# Dimensional Analysis

Alex Schekochihin (and Richard Fern)

## 1 Measurement, Units and Dimensions

A fundamental feature of any advanced (or even not so advanced) technological civilisation is measuring things. For example

- The distance from Oxford to London $\sim$ 80km
- The speed of a car is $\sim$ 60 miles/hour
- The speed of light is $\sim 3 \times 10^8$m/s
- The volume of a bottle is $\sim$ 0.75 litres
- The mass of water is a glass is $\sim$ 200g
- The radius of the Earth is $\sim$ 6371km
- Acceleration due to gravity is $\sim$ 9.8m/s

but what do these things mean? Usually they mean we have chosen some *units* in which to measure a quantity and then compare with some standard. Alternatively, we can define a way to measure some quantity in terms of several units, such as speed in units of length per unit time.

> **Aside 1**
> The definition of a metre, decided in 1795, was to be one ten-millionth the length between the north pole and the equator when travelling via Paris. This was worked out and is now kept in the International Bureau of Weights and Measures in Paris as the distance between two marked lines on a bar of platinum-iridium when measured at 0°C. Due to a small calculation error that distance is about 0.02% shorter than they planned which is why, in the metres we know, the distance is actually about 2km further.

Thus, there are *independent* and *derived* units. For any class of physical phenomena, like mechanics, we can choose a fundamental set of units, like length, time and mass, and then express everything else in terms of these. For example,

$$\text{velocity} = \frac{\text{length}}{\text{time}} \qquad \text{acceleration} = \frac{\text{length}}{\text{time}^2}.$$

In the SI (Système International d'Unités) system we use metres for length, kilograms for mass and seconds for time, making velocity metres per second.

What constitutes an adequate system of units depends on the range of phenomena we're interested in. For example

- Geometry (size of objects) - This requires only length
- Kinematics (moving objects) - This requires length and time
- Dynamics (moving objects subject to forces) - Length, time and mass are needed
- Electromagnetism - Here we have to add a unit of charge (the Coulomb is SI units).

> **Aside 2**
> Note that the choice of units is not unique. For example, we could use velocity and time as our independent units instead of length and time. This would make length a dependent unit, expressed as speed × time. You may even recognise some such cases, for example a 1knot $\sim 2\frac{\text{km}}{\text{hr}}$ is used as an independent unit in the nautical world and in astrophysics, distances are often measured in light years.

Now what if I were to change my units? For example, what if I use truck driver units, kms, tonnes and hours. Then all my quantities previously expressed in m, kg and sec must be multiplied or divided by some conversion factors

$$\text{length} \rightarrow \frac{\text{length}}{L} \qquad L = 10^3 \text{ (number of metres in 1km)}$$

$$\text{time} \rightarrow \frac{\text{time}}{T} \qquad T = 3600 \text{ (number of seconds in 1 hour)}$$

$$\text{mass} \rightarrow \frac{\text{mass}}{M} \qquad M = 10^3 \text{ (number of kg in 1 tonne)}$$

which also changes my dependent units

$$\text{velocity} \rightarrow \frac{\text{velocity}}{L/T}$$

$$\text{acceleration} \rightarrow \frac{\text{acceleration}}{L/T^2}$$

$$\text{density} \rightarrow \frac{\text{density}}{M/L^3}.$$

This allows us to introduce the concept of *dimension*, $[X]$, of some physical quantity, $X$: it is the function that determines the conversion factor by which a physical quantity changes if we change units of measurement. For example,

$$[\text{length}] = L \qquad [\text{time}] = T \qquad [\text{mass}] = M$$

$$[\text{velocity}] = LT^{-1} \qquad [\text{acceleration}] = LT^{-2} \qquad [\text{density}] = ML^{-3}.$$

But what if we used a different system of units, for example where velocity is an independent unit? In that case

$$[\text{velocity}] = V \qquad [\text{time}] = T \qquad [\text{mass}] = M$$

$$[\text{length}] = VT \qquad [\text{acceleration}] = VT^{-1} \qquad [\text{density}] = MV^{-3}T^{-3}.$$

Therefore, we can define units to be independent if we cannot derive their dimensions from each other.

