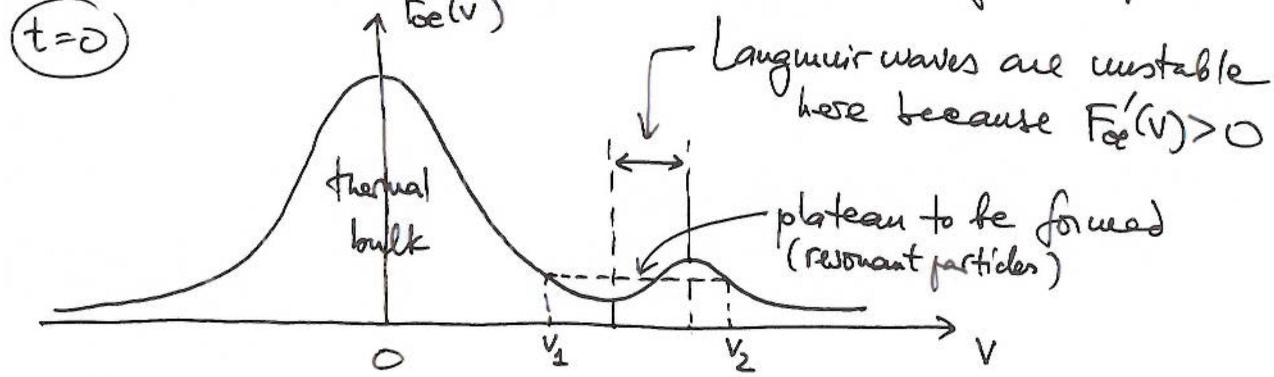
§5. Quasilinear Relaxation in 1D.

Let us consider the original QL problem: saturation of the bump-on-tail instability (or beam instability) in 1D
[Vedenov, Velikhov, Sagdeev, Nucl. Fusion Suppl. 2, 465 (1962)
Drummond & Pines, Nucl. Fusion Suppl. 3, 1049 (1962)].

Let us start with a distribution function of this form:



$$\boxed{\frac{\partial}{\partial t}|\vec{E}_k|^2 = 2\gamma_k |\vec{E}_k|^2}, \quad \gamma_k = \frac{\pi}{2}\frac{\omega_k^3}{k^2}\frac{1}{n_{0e}} F_{0e}'\left(\frac{\omega_k}{k}\right) \tag{45}$$

$$\omega_k^2 = \omega_{pe}^2 = \frac{4\pi e^2 n_{0e}}{m_e}$$ [see eq. (17) & (19), pp. 10–11]

In the regions where $F_{0e}'(v)<0$, fluctuations with $k = \frac{\omega_{pe}}{v}$ will be decaying away via Landau damping

— " — $F_{0e}'(v)>0$, — " —

will be unstable and growing.

In order to have a steady state, we have to have

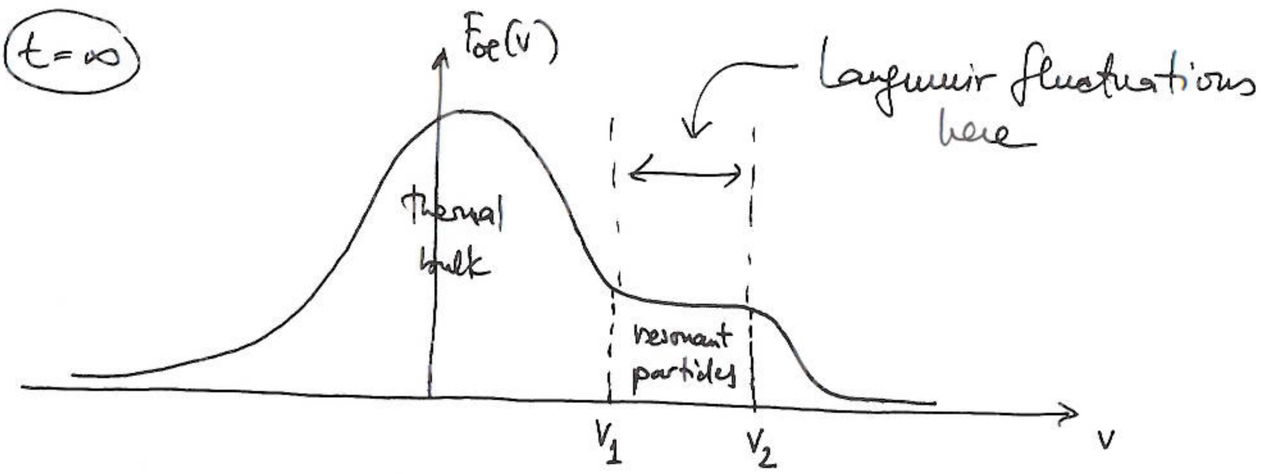
$$2\gamma_k|\vec{E}_k|^2 = 0, \text{ i.e. either } |\vec{E}_k|^2 = 0$$

