Questions on Dimensional Analysis

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1. G. I. Taylor and the Bomb

In the early autumn of 1940, during some of the most desperate days of the Battle of Britain, a Cambridge Professor of fluid dynamics G. I. Taylor was invited to lunch by an Imperial College Professor and Nobel-prize winner George Thomson, who was then chairman of the MAUD committee (MAUD = "Military Application of Uranium Detonation"). G. I. Taylor was told that it might be possible to produce a bomb in which a very large amount of energy would be released by nuclear fission (this was to become the atomic bomb). The crucial question was what would be the mechanical effect of such an explosion? G. I. Taylor's subsequent solution of this problem may be the most famous example of the application of dimensional analysis of all time. In this problem, you will work through some of his calculation.

Let us simplify the problem by assuming that

— a finite amount of energy E is released instantaneously at a point (i.e., we will ignore the radius r_0 of the volume where the initial energy release occurs at time $t = 0$, it will not be a relevant parameter);

— there results a *spherically symmetric* shock wave, with its front propagating according to some law $r_f(t)$, where r_f is the radius of the front.

Find $r_f(t)$ as a function of time t. Find also the velocity of the front $u_f(t)$ and the pressure $p_f(t)$ in the surrounding air just outside the front. The density ρ_0 of air before the explosion is given. *If you identify correctly what the governing parameters are (all of them are mentioned above), you should be able to use dimensional analysis to work out r_f, u_f and p_f with only constant dimensionless prefactors left undetermined.*

Based on the result you have obtained, will, in your opinion, making the bomb bigger (say doubling its size) makes much of a difference?

If you did not know the energy of the explosion E (classified!), but had a movie of the fireball, how would you estimate E? (When the Americans tested the bomb and released a series of high-speed photographs, G. I. Taylor estimated E and published the result, which caused much embarassment in the American government circles.)

2. Poiseuille Flow

Attempt this question after the second lecture on Dimensional Analysis

This example is also famous, and much more peaceful than the previous one. It was first worked out experimentally by H. Hagen (1839) and J. L. M. Poiseuille (1840) (working independently of each other) and later theoretically explained by G. G. Stokes (1845).

Consider a pipe of length l and diameter d. A pressure drop between the ends of the pipe, $p_1 - p_2$, is maintained to pump an incompressible fluid of viscosity μ through the pipe. Find the volumetric flow rate Q, i.e., the volume of the fluid that passes through any cross-section of the pipe per unit time.

If I double the diameter of the pipe, by what factor will Q change? What if I double the pressure contrast? And what if I double viscosity? Does the answer make sense? (Why does viscosity matter?) What if I double viscosity *and* cut the pipe length by half?

Hint. A judicious choice of governing parameters in this problem is d, μ and $(p_1 - p_2)/l$ — the pressure drop per unit length (think about why that is).

Now find the velocity U at which the fluid flows through the pipe.

Why do you think the density of the fluid does not matter here? Under what conditions would you expect it to start being an important parameter? (Think about the discussion in the lectures — what is the dimensionless number that controls this?)

Further Reading

- P. W. Bridgman, *Dimensional Analysis* (there is a very cheap Amazon reprint of this classic 1920 text — a bit dated, but still quite readable)
- G. I. Barenblatt, *Scaling* (this is quite advanced, but you can read the first couple of chapters; this is the book from which I lifted the G. I. Taylor example — but don't look until you have attempted to solve it unaided! It also contains several other intesting examples of dimensional analysis, including one giving important insights into rowing, which you will find fascinating in the unlikely case that you are into that kind of thing, despite being a Merton physicist — and if you are not into it, this will give you the satisfying feeling that you know more of the physics of it than those who are)
- L. Landau and E. Lifshitz, *Fluid Mechanics* (Vol. 6 of their *Course of Theoretical Physics*, all of the ten volumes of which every self-respecting physicist should keep on his/her desk at all times; you will find there an account of some rather complicated issues that arise in the Poiseuille problem at large Reynolds numbers; you can also read there how bubbles rise, bodies move through fluid etc.)