Consider an example in Newton's law,

$$f = ma$$

and ask what are the dimensions of force? Well,

$$f \to \underset{M}{m} \underset{LT^{-2}}{a} \qquad \Longrightarrow \qquad [f] = MLT^{-2}.$$

This is an important point. *Physical laws are independent of the units* and so both sides of equations that express them must have the same the same dimensions. This is a key principle that will allow us to discover some amazing things with very little work and arises from the fact that it should in no way matter to the outcome of a process whether we are measuring in metres, centimetres or light years. Nature has worked for billions of years without the definition of a metre.

We may wish to ask the question how many dimensionally independent quantities are there in a particular set of variables? Consider pressure, $p$, density, $\rho$ and velocity, $v$. Well,

$$[v] = LT^{-1} \qquad [\rho] = ML^{-3} \qquad [p] = \left[\frac{f}{L^2}\right] = ML^{-1}T^{-2}.$$

Therefore,

$$\left[\frac{p}{\rho}\right] = \frac{L^2}{T^2} = [v^2]$$

which means that $\sqrt{\frac{p}{\rho}}$ has the units of velocity. What is this velocity? Significantly more complicated physics tells us that

$$c_s = \sqrt{\gamma\frac{p}{\rho}} \qquad \text{is the speed of sound in a fluid or gas.}$$

The constant, $\gamma$, here is 1.4 in air at room temparature and cannot be determined from dimensional analysis. However, by considering only dimensions we have been able to discover that a fluid or a gas has a special speed associated with it!

This was the first example of *dimensional analysis*. We could have played in reverse, asking the question what is the speed of sounds in any given medium? Clearly the only things this could depend on are the pressure of the fluid, $p$, and the density, $\rho$. We then construct some quantity with dimensions of speed, in this case $\sqrt{\frac{p}{\rho}}$, and this must be the speed of sound

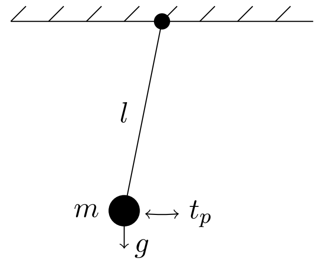
$$c_s = \text{constant}\sqrt{\frac{p}{\rho}}$$

where this constant could be found with just one good experimental measurement.

You can be confident of this relationship between $c_s$, $\rho$ and $p$ because it is a physical law which cannot change if we vary the units we measure $p$ and $\rho$ in. Any change in units on the right hand side (RHS) must produce the same scaling factor on the left hand side (LHS). Note also that we didn't have to solve any equations of motion or wave propagation to get this result. And since any result in physics cannot depend upon the dimensions we measure in, we can use this technique to check any results or any calculation by checking the dimensions of our answer. For example, if I solve $f = ma$, looking for the velocity of some particle, and my answer comes out with dimensions of length, I have done something wrong.

**Example 1**

Consider a pendulum. What is the period, $t_p$, of small oscillations of a pendulum? We can find this result without solving any equations. First we must ask, what can $t_p$ depend on? Consider the diagram. The key variables are the length of pendulum, $l$, the mass, $m$, and the gravitational constant, $g$. Then




$$[l] = L \qquad [m] = M$$

$$[g] = LT^{-2} \qquad [t_p] = T.$$

Therefore, we can define the dimensionless quantity

$$\Pi = \frac{t_p}{\sqrt{l/g}}$$

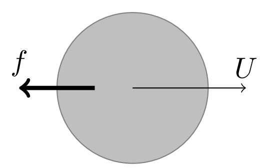
such that $\Pi$ will be exactly the same no matter what units we choose.

In principle, it is entirely possible that this quantity $\Pi$ can depend upon the parameters of the system, $l$, $g$ and $m$. However, if this were the case and we changed the units of, for example, $m$, then $\Pi$ would have to change also. However, looking at the RHS we know that changing the units of mass cannot change anything. Therefore, $\Pi$ must be a constant and so

$$t_p = \Pi\sqrt{\frac{l}{g}}$$

where we can make some measurements to find $\Pi = 2\pi$. We have just solved an interesting physics problem from nothing!

