CPSC 4278

FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP4: DIFFERENTIAL EQUATIONS, WAVES AND OPTICS

Thursday 5 June 2008, 2.30 pm – 5.00 pm

Time allowed: $2\frac{1}{2}$ hours

Answer all *of Section A and* three *questions from Section B.*

Start the answer to each question on a fresh page.

A list of physical constants, mathematical formulae and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. Show that

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i},$$

[2]

and hence prove that

$$\sin^5\theta = \frac{1}{2^4}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta).$$

[5]

2. Find the locus of points on the Argand plane satisfying

$$\arg(z-1) = \frac{\pi}{4}.$$

[4]

3. Find all possible solutions to the equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

of the form, $u = g(t)f(x)$, where c is a real constant. [5]

4. Solve the differential equation

$$x(x+1)\frac{\mathrm{d}y}{\mathrm{d}x} + y = x(x+1)^2\mathrm{e}^{-x^2}.$$

[6]

5. Find the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \mathrm{e}^{-x}.$$

[6]

6. The propagation of transverse waves on a stretched string is described by the wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$$

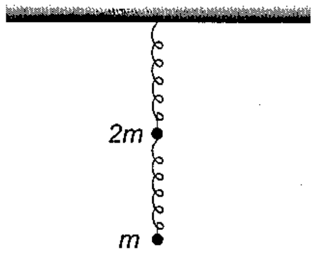
where z is the transverse displacement at point x at time t and c is the speed of propagation.

A string is made of two semi-infinite pieces joined at the origin. For $x < 0$, the speed is c_1; for $x > 0$, the speed is c_2. The wave $z = \cos(\omega t - k_1 x)$ is incident on the boundary, where $k_1 = \omega/c_1$. Find the amplitudes of the reflected and the transmitted waves. [6]

7. Show by means of ray diagrams how i) real and ii) virtual images of an object can be formed by a thin convex lens of focal length f. Find the linear magnification in each case. [6]

Section B

8. Two massless springs each have spring constant k. Masses $2m$ and m are attached as shown in the figure.



The masses make small vertical oscillations about their equilibirium positions. Show that the respective displacements x and y of the masses $2m$ and m satisfy the coupled differential equations

$$\frac{d^2x}{dt^2} = \frac{k}{2m}(y - 2x)$$
$$\frac{d^2y}{dt^2} = \frac{k}{m}(x - y)$$

and explain why there is no term involving the acceleration due to gravity. [7]

Find expressions for the normal frequencies for small oscillations of the masses. [7]

Find the ratio of the amplitudes for each normal mode. [6]

9. a) Prove the identity

$$\sum_{n=0}^{N-1} \cos nx = \frac{\sin(Nx/2)}{\sin(x/2)} \cos((N-1)x/2).$$

[7]

b) Find all the solutions of the equation

$$\left(\frac{z+\mathrm{i}}{z-\mathrm{i}}\right)^n = -1,$$

[8]

and solve the equation

$$z^4 - 10z^2 + 5 = 0.$$

[5]

10. Consider the coupled differential equations

$$\begin{aligned}\frac{du}{dt} + au - bv &= f \\ \frac{dv}{dt} + av + bu &= 0\end{aligned}$$

where a, b and f are constants.

i) Solve them for $f = 0$, subject to the boundary conditions $u = 0$ and $v = v_0$ when $t = 0$. [10]

ii) Solve them for $f \neq 0$, subject to the boundary conditions $u = v = 0$ when $t = 0$, and write down the steady state solutions. [10]

11. Light from a narrow source slit falls on two narrow parallel slits, separated by a distance d in a plane normal to the incoming light. The interference pattern is observed on a screen at a distance D from the two slits. Deduce the intensity distribution on the screen in the case of illumination by monochromatic light of wavelength λ. If $d = 0.5\,\mathrm{mm}$, $D = 0.5\,\mathrm{m}$ and $\lambda = 600\,\mathrm{nm}$ calculate the spacing of the fringes. [8]

a) By what distance does the centre of the fringe pattern move if one slit is covered by a glass slide thickness 0.1 mm with refractive index 1.6? [5]

b) If the slits are illuminated by two wavelengths, $\lambda = 600\,\mathrm{nm}$ and $\lambda' = 500\,\mathrm{nm}$, determine the distance from the centre of the pattern for which a minimum for λ coincides with a maximum for λ', and the corresponding order of interference. [7]

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FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP3: MATHEMATICAL METHODS

also

Moderations in Physics and Philosophy

Wednesday 4 June 2008, 2.30 pm – 5.00 pm

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a fresh page.