$$\text{or } \gamma_k = 0$$

The latter condition means $F_{0e}'(v) = 0$, i.e. a plateau must form in the distribution function:

The saturated solution should look like this:

$(t=\infty)$



and we should find that the fluctuation spectrum is only non-zero for $k \in \left[\frac{\omega_{pe}}{v_2}, \frac{\omega_{pe}}{v_1}\right]$.

Let us now calculate the new distribution function and the spectrum of fluctuations at $t=\infty$.

QL evolution of the distribution function is, from eq. (38),

$$\boxed{\frac{\partial F_{0e}}{\partial t} = \frac{\partial}{\partial v} D_{0e}(v) \frac{\partial F_{0e}}{\partial v}}, \tag{46}$$

where [eq. (39)]

$$D_e(v) = \frac{e^2}{m_e^2} \sum_k \underbrace{k^2 |\varphi_k|^2}_{= |\vec{E}_k|^2} \underbrace{\frac{\gamma_k}{(\omega_k - kv)^2 + \gamma_k^2}}_{= \operatorname{Im} \frac{1}{\omega_k - kv - i\gamma_k}} =$$

$$= \frac{e^2}{m_e^2} \sum_k |\vec{E}_k|^2 \Bigg[\underbrace{\pi \delta(\omega_k - kv)}_{\substack{\uparrow \\ \text{resonant particles}}} + \underbrace{P \frac{\gamma_k}{(\omega_k - kv)^2 + \gamma_k^2}}_{\substack{\uparrow \\ \text{nonresonant particles}}}\Bigg] \tag{46a}$$

Let us first consider velocities for which $|\vec{E}_{k=\omega_{pe}/v}|^2 \neq 0$ $\Rightarrow$ the nonresonant term is small,

$$D_e(v) \approx \frac{e^2}{m_e^2} \sum_k |\vec{E}_k|^2 \pi \delta(\omega_k - kv) =$$

$$= \frac{e^2}{m_e^2}\frac{L}{2\pi}\int dk\,\pi\,\delta(\omega_{pe}-kv)\,|\vec{E}_k|^2 \equiv$$

define spectrum

$$W(k) \equiv \frac{L}{2\pi}|\vec{E}_k|^2$$

$$= \frac{e^2}{m_e^2}\int dk\,\pi\,\delta(\omega_{pe}-kv)\,W(k) = \frac{e^2}{m_e^2}\frac{\pi}{v}W\left(\frac{\omega_{pe}}{v}\right) \qquad (46b)$$