This example was nice and simple. Consider a slightly trickier problem of the drag force on a sphere moving through a gas at high speed. Ignoring friction, the drag will be entirely due to the inertia of the gas as the sphere pushes particles in it's path out of the way. Our parameters of importance are




$$\begin{aligned} &\text{density of gas} \qquad [\rho] = ML^{-3} \\ &\text{pressure} \qquad [p] = ML^{-1}T^{-2} \\ &\text{velocity of the body} \qquad [U] = LT^{-1} \\ &\text{body diameter} \qquad [d] = L \\ &\text{the drag force} \qquad [f] = MLT^{-2}. \end{aligned}$$

So, we can form the dimensionless parameter

$$\frac{f}{\rho U^2 d^2} = \Pi(\rho, p, U, d).$$

However, these arguments are not independent as before! We have already seen that $\sqrt{\frac{p}{\rho}}$ is some velocity and therefore we can define another dimensionless combination, the Mach number

$$\mathrm{Ma} = \frac{U}{c_s} \qquad \text{where recall } c_s = \text{const}\sqrt{\frac{p}{\rho}}.$$

Therefore, we must consider that

$$\frac{f}{\rho U^2 d^2} = \Pi(\rho, U, d, \mathrm{Ma}).$$

Here we apply the same reasoning as before, varying the dimensions of any one of the dimensionful quantities upon which $\Pi$ might depend. Given this would necessitate $\Pi$ to change as well, it cannot be a function of any of them. Thus $\Pi$ depends only on the Mach number, Ma and so

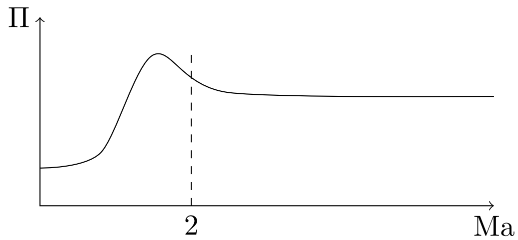
$$f = \rho U^2 d^2 \Pi(\mathrm{Ma})$$

where $\Pi(\mathrm{Ma})$ is some unknown function that depends on the Mach number. We cannot find the form of this function from dimensional analysis.

This result is less conclusive than the previous one but we have reduced the problem hugely from a force, $f$, which could have depended on all four parameters, $\rho, p, U, d$, into a force which can only depend on one, Ma. We have also figured out what matters physically in the problem. Furthermore, we can solve such problems completely in certain limits. For example, consider supersonic motion, where our speed, $U$, is much greater than the sound speed and so $\mathrm{Ma} \gg 1$. If $\Pi(\mathrm{Ma})$ is finite even when $\mathrm{Ma} \to \infty$ then

$$f = \mathrm{const} \cdot \rho U^2 d^2 \qquad \text{as } U \gg c_s.$$

This also works experimentally




Therefore the power to move the body is $P \sim fU \sim \rho U^3 d^2$, which depends very strongly on $U$.

So we can now define a general recipe for all problems:

1. First find the parameters on which the quantity of interest depends. Here one requires physical insight into what is and isn't relevant (for example, we neglected friction in this last problem).

2. Then find the parameters which have independent dimensions (remember this means you can't make the units of any other independent quantity from some combination of units of other independent quantities).

3. Find the dimensionless combinations from the remaining parameters and then

$$\text{Dimensionless combination including quantity of interest } \left(\text{e.g } \frac{f}{\rho U^2 d^2}\right)$$
$$=$$
$$\text{Some function of all the other dimensionless combinations } \Big(\text{e.g } \Pi(\mathrm{Ma})\Big).$$

## 2 The Π Theorem

We can formally generalise the examples from the section above with the following steps.

1. The dimension of a variable is always some power law monomial, i.e,

$$[a] = L^\alpha M^\beta T^\gamma ... \qquad \text{(and other units, such as charge } Q^\delta\text{)}$$

2. A number of such quantities are then independent if none of their dimensions can be expressed as a product of dimensions of others. So if we have a system of $k$ independent units and $k$ quantities, $a_1, a_2, a_3, ..., a_{k-1}, a_k$, with independent dimensions then we can always change to a system of units that have the same dimensions as $a_1, ..., a_k$ and so we can always change units so that any of the $a_i$'s changes by some specified factor while all other $a_i$'s remain unchanged.

   For example, in the drag problem we had length, mass and time ($k = 3$) and found that $a_1 = \rho, a_2 = U$ and $a_3 = d$ were independent. So we could measure everything in units of density, velocity and length and scale these units independently.