The use of calculators is **not** *permitted.*

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Section A

1. Evaluate each of the following limits:

$$\text{a)}\quad \frac{5x^2-23x-42}{x^2-7x+6} \text{ as } x \to 6,$$

$$\text{b)}\quad \frac{x\tan x}{\sinh^2 x} \text{ as } x \to 0,$$

$$\text{c)}\quad 2x^2\sin^2(1/x) \text{ as } x \to \infty.$$

[6]

2. Give the Taylor Expansion for $f(x+h)$ in terms $f(x)$ and its derivatives. State conditions under which such an expansion is valid. [4]

3. Find the differential coefficient $\mathrm{d}y/\mathrm{d}x$ for each of the following:

a) $y = \tanh(x)$,
b) $x = \tanh(y)$,
c) $x = a\cos(\omega t),\ \ y = b\sin(\omega t)$.

[6]

4. A unit vector $\mathbf{e}_1$ is given by

$$\frac{1}{\sqrt{3}}(1,1,1).$$

Find a pair of unit vectors, $\mathbf{e}_2$ and $\mathbf{e}_3$, such that $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ are orthogonal. Check that your choice forms a right-handed set. [4]

Resolve a vector $\mathbf{a} = (\alpha, \beta, \gamma)$ into components along $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. [2]

5. Given that $g(x,y,z) = 0$, derive an expression for $\left(\frac{\partial y}{\partial x}\right)_z$ in terms of the components of ∇g. [4]

6. A vector field in three dimensions may be described as $\mathbf{A}(x,y,z)$. Express the cartesian components of $\nabla^2\mathbf{A}$ in terms of the cartesian derivatives of the components of $\mathbf{A}$. [4]

7. A line is given by the equation

$$\mathbf{r} = 3\mathbf{i} - \mathbf{j} + (2\mathbf{i} + \mathbf{j} - 2\mathbf{k})t$$

where t is a variable parameter and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors along the cartesian axes x, y, z. The equation of the plane containing this line and the point (2,1,0) may be expressed in the form $\mathbf{r}\cdot\mathbf{a} = d$ where $\mathbf{a}$ is a unit vector and d is a constant. Find $\mathbf{a}$ and d, and explain their geometrical meaning. [5]

Find the volume of the tetrahedron with its four corners at: the origin, the point (2,1,0), and the points on the line with $t = 0$ and $t = 1$. [5]

Section B

8. Prove the identity $\nabla \cdot (\nabla \times \mathbf{X}) = 0$ for any field $\mathbf{X}$ in three dimensions. [4]

Consider the line integral of a general vector field $\mathbf{R}$ round a general closed loop $\mathbf{S}$. State how Stokes' Theorem may be used to evaluate this line integral by replacing it with an integral over a surface $\mathbf{A}$ bounded by the loop. There are many such surfaces bounded by the same loop. Explain how the surface might be chosen. [7]

The loop is a circle in the xy-plane, $x^2+y^2 = a^2$, and the field is $\mathbf{R} = 4y\mathbf{i}+x\mathbf{j}+2z\mathbf{k}$. Find the integral when evaluated over the planar circle. [4]

Explicitly evaluate the integral over the surface of the hemisphere bounded by this circle. Comment on your result. [5]

9. i) The three matrices

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are known as the Pauli matrices. Find the square and inverse of each matrix. [5]

Express $AB - BA$, the commutator of A and B, as a linear combination of A, B and C. Do likewise for the commutators, $BC - CB$ and $CA - AC$. [4]

ii) A multiple integral, $\int_R \rho(x, y, z)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$, over a region R may be expressed and evaluated in other coordinates (u, v, w) in place of (x, y, z). Explain carefully how this may be done. [5]

A cone $z^2 = x^2 + y^2$ is truncated by the planes $z = 1$ and $z = 2$. Calculate its mass given that the density is $\rho = (x^2 + y^2 + z^2)^{-1}$. [6]

10. Give the Taylor expansion of the differentiable function $f(x, y)$ about the point (a, b) up to, and including, quadratic terms in $(x - a)$ and $(y - b)$. [4]

This approximate expansion gives the equation of contours passing through (a, b). In particular, if (a, b) is a stationary point of the function, the expansion determines the type of stationary point. Show the conditions that the derivatives of f at (a, b) should satisfy for these contours to represent:

i) a pair of straight lines,

ii) a single point. [8]

For the function $f(x, y) = (sx^2 + ty^2) \exp(-x^2 - y^2)$ find the equations of the contour lines, if any, at the point $(0, 1)$. [8]

11. A volume is enclosed by the plane $z = 0$ and the inverted paraboloid, $z = 6 - r^2$ (expressed in cylindrical coordinates). Find the volume and its surface area. [10]

