, 1)$, $(1, 1, 1)$ and $(1, 1, 0)$.

Vector Calculus

3.4 O is the origin and A, B, C are points with position vectors $\mathbf{a} = (1, 0, 0)$, $\mathbf{b} = (1, 1, 1)$ and $\mathbf{c} = (0, 2, 0)$, respectively. Find the vector area $\mathbf{S}$ of the loop OABCO

(a) by drawing the loop in projection onto the (y, z), (z, x) and (x, y) planes and calculating the components of $\mathbf{S}$;

(b) by filling the loop with (e.g., 2 or 3) plane polygons, ascribing a vector area to each and taking the resultant.

Calculate the projected area of the loop

(i) when seen from the direction which makes it appear as large as possible;

(ii) when seen from the direction of the vector $(0, -1, 1)$.

What are the corresponding answers for the loop OACBO?

3.5 For $\mathbf{A} = (3xyz^2, 2xy^3, -x^2yz)$ and $\phi = 3x^2 - yz$, find: (a) $\nabla\cdot\mathbf{A}$; (b) $\mathbf{A}\cdot\nabla\phi$; (c) $\nabla\cdot(\phi\mathbf{A})$; (d) $\nabla\cdot(\nabla\phi)$.

3.6 A body expands linearly by a factor $1+\alpha$ because of a rise in temperature. The expansion shifts the particle originally at $\mathbf{r}$ to $\mathbf{r} + \mathbf{h}$. Calculate $\nabla\cdot\mathbf{h}$. By what fraction does the volume increase?

3.7 The magnetic field $\mathbf{B}$ at a distance r from a straight wire carrying a current I has magnitude $\mu_0 I/2\pi r$. The lines of force are circles centred on the wire and in planes perpendicular to it. Show that $\nabla \cdot \mathbf{B} = 0$.

3.8 Sketch the vector fields $\mathbf{A} = (x, y, 0)$ and $\mathbf{B} = (y, -x, 0)$. Calculate the divergence and curl of each vector field and explain the physical significance of the results obtained.

3.9 A bucket of water is rotated slowly with angular velocity ω about its vertical axis. When a steady state has been reached the water rotates with a velocity field $\mathbf{v}(\mathbf{r})$ as if it were a rigid body. Calculate $\nabla \cdot \mathbf{v}$ and interpret the result. Calculate $\nabla \times \mathbf{v}$. Can the flow be represented in terms of a velocity potential ϕ such that $\mathbf{v} = \nabla\phi$? If so, what is ϕ?

3.10 To what scalar or vector quantities do the following expressions in suffix notation correspond (sum where possible): $a_i b_j c_i$; $a_i b_j c_j d_i$; $\delta_{ij} a_i a_j$; $\delta_{ij}\delta_{ij}$; $\epsilon_{ijk} a_i b_k$; and $\epsilon_{ijk}\delta_{ij}$.

3.11 Prove, by the most compact calculation you can devise:

(a) $\nabla \times \nabla\phi = 0$;

(b) $\nabla \cdot (\nabla \times \mathbf{A}) = 0$.

3.12 Evaluate

(a) $\nabla \times (\alpha\mathbf{a})$;

(b) $\nabla \times (\nabla \times \mathbf{a})$;

(c) $\nabla(\phi\psi)$;

(d) $\nabla \cdot (\mathbf{a} \times \mathbf{b})$;

(e) $\nabla \times (\mathbf{a} \times \mathbf{b})$;

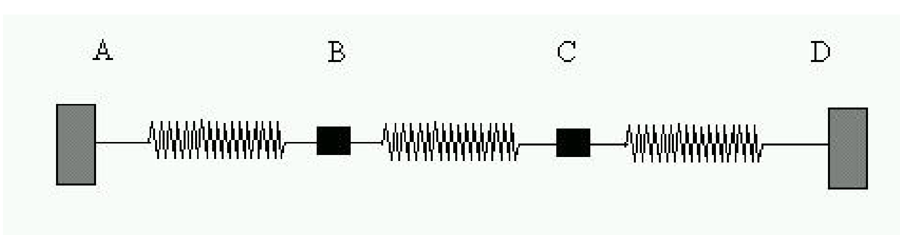
(f) $\nabla \cdot (\nabla\phi \times \nabla\psi)$.

Here α is a constant, ϕ and ψ scalar functions, $\mathbf{a}$ and $\mathbf{b}$ vector functions of position.

3.13 Find a vector field $\mathbf{A}$ such that $\nabla \times \mathbf{A} = (0, 0, B)$, where B is a constant. Is this field unique?

Normal Modes

3.14 AB, BC, and CD are identical springs with negligible mass, and stiffness constant k:



The masses m, fixed to the springs at B and C, are displaced by small distances x_1 and x_2 from their equilibrium positions along the line of the springs, and execute small oscillations. Show that the angular frequencies of the normal modes are $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{3k/m}$. Sketch how the two masses move in each mode. Find x_1 and x_2 at times $t > 0$ given that at $t = 0$ the system is at rest with $x_1 = a$, $x_2 = 0$.

3.15 The setup is as in the previous Problem, except that in this case the springs AB and CD have stiffness constant k_0, while BC has stiffness constant k_1. If C is clamped, B vibrates with frequency $\nu_0 = 1.81\,\mathrm{Hz}$. The frequency of the lower-frequency normal mode is $\nu_1 = 1.14\,\mathrm{Hz}$. Calculate the frequency of the higher-frequency normal mode, and the ratio k_1/k_0.