3. Now consider any given physical problem. It always reduces to finding some relationships of the form

$$a = f(\underbrace{a_1, ..., a_k}_{\text{Independent, like } \rho, U, d}, \overbrace{b_1, ..., b_m}^{\text{Dependent, like } p})$$

   for the desired quantity, $a$. The dependent variables are dependent because their dimensions can be expressed as

$$[b_i] = [a_1]^{\alpha_i}[a_2]^{\beta_i}...$$

   We are then looking for the solution to

$$[a] = [a_1]^\alpha [a_2]^\beta ...$$

   to make everything dimensionally consistent.

So how do we find the exponents? Just try solving a system of simultaneous linea equations. For the drag problem we want a force,

$$[f] = MLT^{-2}$$

as a function of the dependent variables

$$[\rho]^\alpha [U]^\beta [d]^\gamma = (ML^{-3})^\alpha (LT^{-1})^\beta L^\gamma$$

and so we must solve

$$MLT^{-2} = M^\alpha L^{-3\alpha+\beta+\gamma} T^{-\beta}.$$

So $\alpha = 1, \beta = 2$ and $\gamma = 2$ so

$$[f] = [\rho U^2 d^2].$$

This general technique can also find how the pressure's dimensions are dependent upon the independent dimensions

$$[p] = ML^{-1}T^{-2} = [\rho]^{\alpha_1}[U]^{\beta_1}[d]^{\gamma_1} = M^{\alpha_1} L^{-3\alpha_1+\beta_1+\gamma_1} T^{-\beta_1}$$

which can be solved exactly as above to find $\alpha_1 = 1, \beta_1 = 2$ and $\gamma_1 = 0$ and hence

$$[p] = [\rho U^2].$$

So, this means we can introduce $m + 1$ dimensionless combinations

$$\Pi = \frac{a}{a_1^{\alpha} a_2^{\beta} ...}$$

$$\Pi_1 = \frac{b_1}{a_1^{\alpha_1} a_2^{\beta_1} ...}$$

$$\vdots$$

$$\Pi_m = \frac{b_m}{a_1^{\alpha_m} a_2^{\beta_m} ...}.$$

In our drag example these combinations were $\Pi = \frac{f}{\rho U^2 d^2}$ and $\Pi_1 = \frac{p}{\rho U^2}$. We can then recast our physical relationship as

$$\begin{aligned}
\Pi &= \frac{f(a_1, ..., a_k, b_1, ..., b_k)}{a_1^{\alpha} a_2^{\beta} ...} \\
&= \frac{f(a_1, ..., a_k, \Pi_1 a_1^{\alpha_1} a_2^{\alpha_2} ..., ..., \Pi_m a_1^{\alpha_m} a_2^{\alpha_m} ...)}{a_1^{\alpha} a_2^{\beta} ...} \\
\Pi &= \mathcal{F}(a_1, ..., a_k, \Pi_1, ..., \Pi_m).
\end{aligned}$$

But now, since both sides are dimensionless, scaling any of the parameters $a_i$ by an arbitrary factor is equivalent to simply changing units, so shouldn't change the values of $\Pi, \Pi_1, ..., \Pi_m$ because they are dimensionless. Therefore, $\mathcal{F}$ is independent necessarily independent of $a_1, ..., a_k$ and we obtain the $\Pi$-theorem:

$$\Pi = \mathcal{F}(\Pi_1, ..., \Pi_m)$$

$$a = a_1^{\alpha} a_2^{\beta} ... \mathcal{F}(\Pi_1, ..., \Pi_m).$$

For example, in the drag force case we found

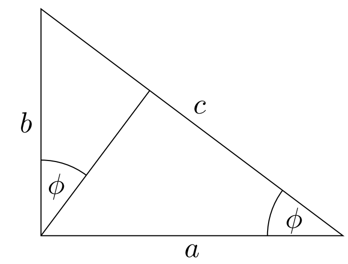
$$f = \rho U^2 d^2 \mathcal{F}(\mathrm{Ma})$$

where the Mach number was $\mathrm{Ma} = \sqrt{\frac{p}{\rho U^2}}$.

So, if we have $k$ independent units in our findamental system of units (e.g, $k = 3$ for a length, mass, time system) and $n$ governing parameters in the problem under scrutiny, we expect to be able to reduce the answer to an undetermined function of $m = n - k$ dimensionless contributions.