Hence, using a suitable linear transformation, show that the volume of the region enclosed between the surfaces $z = ax^2 + by^2$ and $z = 6 - cx^2 - dy^2$ where a, b, c and d are positive constants, is given by

$$\frac{18\pi}{\sqrt{(a+c)(b+d)}}.$$

[10]

CPSC 4278

FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP4: DIFFERENTIAL EQUATIONS, WAVES AND OPTICS

Thursday 7 June 2007, 2.30 pm – 5.00 pm

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

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Section A

1. Find all possible values of $\ln(z) - \ln(-z)$ where z is a complex number. [3]

2. The function

$$z = a^t \exp(ibt)$$

where a and b are real constants, describes a curve in the complex z-plane for real values of t. Sketch the shape of the curve for the case $a = 1.1$, $b = \pi/3$ and t in the range -6 to $+6$. [6]

3. Let $z_1, \ldots z_n$ be the set of n distinct solutions to the equation

$$z^n = a$$

where a is a complex number. By considering successive distinct solutions as the sides of a polygon in an Argand diagram show that these sum to zero. [3]

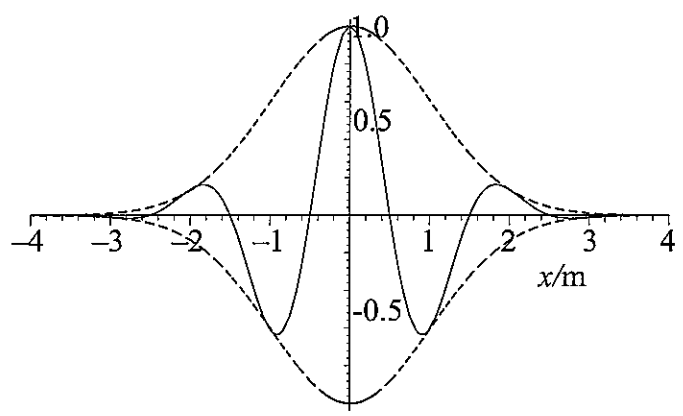
Hence find the sum of the squares of these solutions. For the case $n = 5$ sketch the polygon traced out by these successive squared values in the Argand plane. [4]

4. The variables $\psi(z)$ and $\phi(z)$ obey the simultaneous differential equations

$$\begin{aligned} 3\frac{\mathrm{d}\phi}{\mathrm{d}z} + 5\psi &= 2z \\ 3\frac{\mathrm{d}\psi}{\mathrm{d}z} + 5\phi &= 0. \end{aligned}$$

Find the general solution for ψ. [6]

5.



The figure shows the profile as a function of x of a wave pulse with its accompanying envelope travelling in the positive x direction at time $t = 0$. The phase velocity of the wave is $c = 1.5$ m s^{-1} and its group velocity is $u = 1.0$ m s^{-1}. Sketch the profile of the wave as a function of x at time $t = 1.0$ s. [7]

6. Re-express the wave equation for waves of amplitude $y(x,t)$,

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0,$$

as a differential equation for $y(w,v)$ where $w = x + ct$ and $v = x - ct$. [5]

7. Explain what is meant by the *total internal reflection* of a ray of light and the conditions under which it occurs. [2]

A long transparent cylindrical fibre of refractive index μ is cut with flat faces normal to its axis. Show that, provided that $\mu > \sqrt{2}$, light incident at any angle on one face may be transmitted to the face at the other end. [4]

Section B

8. State what is meant by the *general solution* to a differential equation. What is an *inhomogeneous linear ordinary differential equation*? State how you would recognise whether a given solution to such an equation is the general solution. [5]

Find the general solution to each of the following equations:

(a) $6\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 5\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} + 7y = x$

(c) $y\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{4x+3}$.

[15]

9. The vertical motion $z(t)$ of a mass suspended by a spring is described by the differential equation

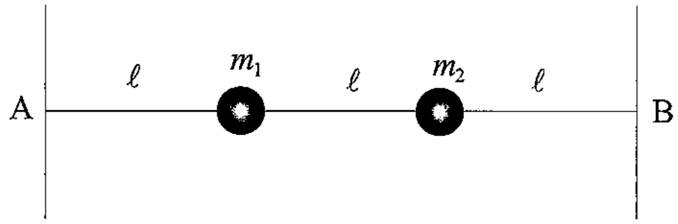
$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}z}{\mathrm{d}t} + \omega_0^2 z = \frac{F}{m}\cos(\omega t).$$

Explain the physical significance of each term. [6]

Why does g, the acceleration due to gravity, not appear in this equation in spite of the fact that it influences the position of the mass? [4]

Find the general solution for the amplitude $z(t)$ when γ is greater than $2\omega_0$. Interpret the value of z that this gives for $\omega = 0$ in the limit of large positive times. [10]

10.