3.16 *Question 1.6 from Christopher Palmer's Problem Set on Normal Modes*

3.17 *Question 1.7 from Christopher Palmer's Problem Set on Normal Modes*

HT IV

Vector Calculus

4.1 Use index notation to find the gradient of the following scalar functions of position $\mathbf{r} = (x, y, z)$:

(a) r^n, where $r = |\mathbf{r}|$;

(b) $\mathbf{a} \cdot \mathbf{r}$, where $\mathbf{a}$ is a constant vector.

4.2 Use index notation to find the divergence and curl of the following vector functions of position $\mathbf{r} = (x, y, z)$:

(a) $\mathbf{r}$;

(b) $r^n \mathbf{r}$, where $r = |\mathbf{r}|$;

(c) $(\mathbf{a} \cdot \mathbf{r})\mathbf{b}$;

(d) $\mathbf{a} \times \mathbf{r}$;

(e) $\mathbf{r} \times (\mathbf{a} \times \mathbf{r})$.

Here $\mathbf{a}$ and $\mathbf{b}$ are constant vectors (independent of $\mathbf{r}$).

4.3 For $\mathbf{a}$ a constant vector, prove that $\nabla[\mathbf{a} \cdot \nabla(r^{-1})] = -\nabla \times [\mathbf{a} \times \nabla(r^{-1})]$, where $r = |\mathbf{r}|$.

Gauss and Stokes Theorems

4.4 The vector $\mathbf{A}$ is a function of position $\mathbf{r} = (x, y, z)$ and has components (xy^2, x^2, yz). Calculate the surface integral $\int \mathbf{A} \cdot \mathrm{d}\mathbf{S}$ over each face of the triangular prism bounded by the planes $x = 0$, $y = 0$, $z = 0$, $x+y = 1$ and $z = 1$. Show that the integral $\int \mathbf{A} \cdot \mathrm{d}\mathbf{S}$ taken outwards over the whole surface is not zero. Show that it equals $\int \nabla \cdot \mathbf{A}\, \mathrm{d}V$ calculated over the volume of the prism. Why?

4.5 Solve Problems 3.1(a,b) and 3.2(c) using Gauss's Theorem.

4.6 The vector $\mathbf{A}(\mathbf{r}) = (y, -x, z)$. Verify Stokes' Theorem for the hemispherical surface $|\mathbf{r}| = 1$, $z \geq 0$.

4.7 Let $\mathbf{A} = (y, -x, 0)$. Find $\int \mathbf{A} \cdot \mathrm{d}\mathbf{l}$ for a closed loop on the surface of the cylinder $(x - 3)^2 + y^2 = 2$. Consider the cases

(i) in which the loop wraps round the cylinder's axis;

(ii) in which the loop does not (so it can be continuously deformed to a point).

Normal Modes

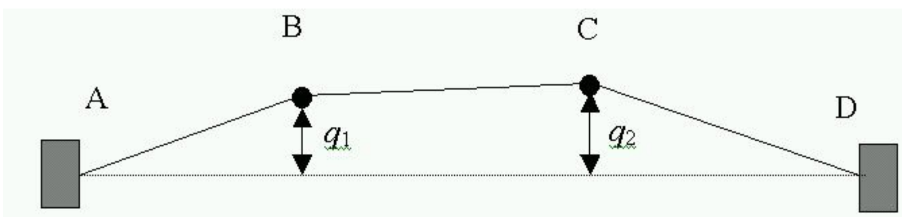
4.8 A stretched massless spring has its ends at $x = 0$ and $x = 3l$ fixed, and has equal masses attached at $x = l$ and $x = 2l$. The masses slide on a smooth horizontal table. Convince yourself that, for small oscillations, it is reasonable to neglect the changes in tension caused by the variation in length of the three sections of the spring resulting from the transverse motion of the masses. Show that the equations of the transverse motion of the masses are approximately

$$m\ddot{y}_1 = \frac{T}{l}(y_2 - 2y_1) \quad \text{and} \quad m\ddot{y}_2 = \frac{T}{l}(y_1 - 2y_2),$$

where T is the tension in the spring.

Find the frequencies and the ratio of amplitudes of the transverse oscillations for the normal modes of the two masses.

4.9 The figure shows two masses m at points B and C of a string fixed at A and D, executing small transverse oscillations. The tensions are assumed to be all equal, and in equilibrium $AB = BC = CD = l$.



If the (small) transverse displacements of the masses are denoted by q_1 and q_2, the equations of motion are

$$m\ddot{q}_1 = -k(2q_1 - q_2) \quad \text{and} \quad m\ddot{q}_2 = -k(2q_2 - q_1),$$

where $k = T/l$, and terms of order q_1^2, q_2^2 and higher have been neglected.

(a) Show that the normal (decoupled) coordinates for this system are

$$Q_1 = \frac{q_1 + q_2}{\sqrt{2}} \quad \text{and} \quad Q_2 = \frac{q_1 - q_2}{\sqrt{2}}.$$

Find the equations they satisfy, then find the general solution of these equations and hence find the general solution for q_1 and q_2.