# 3 Further Examples

## 3.1 Pythagoras Theorem




Amusingly, the area of a right-angle triangle is completely determined by its hypotenuse and one (let's take the smaller one) of it's acute angle, $\phi$. So dimensionally,

$$A = c^2 f(\phi).$$

However, we can go a step further, splitting the triangle into two sub-triangles with hypotenuses $a$ and $b$ with

$$A_1 = a^2 f(\phi) \qquad\qquad A_2 = b^2 f(\phi).$$

And the sum of these areas must be the same as the total area so

$$A = A_1 + A_2 \qquad \implies \qquad c^2 f(\phi) = a^2 f(\phi) + b^2 f(\phi)$$

from which we can cancel $f(\phi)$ as this smallest angle is always the same and the function is universal. Thus

$$a^2 + b^2 = c^2.$$

Note this is only true in a flat space. If we were in a curved space, for example a triangle on the surface of a sphere, there would be another parameter, the radius of the sphere, $r$, and we would have

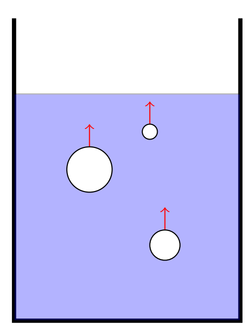
$$A = c^2 f\left(\phi, \frac{c}{r}\right)$$

where $\frac{c}{r}$ is a dimensionless parameter. Then, splitting the triangle,

$$c^2 f\left(\phi, \frac{c}{r}\right) = a^2 f\left(\phi, \frac{a}{r}\right) + b^2 f\left(\phi, \frac{b}{r}\right)$$

from which we cannot cancel $f$. However, remember it's useful to think in certain limits, such as when the sphere is enormous, much larger than the triangles, such that $\frac{c}{r}, \frac{a}{r}, \frac{b}{r} \ll 1$ and so we take the limit where all areas are $A = c^2 f(\phi, 0)$, which can then cancel to reproduce the previous result.

## 3.2 Rising Bubbles



How fast do bubbles rise depending on their size? We want to find velocity $U$ as a function of bubble diameter $d$. Let's first try a quick solution. Given that

$$[U] = \frac{L}{T}$$

we look for just two independent dimensions involving $L$ and $T$. These are

| | |
|---|---|
| bubble size | $[d] = L$ |
| gravity | $[g] = \frac{L}{T^2}$ |

and so we can immediately say

$$U = \text{const}\sqrt{gd}.$$

Does this make sense? Well, it's just force balance. The Archimedian force of buoyancy is

$$F_B \sim \rho d^3 g$$

which must equal the drag force

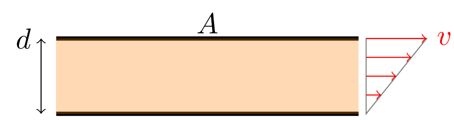
$$F_D \sim \rho U^2 d^2.$$

These forces do indeed balance when

$$U^2 \sim gd.$$

However when we calculated the drag we assumed we were moving at high-speeds, a result which neglects the viscosity of the fluid. This seems like a very dodgy assumption, especially for bubbles which rise rather slowly and at very different speeds in fluids of varying viscosity (think lemonade versus honey). Thus, we must include the effect of viscosity, so we need to introduce some quantity that characterises the viscosity of a fluid, a quantity that could be measured for any given fluid.

**Aside 3**




Viscosity is basically a measure of how difficult it is to move a fluid differentially with respect to itself. In the figure the plate above moves at velocity $v$ due to the action of a constant force $f$. This takes a force because fluid sticks to the surfaces of the two plates, creating a velocity gradient which friction opposes. Empirically it is known that

$$f \propto \frac{v}{d} A$$

where $A$ is the area of the plates and $d$ the distance between them. The constant of proportionality (which is dimensionful!) is independent of $v$, $d$ or $A$ in most fluids so we let

$$f = \mu \frac{vA}{d}$$

where $\mu$ is the viscosity. It has units

$$[\mu] = \left[\frac{fd}{vA}\right] = \frac{M}{LT}.$$

Physically, this viscous force is relevant to determining the drag force on a moving object because there are two sets of forces opposing motion

1. Inertial forces - The object pushes the medium apart as it moves
2. Viscous forces - Fluid in the immediate vicinity of the object is pulled along at the speed of the object while fluid far away is at rest. Thus, the fluid sets up a differential flow.