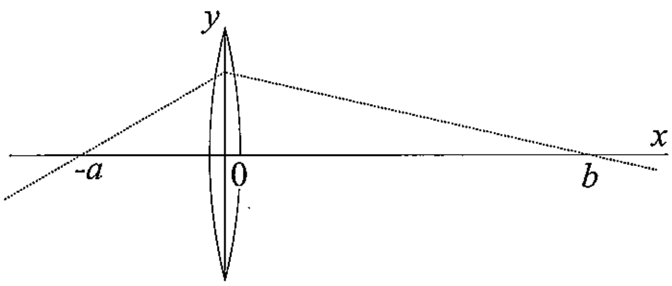


Two masses m_1 and m_2 are connected together by a massless elastic string at tension T between two fixed points A and B, as shown. Write down the equations of motion for x_1 and x_2, the transverse position in one plane from equilibrium of the masses, neglecting the effects of gravity and damping. [4]

Deduce values for the frequencies and related amplitude ratios of the two normal modes for $m_1 = 1.5\,\mathrm{kg}$, $m_2 = 0.8\,\mathrm{kg}$, $\ell = 1\,\mathrm{m}$, $T = 3\,\mathrm{N}$. [8]

The masses are initially in equilibrium at rest. At time $t = 0$ the mass m_2 is given a sudden transverse velocity $0.5\,\mathrm{m\,s^{-1}}$. Find the subsequent displacement of x_1 as a function of time. [8]

11.



The diagram shows a thin lens at the origin with its axis of symmetry along the x-axis. A light ray that crosses the x-axis at $x = -a$ passes through the lens at $y = d$ and is refracted so as to cross the x-axis again at $x = b$. The refractive index of the lens is μ and its thickness t, depending on y as $t = t_0 - \alpha y^2$. Find an expression for the time for the ray to pass from $x = -a$ to b along this path as a function of the distance d. [You should assume that d is small relative to a or b, and that t is sufficiently small that the ray within the lens may be treated as travelling parallel to the x-axis.] [8]

Hence show that in this case the focal length of the lens is given by

$$f = \frac{1}{2\alpha(\mu - 1)}.$$ [7]

Comment on the description of the focussing action of a general lens in terms of waves. [5]

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FIRST PUBLIC EXAMINATION

Trinity Term

Preliminary Examination in Physics

Paper CP3: MATHEMATICAL METHODS

also

Moderations in Physics and Philosophy

Wednesday 6 June 2007, 2.30 pm – 5.00 pm

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

Start the answer to each question on a fresh page.

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Section A

1. Calculate the matrix element $(A^{-1})_{1,2}$ for the matrix

$$A = \begin{pmatrix} 1 & -3 & 2 \\ -d & -1 & 2 \\ d & 3 & -4 \end{pmatrix},$$

given that the matrix satisfies $\det(A) = 2$. [3]

2. A line intersects a plane at an angle $\alpha = 2\pi/3$. The line is defined by $\mathbf{r} = \mu\hat{\mathbf{n}}$ and the plane by $\mathbf{r}\cdot\hat{\mathbf{m}} = 0$, with $\hat{\mathbf{n}}$ and $\hat{\mathbf{m}}$ unit vectors. Calculate the shortest distance from the plane to the point on the line with $\mu = 2$. [4]

3. State the first four terms of the Taylor series of $f(x) = \sin(x - \pi/2)$ about the point $x = 0$. Use this expansion to evaluate $\cos(0.1)$ to three digits. Why is this series expansion not suitable for evaluating $\sin(0.1)$? [4]

4. What is meant by an *exact differential*? Evaluate the integral

$$\int_a^b (2xyz\mathrm{d}x + x^2z\mathrm{d}y + x^2y\mathrm{d}z)$$

for an arbitrary path between the points a and b with coordinates $(x, y, z) = (-1, -1, -1)$ and $(0, 0, 0)$ respectively. [4]

5. For $x \neq 1$, the function f is defined by

$$f(x) = \frac{x^3 - ax^2 + x}{(x-1)^2},$$

with a a real number. Find the value of a for which the limit $\lim_{x\to 1} f(x)$ is finite. For this value of a define $f(1)$ such that f is a continuous function. [4]

6. A plane is tangential to the surface $x\exp(-x^2 - y^2) - z = 0$ at the point where z takes on its maximum value. Find the equation of the plane. [6]

7. Use the divergence theorem to determine the surface integral $\int_S \mathbf{F}\cdot \mathrm{d}\mathbf{S}$ over the closed surface $x^2 + y^2 = a^2 - z$, $z > 0$ for $\mathbf{F} = \mathbf{r}$. [7]