(b) The forces on the RHS of the equations of motion may be interpreted in terms of a potential energy function $V(q_1, q_2)$, viz., the equations of motion may be rewritten as

$$m\ddot{q}_1 = -\frac{\partial V}{\partial q_1} \quad \text{and} \quad m\ddot{q}_2 = -\frac{\partial V}{\partial q_2}$$

(generalising "$m\ddot{x} = -\partial V/\partial x$"). Find an appropriate expression for $V(q_1, q_2)$ — so that correct equations of motion are recovered.

Derive the same result for V by considering the work done in giving each section of the string its deformation [e.g., for AB the work done is equal to $T(\sqrt{l^2+q_1^2}-l)$], and expanding in powers of q_1^2/l^2.

Show that, when written in terms of the variables Q_1 and Q_2, V becomes

$$V = \frac{1}{2}m\omega_1^2 Q_1^2 + \frac{1}{2}m\omega_2^2 Q_2^2,$$

where $\omega_1 = \sqrt{k/m}$ and $\omega_2 = \sqrt{3k/m}$.

(c) Calculate the kinetic energy K of the masses and hence show that their total energy, written in terms of Q_1, Q_2, $\dot{Q}_1$ and $\dot{Q}_2$, can be represented as

$$H = K + V = H_1 + H_2,$$

where H_1 is the total energy of 'oscillator' Q_1 with frequency ω_1, and similarly for H_2.

What is the expression for the total energy when written in terms of q_1, q_2, $\dot{q}_1$, and $\dot{q}_2$? Discuss the similarities and differences.

(d) Show that the equations of motion for Q_1 and Q_2 have the form

$$m\ddot{Q}_1 = -\frac{\partial H}{\partial Q_1} \quad \text{and} \quad m\ddot{Q}_1 = -\frac{\partial H}{\partial Q_2}.$$

Waves

4.10 (a) What is the difference between a travelling wave and a standing wave?

(b) Convince yourself that

$$y_1 = A\sin(kx - \omega t)$$

corresponds to a travelling wave. Which way does it move? What are the amplitude, wavelength, frequency, period and velocity of the wave?

(c) Show that y_1 satisfies the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

provided that ω and k are suitably related (this is called the "dispersion relation").

(d) Write down a wave y_2 of equal amplitude travelling in the opposite direction. Show that $y_1 + y_2$ can be written in the form

$$y_1 + y_2 = f(x)g(t),$$

where $f(x)$ is a function of x only, and $g(t)$ is a function just of t. Convince yourself that the combination of two travelling waves is a standing wave. By determining $f(x)$ and $g(t)$ explicitly, find the wavelength and frequency of $y_1 + y_2$. Comment on the velocity of the waves.

4.11 (a) A string of uniform linear density ρ is stretched to a tension T, its ends being fixed at $x = 0$ and $x = L$. If $y(x,t)$ is the transverse displacement of the string at position x and time t, show that

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

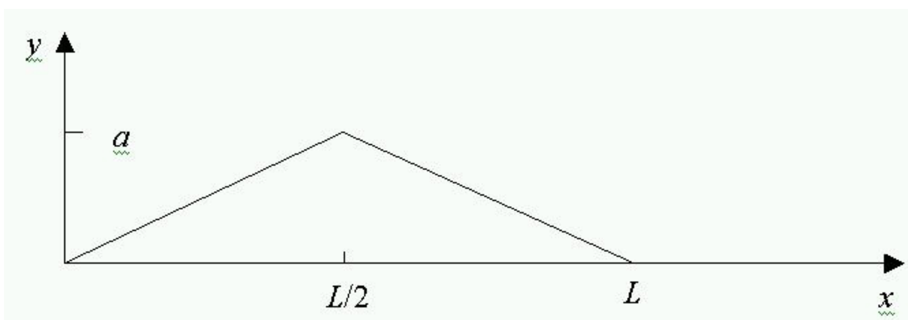
where $c^2 = T/\rho$. What is meant by the statement that this equation is 'linear'?

Verify that

$$y(x,t) = A_r \sin\left(\frac{r\pi x}{L}\right)\sin\left(\frac{r\pi ct}{L}\right) \quad \text{and} \quad y(x,t) = B_r \sin\left(\frac{r\pi x}{L}\right)\cos\left(\frac{r\pi ct}{L}\right),$$

where r is any integer, are both solutions of this equation, obeying the boundary conditions $y(0,t) = y(L,t) = 0$. Explain why sums of such solutions are also solutions.

(b) The string is such that at $t = 0$, $\partial y/\partial t = 0$ for all x, and $y(x,0)$ has the shape



i.e., the mid-point is drawn aside a small distance a. Explain why the solution after the mid-point is released has the form

$$y(x,t) = \sum_{r=1}^{\infty} B_r \sin\left(\frac{r\pi x}{L}\right)\cos\left(\frac{r\pi ct}{L}\right)$$

and find B_r [N.B.: There are infinitely many constants $B_1, B_2, \ldots$ in this expression. They can be determined from the initial displacement of the string by the technique of Fourier Analysis:

$$y(x,0) = \sum_{r=1}^{\infty} B_r \sin\left(\frac{r\pi x}{L}\right) \quad \Rightarrow \quad B_r = \frac{2}{L}\int_0^L \mathrm{d}x\, y(x,0)\sin\left(\frac{r\pi x}{L}\right).$$

Fourier Analysis is now on the 2nd-year course.]