From (46), we have

$$\frac{\partial F_{oe}}{\partial t} = \frac{\partial}{\partial v}D_e(v)\underbrace{\frac{\partial F_{oe}}{\partial v}}_{=\left[\frac{2}{\pi}\frac{k^2}{\omega_{pe}^3}n_{oe}\gamma_k\right]_{k=\omega_{pe}/v}} = \frac{\partial}{\partial v}\frac{e^2}{m_e^2}\frac{\cancel{\pi}}{v}\frac{\cancel{2}}{\cancel{\pi}}\frac{\omega_{pe}^2}{v^2}\frac{1}{\omega_{pe}^3}n_{oe}\underbrace{\gamma_{\omega_{pe}/v}W\left(\frac{\omega_{pe}}{v}\right)}_{=\frac{1}{\cancel{2}}\frac{\partial}{\partial t}W\left(\frac{\omega_{pe}}{v}\right)\ \text{eq. (45)}}$$

So,

$$\frac{\partial}{\partial t}\left[F_{oe} - \frac{\partial}{\partial v}\frac{e^2 n_{oe}}{m_e^2\omega_{pe}v^3}W\left(\frac{\omega_{pe}}{v}\right)\right] = 0$$

$$F_{oe} - \frac{\partial}{\partial v}\frac{e^2 n_{oe}}{m_e^2\omega_{pe}v^3}W\left(\frac{\omega_{pe}}{v}\right) = \text{const (in time)} = F_{oe}(t=0,v) \qquad (47)$$

(assuming very small initial wave energy)

Then, at $t=\infty$,

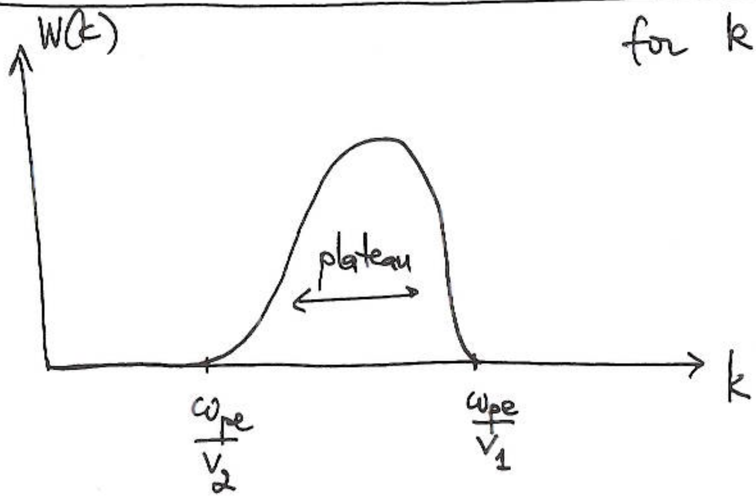
$$W\left(\frac{\omega_{pe}}{v}\right) = \frac{m_e^2\omega_{pe}v^3}{e^2 n_{oe}}\int_{v_1}^{v} dv'\left[F_{oe}(t=\infty,v') - F_{oe}(t=0,v')\right] \qquad (48)$$

↑ const for res. particles

↑ lower boundary of the plateau

$$\boxed{W(k) = \frac{m_e^2\omega_{pe}^4}{e^2 n_{oe}}\frac{1}{k^3}\int_{v_1}^{\omega_{pe}/k} dv\left[F_{oe}(\infty) - F_{oe}(0,v)\right]} \qquad (49)$$

for $k \in \left[\frac{\omega_{pe}}{v_2}, \frac{\omega_{pe}}{v_1}\right]$



and $W(k)=0$ for all other k.

We must have

$$W\left(\frac{\omega_{pe}}{v_1}\right) = W\left(\frac{\omega_{pe}}{v_2}\right) = 0$$

In (49), we must impose that $W\left(\frac{\omega_{pe}}{V_2}\right)=0$, which gives us res. particle conservation.

- Number of resonant particles: $\int dv$ (47)

$$\int_{V_1}^{V_2} dv\, F_{oe} - \underbrace{\left[\frac{e n_{oe}}{m_e^2 \omega_{pe} V^3} W\left(\frac{\omega_{pe}}{V}\right)\right]_{V_1}^{V_2}}_{0} = \int_{V_1}^{V_2} dv\, F_{oe}(t=0,v)$$

conserved

this determines plateau parameters

so, at $t=\infty$, $F_{oe}(\infty) = \frac{1}{V_2 - V_1}\int_{V_1}^{V_2} dv\, F_{oe}(t=0,v)$

and, in fact, also

$$F_{oe}(\infty) = F_{oe}(t=0,V_1) = F_{oe}(t=0,V_2)$$

(50)

These 3 relations determine V_1, V_2 and $F_{oe}(\infty)$.