8. Sketch the curve $y^2 = (x+1)^2(2-x)$ in the xy-plane and find the area enclosed by the loop of the curve. [8]

Section B

9. Give geometric interpretations of cases where the solution of a set of three linear equations in three variables (i) does not exist, (ii) is a line, and (iii) is a plane. [5]

A set of linear equations in x, y and z is given by

$$\begin{aligned} 2x + ay + z &= 1 \\ -2x + by + 2z &= 3 \\ 4ax + 2y + 2z &= b. \end{aligned}$$

(a) Setting $b = 3$ calculate all values of a for which the solution is not unique. [3]

(b) Show that for $a = 1$ a unique solution does not exist whatever the value of b. [3]

(c) Setting $a = 1$ find the value of b for which solutions exist and derive the equation which relates all possible solutions. [9]

10. State the vector condition for two surfaces $v(x, y, z) = 0$ and $w(x, y, z) = 0$ to cut orthogonally. [3]

Prove that

$$\nabla v \cdot \nabla w = \frac{1}{2}\left(\nabla^2(vw) - v\nabla^2 w - w\nabla^2 v\right).$$ [5]

Hence or otherwise calculate all values of the constant a for which the surfaces $ay - x = 0$ and $5x^2 + y^2 + 3xy + z^2 = 25$ cut orthogonally. Sketch and describe a geometric interpretation for each of these values of a. [12]

11. A particle moves at constant speed v in the xy-plane. Explain why its velocity can be parameterized by

$$\dot{\mathbf{r}}(t) = v \begin{pmatrix} \cos\phi(t) \\ \sin\phi(t) \\ 0 \end{pmatrix},$$

with $\phi(t)$ a continuous function of time. Determine $\phi(t)$ for a particle moving along the x-axis. [5]

Consider a particle which moves in three dimensions with velocity $\mathbf{v}(t)$ such that

$$\dot{\mathbf{v}}(t) = \gamma t\, \mathbf{k} \wedge \mathbf{v}(t),$$

where γ is a constant and the unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ have their usual meaning. At time $t = 0$ the particle is located at the origin and moves with velocity $\mathbf{v}(0) = w\mathbf{i}$. Show that the speed of the particle is constant and that it moves in the xy-plane. Hence determine its velocity $\mathbf{v}(t)$. [10]

Interpret this result in terms of rotational motion by working out the instantaneous angular velocity $\omega(t)$ and the instantaneous radius of curvature $R(t)$. [5]

12. The function $z = f(x, y)$ defines a continuous surface above the xy-plane. Derive an expression for $\left(\frac{\partial x}{\partial y}\right)_z$ in terms of $\left(\frac{\partial z}{\partial y}\right)_x$ and $\left(\frac{\partial z}{\partial x}\right)_y$. Give a geometrical interpretation of $\left(\frac{\partial x}{\partial y}\right)_z$ in terms of properties of contour lines of constant z. [6]

Expand $f(x, y)$ around a stationary point (x_0, y_0) into a Taylor series keeping terms up to second order. [4]

For $\frac{\partial^2 z}{\partial y \partial x} = 0$ at $(x, y) = (x_0, y_0)$ show that contour lines near a maximum or a minimum are described by ellipses of the form

$$a(x - x_0)^2 + b(y - y_0)^2 = c^2.$$

Determine a, b and c from the Taylor expansion. [4]

What is the shape of contour lines passing through a saddle point near the saddle point? Find the slopes of the contour lines through the saddle point of $f(x, y) = 2x^2 - y^2$ at $(x_0, y_0) = (0, 0)$. Show that these agree with the possible values of $\left(\frac{\partial x}{\partial y}\right)_z$ at $z = 0$. [6]

CPSC 4278

FIRST PUBLIC EXAMINATION

TRINITY TERM

Preliminary Examination in Physics

Paper CP4: DIFFERENTIAL EQUATIONS, WAVES AND OPTICS

Thursday, 8 June 2006, 2.30 pm – 5.00 pm

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

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Section A

1. Draw the locus of the points $z = r\exp(\mathrm{i}\theta)$ with $r = |\sin(2\theta - \pi/3)|$ in an Argand diagram. For which angles θ is $|z|$ maximal? [5]

2. Find the general solution of the differential equation

$$\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x^2} = \sin x.$$ [4]

3. Express $\tan 3\theta$ in terms of $\tan\theta$. [5]

4. Solve the differential equation

$$3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 4y = 0$$

with initial conditions $y(0) = 0$ and $\frac{\mathrm{d}y}{\mathrm{d}x}(0) = 7$. [5]

5. The motion of an infinite string is described by the wave equation

$$\frac{\partial^2 y}{\partial x^2} - \frac{\rho}{T}\frac{\partial^2 y}{\partial t^2} = 0,$$

where y is the transverse displacement. The constants T and ρ are the tension and the linear mass density of the string, respectively. For which value of c is $y(x,t) = A\exp[-(x-ct)^2]$,with A a constant, a solution of this wave equation? Give a physical interpretation of c and A. Show that the potential and kinetic energy of the wave packet $y(x,t)$ are equal. [6]