4.12 (a) Standing waves $y = f(x)g(t)$ exist on a string of length L, as in the previous problems. Given that the x dependence is

$$f(x) = A\sin(kx),$$

what is $g(t)$? [This involves 2 arbitrary constants.]

(b) At $t = 0$, the displacement is

$$y(x,0) = \sin\left(\frac{\pi x}{L}\right) + 2\sin\left(\frac{2\pi x}{L}\right)$$

and the string is instantaneously stationary. Find the displacement at subsequent times. Make rough sketches of $y(x,t)$ at the following times: 0, $L/4c$, $L/2c$, $3L/4c$, L/c.

4.13 Outline the solution of the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

using the method of separation of variables.

Assuming that transverse waves exist on a string fixed between the points $x = 0$ and $x = l$ and that at time $t = 0$, the string is in its equilibrium position, but has a velocity $[\partial y/\partial t](x, t = 0) = v(x)$, find, by the method of separation of variables, the general solution for the subsequent motion of the string.

HT V

Vector Calculus

5.1 Prove that

$$\text{(i)} \qquad \int_{\partial V} p\, \mathrm{d}\mathbf{S} \;=\; \int_V \nabla p\, \mathrm{d}^3\mathbf{r}$$

$$\text{(ii)} \qquad \int_{\partial V} \mathrm{d}\mathbf{S} \times \mathbf{a} \;=\; \int_V \nabla \times \mathbf{a}\, \mathrm{d}^3\mathbf{r},$$

where ∂V is the closed surface bounding the volume V.

5.2 If $\mathbf{G} = (\mathbf{a} \cdot \mathbf{r})\mathbf{a}$, where $\mathbf{a}$ is a constant vector, and S is a closed surface, show that

$$\int_S \mathbf{G} \times \mathrm{d}\mathbf{S} = 0,$$

where S is any closed surface.

Verify this result for the special case in which $\mathbf{a} = (0, 0, 1)$ and S is the bounding surface of the hemisphere $r \leq 1$, $z \geq 0$.

5.3 Prove that

$$\oint_{\partial S} \phi \mathrm{d}\mathbf{l} = \int_S \mathrm{d}\mathbf{S} \times \nabla\phi,$$

where ∂S is the closed curve bounding the surface S.

Verify this relation for the function $\phi = x^3$, with ∂S the circle $x^2 + y^2 = a^2$, $z = 0$, and S the part of the plane $z = 0$ enclosed by ∂S.

5.4 $\mathbf{A}(\mathbf{r})$ is defined by

$$\mathbf{A}(\mathbf{r}) = \int_V \nabla_{\mathbf{r}'} f(|\mathbf{r} - \mathbf{r}'|)\mathrm{d}^3\mathbf{r}',$$

where $\nabla_{\mathbf{r}'}$ denotes the gradient with respect to $\mathbf{r}'$, $\mathbf{r}$ remaining fixed, f is a well-behaved function of a single variable and the volume V is fixed. Show that $\nabla \times \mathbf{A} = 0$. [Hint: Find a function of which $\mathbf{A}$ is the gradient.]

Hence find $\mathbf{A}$ when $f(x) = x^4$ and V is the sphere $r = 1$.

5.5 The region V is bounded by a simple, closed surface ∂V. Prove that

$$\int_{\partial V} \psi \nabla\phi \cdot \mathrm{d}\mathbf{S} = \int_V \psi \nabla^2\phi \mathrm{d}^3\mathbf{r} + \int_V (\nabla\psi \cdot \nabla\phi)\mathrm{d}^3\mathbf{r}.$$

Let $\nabla^2\phi = 0$ in V and $\phi(\mathbf{r}) = g(\mathbf{r})$ on ∂V. A function $f(\mathbf{r})$ is chosen so that it also satisfies $f(\mathbf{r}) = g(\mathbf{r})$ on ∂V. By writing $\psi = f - \phi$ show that

$$\int_V |\nabla\phi|^2 \mathrm{d}^3\mathbf{r} \leq \int_V |\nabla f|^2 \mathrm{d}^3\mathbf{r}.$$

If ∂V is the sphere $x^2 + y^2 + z^2 = a^2$ and in polar coordinates (r, θ, ϕ), $g = a\cos\theta$, show that

$$\int_V |\nabla\phi|^2 \mathrm{d}^3\mathbf{r} \leq V = \frac{4}{3}\pi a^3.$$

5.6 If S is a surface bounded by the closed curve ∂S, prove that

$$\oint_{\partial S} \mathbf{r}(\mathbf{r}\cdot \mathrm{d}\mathbf{l}) = \int_S \mathbf{r} \times \mathrm{d}\mathbf{S}.$$

Verify this formula when ∂S is

(i) the intersection of the cylinder $x^2 - x + y^2 = 2$ with the plane $z = 0$,

(ii) the intersection of the same cylinder with the sphere $x^2 + y^2 + z^2 = 9$.

What is the value of the right-hand integral for the portion of the cylinder cut off by the plane and the sphere?