- Energy of resonant particles: $\int dv \frac{m_e V^2}{2}$ (47)

$$\int_{V_1}^{V_2} dv \frac{m_e V^2}{2} F_{oe}(\infty) - \int_{V_1}^{V_2} dv \frac{m_e V^2}{2} F_{oe}(t=0,v) =$$

$$= \int_{V_1}^{V_2} dv \frac{m_e V^2}{2} \frac{\partial}{\partial V} \frac{e^2 n_{oe}}{m_e^2 \omega_{pe} V^3} W\left(\frac{\omega_{pe}}{V}\right) =$$

$$= -\int_{V_1}^{V_2} dv \frac{e^2 n_{oe}}{m_e \omega_{pe} V^2} W\left(\frac{\omega_{pe}}{V}\right) = -\int_{\omega_{pe}/V_2}^{\omega_{pe}/V_1} dk\, \underbrace{\frac{e^2 n_{oe}}{m_e \omega_{pe}^2}}_{=\frac{1}{4\pi}} W(k) =$$

$$\boxed{\begin{aligned} k &= \frac{\omega_{pe}}{V} \\ dk &= -\frac{\omega_{pe}}{V^2} dv \end{aligned}}$$

$$= -\frac{1}{4\pi}\int_{\omega_{pe}/V_2}^{\omega_{pe}/V_1} dk\, W(k) = -2\,\mathcal{E}_{waves}(\infty)$$

So, we find that

$$\mathcal{E}_{res}(\infty) - \mathcal{E}_{res}(0) = -2\,\mathcal{E}_{waves}(\infty)$$

or

$$\boxed{\frac{\mathcal{E}_{res}(0) - \mathcal{E}_{res}(\infty)}{2} = \mathcal{E}_{waves}(\infty)}, \qquad (51)$$

i.e. $\frac{1}{2}$ the energy lost by the resonant particles goes into the waves. Where does the rest go?

Into nonresonant thermal particles! (bulk of the distribution)

Let us consider them now.

The velocities in the thermal bulk are $v \ll v_1$ (assuming the beam was far out on the tail).

The diffusion coefficient at these velocities [Eq. (46a)] is small – contribution from nonresonant particles is subdominant. However, there are many more of these particles than of the resonant ones, so they cannot be neglected. We have:

$$D_e(v) \approx \frac{e^2}{m_e^2} \sum_k |\vec{E}_k|^2\, \mathcal{P} \frac{\gamma_k}{(\omega_k - kv)^2 + \gamma_k^2} \underset{\substack{\omega_k = \omega_{pe} \gg kv \\ \text{in thermal bulk}}}{\approx} \frac{e^2}{m_e^2} \underbrace{\sum_k \gamma_k |\vec{E}_k|^2}_{= \frac{\partial}{\partial t}\frac{1}{2}|\vec{E}|^2} \frac{1}{\omega_{pe}^2} =$$

$$= \frac{e^2}{2m_e^2\omega_{pe}^2} \frac{\partial |\vec{E}|^2}{\partial t} \quad \text{independent of } v. \qquad (52a)$$

From (46), we have

$$\frac{\partial F_{0e}}{\partial t} = \frac{e^2}{2m_e^2\omega_{pe}^2} \frac{\partial |\vec{E}|^2}{\partial t} \frac{\partial^2 F_{0e}}{\partial v^2} \qquad (52)$$

- Number of these particles is again conserved, $\int dv$ (52), as it should be, so they do not interfere with the resonant ones.