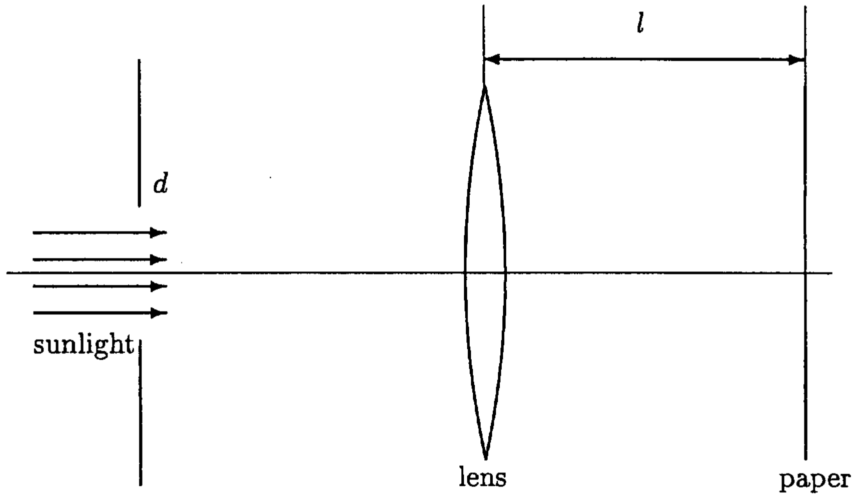
6. A point like object moves away from a concave mirror with focal length f along the principal axis at constant velocity v_o. It starts at the mirror surface at time $t = 0$. Calculate and briefly discuss how the velocity and position of its image change as a function of time. [6]

7. An optical system produces an inverted image of an object O. A thin lens of focal length f is to be used to transform this image into an upright one and magnify it by a factor of 2. Which type of lens is needed? Draw an optical setup which can be used to achieve this. Find the distances between the inverted image, the upright magnified image and the lens. Check your answer geometrically with the help of your drawing. [5]

8. A normally illuminated diffraction grating with $N = 500$ lines per mm produces multiple diffraction maxima the first being at an angle of $\theta = 10°$. Find the angular frequency ω of the light used to illuminate the grating. [4]

Section B

9. Light passing through a convex lens of focal length $f = 20$cm is used to illuminate a sheet of paper distance l away from the lens. The incoming light beam has diameter $d = 1$cm and is arriving from the sun.



Explain why the light from the sun may be assumed to be collimated. The light intensity coming from the sun is $S = 1{\cdot}37\text{kW/m}^2$. Calculate the light intensity I arriving at the sheet of paper and the illuminated area A as a function of the distance l. [5]

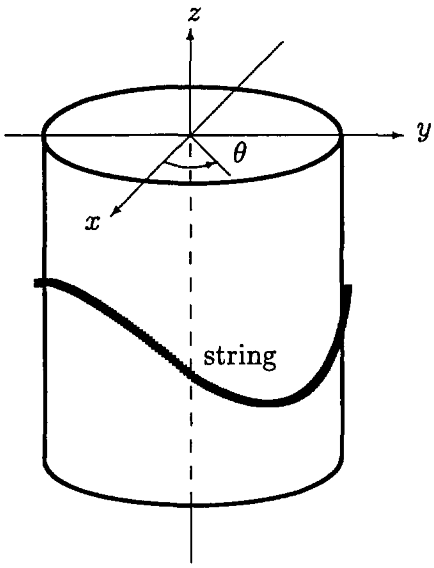
The temperature T of the illuminated area A changes with time t according to

$$c\frac{\mathrm{d}T}{\mathrm{d}t} = I - \alpha(T - T_0)\,,$$

where $T_0 = 293$K is the ambient temperature and $c = 170\text{J/Km}^2$, $\alpha = 250\text{W/Km}^2$ are constants. Assuming that the ignition point of the paper is $T_i = 506$K find the range of distances l for which the paper will start to burn. How long will the light need to heat up the paper to T_i? [9]

Explain what happens when $l = f$. Give a realistic estimate of the maximally achievable intensity I for a mean wavelength of $\lambda = 500$nm taking into account the resolution of the lens. What is the shortest possible time needed to heat the paper to T_i? [6]

10. A circular closed string of linear mass density ρ is held under tension T on a smooth cylinder of radius R as shown in the figure. The string can undergo small transverse vibrations on the cylinder surface in the z-direction. Any effects caused by the curvature of the cylinder may be neglected. Write down the wave equation for transverse vibrations of displacement $z(\theta, t)$ with $0 \leq \theta \leq 2\pi$ denoting the polar angle and t the time. State the boundary conditions obeyed by the string displacement z at $\theta = 0$ and $\theta = 2\pi$. [5]