Waves: Dispersion

5.7 What is meant by (a) a dispersive medium, and (b) the phase velocity v? Explain the relevance of group velocity g for the transmission of signals in a dispersive medium. Justify the equation

$$g = \frac{\mathrm{d}\omega}{\mathrm{d}k}.$$

Show that for electromagnetic waves alternative expressions for g are

$$g = v + k\frac{\mathrm{d}v}{\mathrm{d}k} \quad \text{or} \quad g = v - \lambda\frac{\mathrm{d}v}{\mathrm{d}\lambda} \quad \text{or} \quad g = \frac{c}{\mu}\left(1 + \frac{\mathrm{d}\ln\mu}{\mathrm{d}\ln\lambda}\right),$$

where μ is the refractive index for waves of wavelength λ and wavenumber k (in the medium).

Show that

$$g = v\left(1 - \frac{1}{1 + \mathrm{d}\ln\lambda'/\mathrm{d}\ln v}\right),$$

where λ' is the wavelength in vacuum.

5.8 In quantum mechanics, a particle of momentum p and energy E has associated with it a wave of wavelength λ and frequency ν given by

$$\lambda = h/p \quad \text{and} \quad \nu = E/h,$$

where h is Planck's constant. Find the phase and group velocities of these waves when the particle is

(a) non-relativistic, given that

$$p = m_0 v \quad \text{and} \quad E = \frac{1}{2}m_0 v^2;$$

(b) relativistic, in which case

$$p = \frac{m_0 v}{\sqrt{1 - v^2/c^2}} \quad \text{and} \quad E = \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}},$$

where m_0 is the particle's rest mass.

Comment on your answers.

5.9 In a certain dispersive medium a disturbance ϕ obeys the equation

$$\tau \frac{\partial}{\partial t}\left(\frac{\partial^2 \phi}{\partial t^2} - c_1^2 \frac{\partial^2 \phi}{\partial x^2}\right) + \frac{\partial^2 \phi}{\partial t^2} - c_0^2 \frac{\partial^2 \phi}{\partial x^2} = 0.$$

Show that a disturbance with frequency $\omega \ll 1/\tau$ travels with phase velocity c_0, and that its amplitude decreases by a factor

$$\simeq \exp\left[-\pi\omega\tau\left(\frac{c_1^2}{c_0^2} - 1\right)\right]$$

in each wavelength. [Assume $c_1 > c_0$.]

Vector Calculus and Waves: Electromagnetism

5.10 Microscopic *Maxwell's equations* in integral form are, if written in Gauss units (I leave it to you to figure out the conversion to the abominable SI system):

$$\int_{\partial V} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = 4\pi \int_V \mathrm{d}^3\mathbf{r}\, \rho \quad \text{(Gauss's law)}, \tag{1}$$

$$\int_{\partial V} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0, \tag{2}$$

$$\oint_{\partial S} \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{1}{c}\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} \quad \text{(Faraday's law)}, \tag{3}$$

$$\oint_{\partial S} \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \frac{1}{c} \int_S \left(4\pi \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t}\right) \cdot \mathrm{d}\mathbf{S} \quad \text{(Ampère-Maxwell law)}, \tag{4}$$

where $\mathbf{E}$ is electric field, $\mathbf{B}$ magnetic field, ρ charge density, $\mathbf{j}$ current density and c the speed of light. The integrals are over an arbitrary volume V (whose bounding surface is ∂V) or arbitrary surface S (whose bounding loop is ∂S).

(a) Use Gauss's and Stokes' theorems and the fact that V and S are arbitrary and can be made infinitesimal to derive Maxwell's equations in differential form (do make sure you get the right result: look it up!).

(b) Using Gauss's and Ampère-Maxwell laws in their differential form, show that

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0.$$

Integrate this over some volume V, use Gauss's theorem and use the result to argue that this equation expresses *conservation of charge.*

(c) The energy density of the electromagnetic field is

$$\varepsilon = \frac{E^2 + B^2}{8\pi}.$$

Use Maxwell's equations to show that

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot \mathbf{P} = -\mathbf{E} \cdot \mathbf{j},$$

where $\mathbf{P} = c\mathbf{E} \times \mathbf{B}/4\pi$ is called the *Poynting vector*. Again integrate over some volume V and argue that the above equation expresses *conservation of energy*. Give physical interpretation of $\mathbf{P}$ and explain what the right-hand side of the above equation represents.

5.11 Consider electromagnetic field in vacuo: $\rho = 0$, $\mathbf{j} = 0$.

(a) Use Maxwell's equations to show that $\mathbf{E}$ and $\mathbf{B}$ satisfy the wave equation:

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = c^2 \nabla^2 \mathbf{E}$$

and similarly for $\mathbf{B}$. These are *electromagnetic waves* (light).

(b) Let $\mathbf{E}$ and $\mathbf{B}$ both be $\propto \exp(-i\omega t + i\mathbf{k} \cdot \mathbf{r})$.

Show that in order for this to be a solution, we must have $\omega = \pm ck$.

Using Maxwell's equations, show that $\mathbf{E}$ and $\mathbf{B}$ are perpendicular to each other and to the direction of the propagation of the wave.

(c) Show that the Poynting vector for light waves is

$$\mathbf{P} = \pm c \, \frac{\mathbf{k}}{k} \, \varepsilon$$

and explain what this result means physically.