- Energy of thermal particles: $\int dv \frac{m_e v^2}{2}$ (52)

$$\frac{\partial}{\partial t}\int dv \frac{m_e v^2}{2} F_{oe} \equiv \frac{\partial \mathcal{E}_{th}}{\partial t} = \frac{e^2}{2m_e^2\omega_{pe}^2}\frac{\partial |\vec{E}|^2}{\partial t}\underbrace{\int dv \frac{m_e v^2}{2}\frac{\partial^2 F_{oe}}{\partial v^2}}_{=\, m_e \int dv F_{oe} \simeq m_e n_{oe}} =$$

(assuming $n_{oe}^{(res)} \ll n_{oe}^{(th)}$)

$$= \underbrace{\frac{e^2 n_{oe}}{2m_e\omega_{pe}^2}}_{=\frac{1}{8\pi}}\frac{\partial |\vec{E}|^2}{\partial t} = \frac{\partial}{\partial t}\frac{|\vec{E}|^2}{8\pi}$$

So
$$\boxed{\mathcal{E}_{th}(\infty) - \mathcal{E}_{th}(0) = \mathcal{E}_{waves}(\infty) \overset{\text{eq.(51)}}{=} \frac{\mathcal{E}_{res}(0) - \mathcal{E}_{res}(\infty)}{2}} \quad (53)$$

and we have shown that half the energy of the resonant particles goes into the thermal bulk (the other half goes into waves, so $\mathcal{E}_{th}(\infty) + \mathcal{E}_{res}(\infty) + \mathcal{E}_{waves} = \mathcal{E}_{th}(0) + \mathcal{E}_{res}(0)$)

Note. There is, btw, a nice ~~calculation~~ demonstration one can do to show how the heating by waves is happening (this also applies to the nonresonant particle behaviour in many other problems). From (52), we get

$$\frac{\partial F_{oe}}{\partial t} = \frac{1}{m_e n_{oe}}\frac{\partial}{\partial t}\frac{|\vec{E}|^2}{8\pi}\frac{\partial^2 F_{oe}}{\partial v^2}$$

Let $\tau = \frac{|\vec{E}|^2}{8\pi m_e n_{oe}} \Rightarrow \frac{\partial F_{oe}}{\partial \tau} = \frac{\partial^2 F_{oe}}{\partial v^2}$ diffusion equation

$$F_{oe}(\tau, v) = \int_{-\infty}^{+\infty} dv' F_{oe}(0,v')\frac{e^{-\frac{(v-v')^2}{4\tau}}}{\sqrt{4\pi\tau}} = \int_{-\infty}^{+\infty} dv' \frac{n_{oe}\, e^{-\frac{v'^2}{v_{the}^2} - \frac{(v-v')^2}{4\tau}}}{\sqrt{\pi v_{the}^2 \cdot 4\pi\tau}} =$$

start with a Maxwellian

$$= \int_{-\infty}^{+\infty} \frac{dv' n_{oe}}{\sqrt{\pi^2 v_{the}^2 4\tau}} \, e^{-\left[v'\sqrt{\frac{1}{v_{the}^2}+\frac{1}{4\tau}} - \frac{v}{4\tau\sqrt{\frac{1}{v_{the}^2}+\frac{1}{4\tau}}}\right]^2 + \underbrace{\frac{v^2}{4\tau + \frac{16\tau^2}{v_{the}^2}} - \frac{v^2}{4\tau}}_{= \frac{v^2}{\cancel{4\tau}\left(1+\frac{4\tau}{v_{the}^2}\right)}\left[\cancel{1} - \cancel{1} - \frac{\cancel{4\tau}}{v_{the}^2}\right]}}$$

$$= \frac{n_{oe}}{\sqrt{\pi^2 v_{the}^2 4\tau}} \frac{\sqrt{\pi}}{\sqrt{\frac{1}{v_{the}^2}+\frac{1}{4\tau}}} e^{-\frac{v^2}{v_{the}^2+4\tau}} = \frac{n_{oe}}{\sqrt{\pi\left(v_{the}^2+4\tau\right)}} e^{-\frac{v^2}{v_{the}^2+4\tau}}$$