For which values of k and $\omega > 0$ is the complex function $z(\theta, t) = A \exp[\mathrm{i}(k\theta - \omega t)]$, with A the constant complex amplitude, a solution of the wave equation which obeys the boundary conditions? Give a physical interpretation of ω and k. Write down the two smallest allowed values ω_1 and $\omega_2 > \omega_1$ and the associated values of k explicitly. Find a superposition of solutions with $\omega = \omega_2$ for which $z(\pi/3, t) = 0$ at all times. At which other values of θ will the string not be displaced at any time? [12]

Is $z(\theta) = A$ for $\omega = 0$ and $k = 0$ a valid solution? Find the general solution with $\partial^2 z / \partial t^2 = 0$. What kind of motion does this solution describe? [3]

11. The angular frequency of waves in a dispersive medium is given by

$$\omega(k) = \sqrt{gk \tanh(kh)}$$

where k is the wave number and g and h are constants. Find expressions for the phase velocity v_p and the group velocity v_g. Simplify these expressions for v_p and v_g in the limiting cases $h \gg 1/k$ and $h \ll 1/k$. [6]

By considering the superposition of two waves of different wavelength explain how wave packets will disperse in these two limiting cases, respectively. [10]

Calculate the time that the peak of a wave packet with mean frequency 0·2Hz needs to travel a distance of 1000km for $h = 1$km and $g = 10\text{m/s}^2$. [4]

12. Two identical masses $m_1 = m_2 = m$ are connected by a massless spring with spring constant k. Mass m_1 is attached to a support by another massless spring with spring constant $2k$. The masses and springs lie along the horizontal x-axis on a smooth surface. The masses and the support are allowed to move along the x-axis only. The displacement of the support in the x-direction at time t is given by $f(t)$ and is externally controlled. Write down a system of differential equations describing the evolution of the displacements x_1 and x_2 of the masses from their equilibrium positions. [5]

Determine the frequencies of the normal modes and their amplitude ratios. [8]

The displacement of the support is given by $f(t) = A\sin(\omega t)$ with $\omega^2 = k/m$ and constant amplitude A. Find expressions for $x_1(t)$ and $x_2(t)$ assuming that any transients have been damped out by a small, otherwise negligible, damping term. [7]

CPSC 4277
BPHP 4277

FIRST PUBLIC EXAMINATION

TRINITY TERM

Preliminary Examination in Physics

Paper CP3: MATHEMATICAL METHODS

also

Moderations in Physics and Philosophy

Wednesday, 7 June 2006, 2.30 pm – 5.00 pm

Time allowed: $2\frac{1}{2}$ hours

Answer **all** *of Section A and* **three** *questions from Section B.*

The use of calculators is **not** *permitted.*

A list of physical constants, mathematical formulae and conversion factors accompanies this paper.

The numbers in the margin indicate the weight which the Moderators expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Section A

1. Given that the (2×2) matrix $\mathbf{A}$ satisfies

$$\mathbf{A}\begin{pmatrix} -1 & 5 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix}$$

find the matrix $\mathbf{A}^{-1}$. [3]

2. The points $\mathbf{A} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\mathbf{A}' = \begin{pmatrix} x' \\ y' \end{pmatrix}$ are related by the equation

$$\mathbf{A}' = \mathbf{R}\mathbf{A}$$

where $\mathbf{R}$ is the (2×2) matrix, $\mathbf{R} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. By expressing $x^2 + y^2$ and $x'^2 + y'^2$ in terms of $\mathbf{A}$ and $\mathbf{A}'$ respectively show that $x^2 + y^2 = x'^2 + y'^2$ and hence find the inverse, $\mathbf{R}^{-1}$, of $\mathbf{R}$. [4]

3. The quantity I of a uniform solid body of density ρ is given by the integral

$$I = \int_V (x^2 + y^2 + z^2)\rho \, \mathrm{d}V$$

over the volume, V, of the body. Find I for a sphere of radius r and of uniform density ρ. [4]

4. Evaluate the surface integral $\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ over the hemispherical surface $x^2 + y^2 + z^2 = a^2$, $z \geq 0$, for the case $\mathbf{F} = -x\,\mathbf{i} + z(\mathbf{j} + \mathbf{k})$ in two ways:

(a) directly, and

(b) by use of the divergence theorem. [6]