Vector Calculus and Waves: Hydrodynamics

5.12 Consider a fluid or gaseous medium with density $\rho(t, \mathbf{r})$ and velocity $\mathbf{u}(t, \mathbf{r})$.

(a) Taking an arbitrary volume V within the fluid, we can express the *conservation of mass* as follows

$$\frac{\partial}{\partial t} \int_V \mathrm{d}^3\mathbf{r} \, \rho = -\int_{\partial V} \mathbf{F}_{\mathrm{mass}} \cdot \mathrm{d}\mathbf{S},$$

where the flux of mass is $\mathbf{F}_{\mathrm{mass}} = \rho \mathbf{u}$ (density ρ flowing with velocity $\mathbf{u}$). Use Gauss's Theorem and the fact that V is entirely arbitrary and can be taken to be as small as we like to derive the *continuity equation*:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \qquad (5)$$

Note that this can be written as

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \rho = -\rho \nabla \cdot \mathbf{u}.$$

The left-hand side is the so-called *convective time derivative* of ρ — the rate of change of density in a fluid element moving with velocity $\mathbf{u}$. The above equation then means that negative divergence of the fluid flow $\nabla \cdot \mathbf{u} < 0$ implies local compression and positive divergence $\nabla \cdot \mathbf{u} > 0$ local rarefaction.

(b) Now apply similar logic to the *conservation of momentum*. The momentum density is $\rho\mathbf{u}$. The rate of change of momentum in a volume V is

$$\frac{\partial}{\partial t}\int_V \mathrm{d}^3\mathbf{r}\,\rho\mathbf{u} = -\int_{\partial V} \rho\mathbf{u}\mathbf{u}\cdot \mathrm{d}\mathbf{S} - \int_{\partial V} p\mathrm{d}\mathbf{S}.$$

The first term on the right-hand side is the flux of momentum (which is a tensor quantity: vector flux of each component ρu_i is $\rho u_i \mathbf{u}$). The second term is the force on the boundary of the volume V due to pressure p (pressure times area; minus because $\mathrm{d}\mathbf{S}$ points outward). Use the above equation and Eq. (5) to derive *Euler's Equation* for the velocity of the fluid:

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p. \qquad (6)$$

(c) Finally we deal with the *conservation of energy*. The energy density of the fluid is

$$\varepsilon = \frac{\rho u^2}{2} + \frac{p}{\gamma - 1},$$

where the first term is kinetic-energy density and the second term is internal-energy density (γ is a constant and depends on the nature of the medium; e.g., for monatomic ideal gases, it is 5/3). Express the rate of change of energy in a volume V as

$$\frac{\partial}{\partial t}\int_V \mathrm{d}^3\mathbf{r}\,\varepsilon = \text{flux of energy through } \partial V + \text{work done by pressure on } \partial V \text{ per unit time.}$$

(the first term is analogous to the other flux terms you have encountered above; the second term you can work out by considering that if the fluid is flowing through the boundary of V at velocity $\mathbf{u}$, it must be pushing against pressure p and so doing work against force $p\mathrm{d}\mathbf{S}$). From the resulting equation, prove, using also Eqs. (5) and (6), that p satisfies

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\frac{p}{\rho^\gamma} = 0. \qquad (7)$$

Congratulations, you have derived the *equations of compressible hydrodynamics* — Eqs. (5–7) for ρ, $\mathbf{u}$ and p, a closed system.

5.13 The equations derived in the previous problem have a simple solution: $\rho = \rho_0$, $p = p_0$ and $\mathbf{u} = 0$, where p_0 and ρ_0 are constants independent of t or $\mathbf{r}$. Now consider small disturbances of this static homogeneous state:

$$\rho = \rho_0 + \delta\rho, \quad p = p_0 + \delta p, \quad \mathbf{u} = \delta\mathbf{u},$$

where $\delta\rho$, δp and $\delta\mathbf{u}$ are all infinitesimally small. Since they are infinitesimally small, we can substitute the above expressions for ρ, p, $\mathbf{u}$ into Eqs. (5-7) and neglect all terms where these small perturbations enter quadratically.

(a) Show therefore that $\delta\rho$, δp and $\delta\mathbf{u}$ satisfy

$$\frac{\partial\delta\rho}{\partial t} + \rho_0\nabla\cdot\delta\mathbf{u} = 0, \quad \rho_0\frac{\partial\delta\mathbf{u}}{\partial t} = -\nabla\delta p, \quad \frac{\partial}{\partial t}\left(\delta p - \gamma\,\frac{p_0}{\rho_0}\,\delta\rho\right) = 0. \tag{8}$$

(b) Hence show that the perturbations satisfy the wave equation

$$\frac{\partial^2\delta\rho}{\partial t^2} = c_s^2\nabla^2\delta\rho$$

and determine c_s. These are *sound waves.*

(c) Let $\delta\rho$, δp and $\delta\mathbf{u}$ all be $\propto \exp(-i\omega t + i\mathbf{k}\cdot\mathbf{r})$.

What is the relationship between ω and $\mathbf{k}$?

Using Eqs. (8), show that

$$\delta\mathbf{u} = \pm c_s\frac{\mathbf{k}}{k}\frac{\delta\rho}{\rho_0} \quad \text{and} \quad \frac{\delta p}{p_0} = \gamma\,\frac{\delta\rho}{\rho_0}.$$

If you have time and want more hydro action, solve Problem 7.10 now.

HT VI

Travelling Waves

6.1 By transforming the wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

to the variables $u = x - ct$ and $v = x + ct$, show that its general solution is the d'Alembert solution

$$y = f(x - ct) + g(x + ct),$$

where f and g are arbitrary functions.