– a Maxwellian with a new effective temperature:

$$v_{the}^2 + 4\tau = \frac{2T_e}{m_e} + \frac{|\vec{E}|^2}{2\pi m_e n_{oe}} = \frac{2}{m_e}\Big(\underbrace{T_e + \frac{|\vec{E}|^2}{4\pi n_{oe}}}_{= T_e^{(eff)}}\Big)$$

So

$$T_e^{(eff)} = T_e^{(particles)} + T^{(waves)}, \qquad (54)$$

where $T^{(waves)} = \dfrac{|\vec{E}|^2}{4\pi n_{oe}}$ [in 3D, this is $\dfrac{2}{3}\cdot\dfrac{|\vec{E}|^2}{8\pi n_{oe}}$]

Finally, how about momentum conservation?

- Momentum of resonant particles: $\int dv\, m_e v$ (47)

$$\int_{v_1}^{v_2} dv\, m_e v\, F_{oe}(\infty) - \int_{v_1}^{v_2} dv\, m_e v\, F_{oe}(0,v) = \int_{v_1}^{v_2} dv\, m_e v \frac{\partial}{\partial v}\frac{e^2 n_{oe}}{m_e^2 \omega_{pe} v^3} W\left(\frac{\omega_{pe}}{v}\right)$$

$$= -\int_{v_1}^{v_2} dv \frac{e^2 n_{oe}}{m_e \omega_{pe} v^3} W\left(\frac{\omega_{pe}}{v}\right) \underset{k=\frac{\omega_{pe}}{v}}{=} -\int_{\omega_{pe}/v_2}^{\omega_{pe}/v_1} dk \underbrace{\frac{e^2 n_{oe}}{m_e \omega_{pe}^2}}_{=\frac{1}{4\pi}} \frac{k}{\omega_{pe}} W(k) =$$

$$= -\frac{1}{4\pi}\int_{\omega_{pe}/v_2}^{\omega_{pe}/v_1} dk \left.\frac{kW(k)}{\omega_{pe}}\right|_{t=\infty} = P_{res}(\infty) - P_{res}(0) \qquad (55)$$

In order to calculate the momentum of the thermal particles, we have to go to higher order in our expansion of the diffusion coefficient on p. 35:

$$D_e(v) \simeq \frac{e^2}{m_e^2}\sum_k |\vec{E}_k|^2 \,\mathcal{P}\,\frac{\gamma_k}{(\omega_k-kv)^2+\gamma_k^2} \simeq \frac{e^2}{m_e^2}\sum_k \underbrace{\gamma_k|\vec{E}_k|^2}_{=\frac{1}{2}\frac{\partial}{\partial t}|\vec{E}_k|^2}\frac{1}{\omega_{pe}^2}\left(1+\frac{2kv}{\omega_{pe}}+\dots\right)$$

$$= \underbrace{\frac{e^2}{2m_e^2\omega_{pe}^2}}_{=\frac{1}{m_e n_{oe}}\frac{1}{8\pi}}\left[\frac{\partial|\vec{E}|^2}{\partial t}+2v\frac{\partial}{\partial t}\underbrace{\sum_k\frac{k}{\omega_{pe}}|\vec{E}_k|^2}_{=\int dk\,\frac{kW(k)}{\omega_{pe}}}\right] =$$

$$= \frac{1}{m_e n_{oe}}\left[\frac{\partial}{\partial t}\frac{|\vec{E}|^2}{8\pi}+v\frac{\partial}{\partial t}\frac{1}{4\pi}\int dk\,\frac{kW(k)}{\omega_{pe}}\right]$$