5. The field $\mathbf{H}$ of a magnetic dipole of moment $\mathbf{M}$ placed at the origin is given by $\mathbf{H} = -\nabla\Omega$, where Ω is the magnetostatic potential given by $\Omega = -\frac{1}{4\pi}\mathbf{M} \cdot \nabla(\frac{1}{r})$. Find an expression for $\mathbf{H}$ in terms of $\mathbf{M}$ and $\mathbf{r}$ where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}|$. [5]

6. The first law of thermodynamics may be expressed as

$$dU = T\,dS - p\,dV$$

where U is the internal energy of the substance and any two of the four quantities, the pressure, p, the volume, V, the temperature, T and the entropy S, can be treated as independent variables. Find $\left(\frac{\partial U}{\partial S}\right)_p$ and $\left(\frac{\partial U}{\partial p}\right)_S$. Assuming that second order differentials do not depend on the order of differentiation prove the relation

$$\left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial V}{\partial S}\right)_p .$$ [5]

7. Derive an expression for the shortest distance between the lines $\mathbf{r}_1$ and $\mathbf{r}_2$ where

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{a}_1 + \lambda_1\,\mathbf{b}_1 \\ \mathbf{r}_2 &= \mathbf{a}_2 + \lambda_2\,\mathbf{b}_2. \end{aligned}$$ [4]

8. Under what condition do the linear equations

$$\begin{aligned} 4x + y + z &= 3 \\ x + \beta y + 2z &= 2 \\ 3x - y - z &= \gamma \end{aligned}$$

have a unique solution? If this condition is **not** satisfied, under what condition is there a solution to these equations? Give a geometrical interpretation of the solution in this case. [5]

9. Find the flux, $\int_S \mathbf{F} \cdot d\mathbf{S}$ passing through a surface, S, bounded by the curve $x^2 + y^2 = 4$ for the case $\mathbf{F} = \nabla \times (-y\mathbf{i} + x\mathbf{j})$. [4]

Section B

10. For a conservative force the integral $\int \mathbf{F}\cdot d\mathbf{l}$ around any closed loop is zero. Show that this is true if $\mathbf{F}$ can be written as the gradient of a potential $\phi(x, y, z)$, i.e. $\mathbf{F} = -\nabla\phi$. [4]

In which of the following cases is $\mathbf{F}$ conservative:

(a) $\mathbf{F} = [\mathbf{i} - \mathbf{j} - 2\mathbf{r}(x - y)]\, e^{-r^2}$

(b) $\mathbf{F} = z\mathbf{k} + \frac{1}{r^2}\mathbf{r} \times \mathbf{k}$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}|$. [6]

If $\mathbf{F}$ is conservative determine the associated potential and evaluate $\int_A^B \mathbf{F}\cdot d\mathbf{l}$ where $A = (x_0, y_0, z_0)$ and $B = (x_1, y_1, z_1)$. [10]

11. A particle with time dependent position vector $\mathbf{r}(t)$ has velocity vector $\mathbf{v}(t)$ $(= \dot{\mathbf{r}}(t))$ and is constrained to move in the xy-plane at constant speed. Show, by computing the direction of the force acting on the particle, that no work is done on it in the absence of frictional forces. [5]

Consider now the case where the particle, initially at the origin, moves in three dimensions such that

$$\frac{d\mathbf{v}(t)}{dt} = \alpha\mathbf{k} \times \mathbf{v}(t).$$

Prove that

$$\frac{dv^2}{dt} = 0.$$

[3]

Determine $\mathbf{r}(t)$ given that $\mathbf{v}(0) = A\mathbf{i} + B\mathbf{k}$. [12]

12. In the xy-plane sketch the area, A, specified by the conditions: $x-1 \leq y \leq x$ and $0 \leq y \leq \frac{1}{x}$. [5]

The variables u and v are defined by

$$\begin{aligned} u &= x-y \\ v &= xy. \end{aligned}$$

Sketch the area A in the uv-plane. [5]

A function $f(x,y)$ of the variables x, y may also be considered to be a function $F(u,v)$ of the variables u, v. Show that

$$\int\int_A \mathrm{d}x\,\mathrm{d}y\,(x+y)f(x,y) = \int\int_A \mathrm{d}u\,\mathrm{d}v\,F(u,v).$$ [4]

Evaluate the integral

$$\int\int_A \mathrm{d}x\,\mathrm{d}y\,(x^2-y^2)\,e^{(x^2-xy+y^2)}.$$ [6]

13. Write down the terms of the Taylor expansion of the function $f(x,y)$ about the point (a,b) up to and including the second order. [3]

Referring to this expansion discuss how the stationary points of the function may be identified and determine the conditions for a stationary point to be a maximum, minimum or saddle point. [7]

Determine the location and nature of the stationary points of the function

$$f(x,y) = 1 + xy\,e^{-x-y^2}.$$

Sketch the contours. [10]