6.2 At time $t = 0$, the displacement of an infinitely long string is:

$$y(x,t) = \begin{cases} \sin(\pi x/a) & \text{for } -a \le x \le a, \\ 0 & \text{otherwise.} \end{cases}$$

The string is initially at rest. Using d'Alembert's solution with phase speed c, sketch the displacement of the string at $t = 0$, $t = a/2c$, and $t = a/c$.

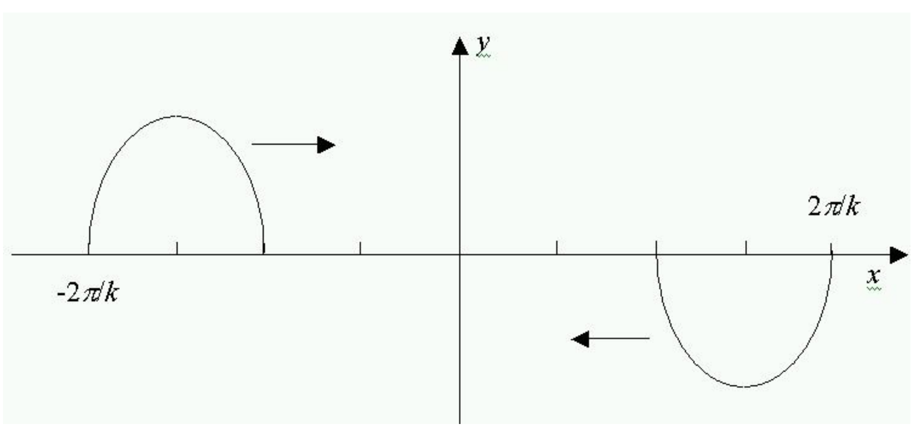
6.3 Calculate the rate of working (power input) of a device which launches small amplitude waves $y(x,t) = A\cos(kx - \omega t)$ into the end of a semi-infinite string by forcing $y(0,t) = A\cos(\omega t)$.

6.4 Show that the kinetic energy U and the potential energy V for a length $\lambda = 2\pi/k$ of a transverse wave on a string of linear density ρ and at tension T are given by

$$U = \int_0^\lambda \mathrm{d}x\, \frac{1}{2}\rho\left(\frac{\partial y}{\partial t}\right)^2 \quad \text{and} \quad V = \int_0^\lambda \mathrm{d}x\, \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2.$$

Evaluate these for the wave $y = A\cos(kx + \omega t + \phi)$ and show that $U = V$.

6.5 Two transverse waves are on the same piece of string. The first has displacement y non-zero only for $kx + \omega t$ between π and 2π, when it is equal to $A\sin(kx + \omega t)$. The second has $y = A\sin(kx - \omega t)$ for $kx - \omega t$ between -2π and $-\pi$, and is zero otherwise. When $t = 0$, the displacement is as shown in the figure.



Calculate the energy of the two waves.

What is the displacement of the string at $t = 3\pi/2\omega$? Calculate the energy at this time.

6.6 Two long strings lie along the x-axis under tension T. They are joined at $x = 0$ so that for $x < 0$ the line density $\rho = \rho_1$, and for $x > 0$, $\rho = \rho_2$. Small transverse oscillations propagate along these strings from $x = -\infty$. Show that at the join

$$\left.\frac{\partial y}{\partial x}\right|_{x=0-} = \left.\frac{\partial y}{\partial x}\right|_{x=0+}.$$

If a train of waves $y(x,t) = \mathrm{e}^{\mathrm{i}(kx-\omega t)}$ is launched into the combined string from $x = -\infty$, find the amplitude and phase of the trains that are (a) reflected from the join, and (b) transmitted through the join.

Using the result of Problem 6.3, verify that energy is conserved.

6.7 The apparatus of the last problem is modified by attaching to the join a particle of mass m which is connected to a fixed support by a light spring of stiffness p. This spring exerts a transverse force on the mass when the latter is displaced from $y = 0$. Show that at the join

$$T\left(\left.\frac{\partial y}{\partial x}\right|_{x=0+} - \left.\frac{\partial y}{\partial x}\right|_{x=0-}\right) = m\frac{\partial^2 y}{\partial t^2} + py$$

is satisfied.

A train of harmonic waves of frequency ω is transmitted from $x = -\infty$. Show that the phase of the transmitted wave lags behind that of the incident wave by an angle

$$\arctan\left(\frac{c_1 c_2(m\omega^2 - p)}{\omega T(c_1 + c_2)}\right),$$

where c_1 and c_2 are the speeds of the waves for $x < 0$ and $x > 0$, respectively.

6.8 A semi-infinite string of density ρ per unit length is under tension T. At its free end is a mass m which slides on a smooth horizontal rod that lies perpendicular to the string. Determine the amplitude reflection coefficient for transverse waves incident on the mass. What is the phase difference between the incident and reflected waves?

6.9 An infinite string lies along the x-axis, and is under tension T. It consists of a section at $0 < x < a$, of linear density ρ_1, and two semi-infinite pieces of density ρ_2. A wave of amplitude A travels along the string at $x > a$, towards the short section.

How many types of waves are there in the various sections of the string? How many boundary conditions need to be satisfied?

Show that, if $a = n\lambda_1$, (where λ_1 is the wavelength on the short section, and n is an integer), the amplitude of the wave that emerges at $x < 0$ is A. What is the amplitude of the wave in the short section?