So, the more precise version of eq. (52) is

$$\frac{\partial F_{oe}}{\partial t} = \frac{1}{m_e n_{oe}}\frac{\partial}{\partial v}\left[\frac{\partial}{\partial t}\frac{|\vec{E}|^2}{8\pi}+v\frac{\partial}{\partial t}\frac{1}{4\pi}\int dk\,\frac{kW(k)}{\omega_{pe}}\right]\frac{\partial F_{oe}}{\partial v} \qquad (56)$$

• <u>Momentum of thermal particles</u>: $\int dv\; m_e v\,(56)$

$$\frac{\partial}{\partial t}\int dv\; m_e v F_{oe} \equiv \frac{\partial \mathcal{P}_{th}}{\partial t} = \frac{1}{n_{oe}}\int dv\, v\frac{\partial}{\partial v}\left[\dots\right]\frac{\partial F_{oe}}{\partial v} =$$

$$= -\frac{1}{n_{oe}}\int dv\left[\cancel{\frac{\partial}{\partial t}\frac{|\vec{E}|^2}{8\pi}}+v\frac{\partial}{\partial t}\frac{1}{4\pi}\int dk\,\frac{kW(k)}{\omega_{pe}}\right]\frac{\partial F_{oe}}{\partial v}$$

vanishes under integration

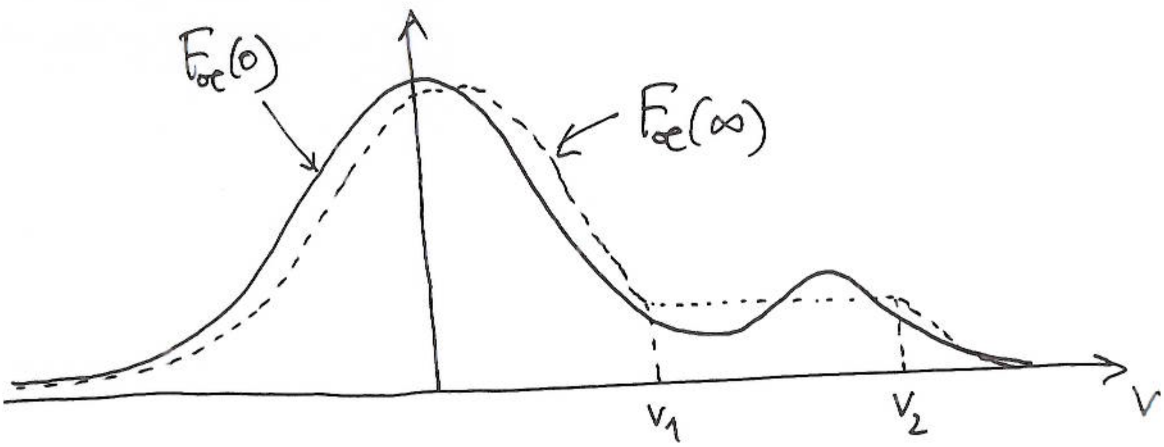
$$= \frac{1}{\cancel{n_{oe}}}\frac{\partial}{\partial t}\frac{1}{4\pi}\int dk\,\frac{kW(k)}{\omega_{pe}}\cancel{\int dv F_{oe}} = \frac{\partial}{\partial t}\frac{1}{4\pi}\int dk\,\frac{kW(k)}{\omega_{pe}}$$

So, using (55),

$$\boxed{\mathcal{P}_{th}(\infty) - \mathcal{P}_{th}(0) = \frac{1}{4\pi}\int dk \left.\frac{k W(k)}{\omega_{pe}}\right|_{t=\infty} = \mathcal{P}_{res}(0) - \mathcal{P}_{res}(\infty)} \quad (57)$$

We have shown that all of the momentum of the resonant particles gets transferred to the thermal bulk.

So, the new distribution is slightly shifted towards the plateau compared to the initial distribution:



NB: In a collisionless plasma (1D), this is the steady state. However, as this saturation is approached, $\gamma_k \to 0$, so the quasi-linear evolution becomes very slow and even a very small collision frequency can become important – it will eventually erode the plateau and return the plasma to a Maxwellian equilibrium.