So now we should repeat our drag force calculation. Our parameters are

$$[f] = \frac{ML}{T^2} \qquad \longleftarrow \text{Desired quantity}$$

$$[\rho] = \frac{M}{L^3} \qquad [p] = \frac{M}{LT^2} \qquad [U] = \frac{L}{T}$$

$$[d] = L \qquad [\mu] = \frac{M}{LT}.$$

Even with all these considerations we are still ignoring a lot. For example, the weight of the air in the bubble, the pressure changes with height in the fluid and associated expansion and the effect of surface tension on the shape of the bubble. However, running without these small effects we have $n = 5$ parameters, $k = 3$ independent parameters and $m = 2$ dimensionless combinations. The first one we already know is

$$\text{Ma} = \frac{U}{\sqrt{p/\rho}}.$$

For the second consider

$$[\mu] = \frac{M}{LT} = [\rho]^\alpha [U]^\beta [d]^\gamma = M^\alpha L^{-3\alpha+\beta+\gamma} T^{-\beta}$$

and so $\alpha = \beta = \gamma = 1$. This defines the Reynolds number

$$\text{Re} = \frac{\rho U d}{\mu}.$$

Now we can apply the $\Pi$-theorem to see that

$$f = \rho U^2 d^2 \mathcal{F}(\text{Ma}, \text{Re}).$$

This tells us what it means to move fast enough to ignore friction. To do so we need $\text{Re} \gg 1$ and assume that $\mathcal{F}(\text{Ma}, \infty)$ is finite.

So, back to our bubbles. Firstly we note that bubbles move much slower than the speed of sound, so let's set $\text{Ma} = 0$, leaving us only to figure out the dependence on Re.

1. The familiar limit we worked in was where $\text{Re} \gg 1$, and so

   $$f = \text{const} \cdot \rho U^2 d^2$$

   where this constant is $\mathcal{F}(0, \infty)$. Physically this situation is the case where the fluid behind the bubble becomes turbulent and viscous forces no longer matter. This is turbulent drag.

2. In the opposite limit, $\text{Re} \ll 1$, it is clear that $\mathcal{F}(0, 0)$ is not finite, or our answer would be independent of viscosity and we know that viscosity is infinite, the viscous force is so large that our bubbles can't even move. So to figure out what $\mathcal{F}(\text{Re})$ looks like at small Re we need some physical insight.

So let's think. If the viscosity is large, let's argue the drag force is independent of density, as the inertia of the fluid is no longer important. Therefore, since

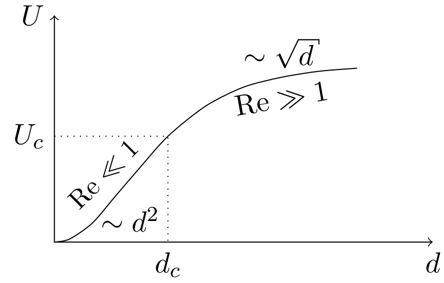
$$f = \rho U^2 d^2 \mathcal{F}(\text{Re})$$

the only way to cancel the dependence on $\rho$ is by setting

$$\mathcal{F}(\text{Re}) = \frac{\text{const}}{\text{Re}}$$

which leaves us with Stoke's formula for drag

$$f = \text{const} \mu U d.$$




This formula applies for slow velocities, large viscosities or low densities/small bubbles.

Thus, we can once again apply the force balance equation with the Archimedes force

$$\rho d^3 g \sim \mu U d \qquad \Longrightarrow \qquad U = \text{const}\frac{\rho g}{\mu} d^2.$$

Therefore, the dependence of bubble velocity on the diameter is as shown.

Note that we go through some intermediate region which can depend on all sorts of complications to do with bubble shape changes, surface tension and pressure changes. The crossover occurs where $\text{Re} \sim 1$ and we can't make either of our previous approximations. Thus

$$\frac{\rho^2 g d^3}{\mu^2} \sim 1$$

when

$$d = d_c \sim \frac{\mu^{2/3}}{\rho^{2/3} g^{1/3}}.$$