HT VII

(vacation work)

Waves

7.1 A bar of uniform cross section A, density ρ and Young's modulus Y transmits longitudinal elastic waves. If a small element at position x is displaced a distance ξ, derive the wave equation for ξ and find the wavelength of a harmonic wave of frequency ω.

7.2 Waves of frequency ω travelling in the bar of the last problem are reflected at an end which has a mass M rigidly attached to it. Find the phase change on reflection and discuss the cases $M = 0$ and $M \to \infty$.

7.3 For the infinite transmission line of capacitance C and indunctance L per unit length, show that the voltage V obeys the wave equation and determine the speed of the waves. Find the characteristic impedance Z (i.e., the ratio of voltage to current) for waves travelling in both the positive and negative x-direction. Why is the characteristic impedance positive for waves travelling to the right, but negative for waves travelling to the left? Isn't that paradoxical, since the circuit is the same for left- and right-travelling waves?

7.4 A semi-infinite transmission line, of capacitance C and inductance L per unit length, is terminated by an impedance Z_T. Find the ratio of the amplitude and the phase difference for the reflected and incident waves if

(a) $Z_T = \sqrt{L/C}$,

(b) $Z_T = 2\sqrt{L/C}$,

(c) Z_T is a capacitor of capacitance C.

In (a) and (b), what type of impedance is required?

7.5 A uniform string of length l and density ρ has its end points fixed so that its equilibrium tension is T. A mass M is attached to its mid-point. Show that the angular frequency ω of small vibrations is given by

$$z \tan z = \frac{\rho l}{M}, \quad \text{where} \quad z \equiv \frac{\omega l}{2c} \quad \text{and} \quad c^2 \equiv T/\rho.$$

7.6 Two uniform wires of densities ρ_1 and ρ_2 and of equal lengths are fastened together and the two free ends are attached to two fixed points a distance $2l$ apart, so that the equilibrium tension is T. Show that the angular frequency ω of small vibrations satisfies

$$c_1 \tan(\omega l/c_1) = -c_2 \tan(\omega l/c_2),$$

where $c_{1,2}^2 \equiv T/\rho_{1,2}$.

7.7 An elastic string of length a consists of two portions, $0 < x < a/2$ of density ρ_1 and $a/2 < x < a$ of density ρ_2. It is stretched to tension T and the end at $x = a$ is fixed. The end at $x = 0$ is then shaken transversely at frequency ω. Show that throughout the motion, the ratio of the displacement at $x = a/2$ to that at $x = 0$ is given by

$$\frac{c_2 \csc(\omega a/2c_1)}{c_2 \cot(\omega a/2c_1) + c_1 \cot(\omega a/2c_2)},$$

where $c_{1,2}^2 \equiv T/\rho_{1,2}$.

Vector Calculus: Hydrodynamics

7.8 *Vorticity* of a fluid is defined $\omega = \nabla \times \mathbf{u}$ and tells you how the fluid circulates (locally), as we are about to see.

(a) Let us assume that the fluid is *barotropic*: $p = p(\rho)$, i.e., pressure depends only on density and has no other variation except via ρ, so $\nabla p = p'(\rho)\nabla\rho$ (this would be the case, for example, if $p = \text{const}\rho^\gamma$, which is clearly a solution of Eq. (7) of Problem 5.12).

Use Eq. (6) of Problem 5.12 and vector calculus to prove that

$$\frac{\partial \omega}{\partial t} = \nabla \times (\mathbf{u} \times \omega). \tag{9}$$

(b) *Circulation* is defined $\Gamma = \oint_C \mathbf{u} \cdot \mathrm{d}\mathbf{l}$, where C is a loop. Show that circulation over a loop is the flux of vorticity through a surface (so a vorticity "line" is a fluid swirl, or vortex).

(c) Let $C(t)$ be a "material" loop that moves with the fluid (i.e., each point on the loop moves at the local instantaneous velocity $\mathbf{u}(t, \mathbf{r})$). Prove *Kelvin's Circulation Theorem:*

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = 0, \quad \text{where} \quad \Gamma(t) = \oint_{C(t)} \mathbf{u} \cdot \mathrm{d}\mathbf{l}$$

(circulation through a loop moving with the fluid is conserved).

Strategy: Work out $\Gamma(t + \mathrm{d}t)$ and $\Gamma(t)$ to calculate the time derivative. Express these circulations as fluxes of vorticity through surfaces $S(t)$ and $S(t + \mathrm{d}t)$ for which $C(t) = \partial S(t)$ and $C(t + \mathrm{d}t) = \partial S(t + \mathrm{d}t)$. As the surface $S(t + \mathrm{d}t)$, it is covenient to choose the surface $S(t)+$ the ribbon traced by the loop $C(t)$ as it moved to become $C(t + \mathrm{d}t)$ (i.e., each of its points moved a distance $\mathbf{u}\mathrm{d}t$ in the direction of the local velocity — this should allow you to calculate the surface element $\mathrm{d}\mathbf{S}$ on the ribbon in terms of $\mathbf{u}$, $\mathrm{d}t$ and the line element $\mathrm{d}\mathbf{l}$ of the loop $C(t)$). Judicious application of Stokes' Theorem and Eq. (9) will lead to the desired result.

(d) Convince yourself that this result means that the field lines of ω "move with the fluid."