§3. Phase Mixing

From the linear theory of §2, we have learned that in a collisionless plasma, small-amplitude (electrostatic) fluctuations will have the character of waves (Langmuir, ion-acoustic) that are either Landau-damped or unstable—depending on the eq. distribution $f_{eq}$.

In the linear theory, $f_{eq}$ was considered to be constant in time—this was OK for small-amplitude fluctuations and only for a limited time.

Clearly, after some time (related to the fluctuation amplitude), the fluctuations will start affecting the equilibrium via the rhs in eq. (7):

$$\frac{\partial f_{eq}}{\partial t} = -\frac{q_s}{m_s} \sum_{k'} \left( q_{k'} \frac{\partial}{\partial V} \frac{\partial f_{eq}^{k'}}{\partial V} \right)$$  (7)

So now we could take the solutions for $q_{k'}$ and $f_{eq}^{k'}$ calculated in the linear theory, substitute into (7), average over time, and get the evolution of $f_{eq}$.

This is the algorithm for deriving quasi-linear theory, but before we do this, I would like to explain some nuances about what it
actually means that fluctuations are "collisionally" damped (or unstable). It is understood that in the process of such damping, energy is transferred from fluctuations to particles (or vice versa for unstable equilibria). Let us examine what actually happens to energy.

Let us go back to the original system of eqs (2-3):

\[
\begin{align*}
\frac{\partial f_s}{\partial t} + \nabla \cdot \vec{v} f_s &= -\frac{q_s}{m_s} (\nabla \psi) \cdot \frac{\partial f_s}{\partial \vec{v}} = \left(\frac{\partial f_s}{\partial \vec{v}}\right)_c \\
-\nabla^2 \psi &= 4\pi \sum_s q_s \int d^3 \vec{v} f_s 
\end{align*}
\]

(2)

(3)

From (3), we have

\[-\nabla^2 \frac{\partial \psi}{\partial t} = 4\pi \sum_s q_s \int d^3 \vec{v} \frac{\partial f_s}{\partial t} =
\]

\[= 4\pi \sum_s q_s \int d^3 \vec{v} \left[ -\nabla \cdot \vec{v} f_s + \frac{q_s}{m_s} (\nabla \psi) \frac{\partial f_s}{\partial \vec{v}} + \left(\frac{\partial f_s}{\partial \vec{v}}\right)_c \right]
\]

\[= -\nabla \cdot 4\pi \sum_s q_s \int d^3 \vec{v} \vec{v} f_s \quad \text{o because} \quad \psi(\pm 0) = 0 \quad \text{o because particle # is constant}
\]

Obviously, for this current density is just \(+\vec{v}[\text{Ampère-Maxwell law with } \vec{B} = 0]\):

\[\frac{\hat{E}}{4\pi} \cdot \frac{\partial \hat{E}}{\partial t} = -4\pi \hat{J}
\]

(22)

\[\frac{\partial}{\partial t} \frac{\hat{E}^2}{8\pi} = -\hat{E} \cdot \hat{J} = -\sum_s q_s \int d^3 \vec{v} \hat{E} \cdot \vec{v} f_s
\]

(23)

work done on the particles

"heating"
Where does all this energy go?

Consider the evolution of the total particle energy:

\[
\frac{dE}{dt} = \frac{d}{dt} \sum_s \int d^3\mathbf{r} \int d^3\mathbf{V} \frac{m^2 v^2}{2} f_s = \sum_s \int d^3\mathbf{r} \int d^3\mathbf{V} \frac{m^2 v^2}{2} \left[ -\nabla \cdot \mathbf{V} f_s - \frac{q_s}{m_s} \mathbf{E} \cdot \frac{df_s}{dV} \right] + \left( \frac{df_s}{dt} \right)_c =
\]

0 if we neglect fluxes through the boundary (OK for inf. or periodic system) 0 because collisions conserve energy

\[
= \sum_s q_s \int d^3\mathbf{r} \int d^3\mathbf{V} \mathbf{E} \cdot \mathbf{V} f_s - \text{OK, here's the energy} \quad (24)
\]

Energy conservation:

\[
\frac{d}{dt} \left( \frac{E}{c^2} + \sum_s \int d^3\mathbf{r} \frac{|\mathbf{E}|^2}{8\pi} \right) = 0 \quad (25)
\]

Now recall that \( E_k = -iE_k \) and \( \phi_k(t) = \sum_i c_i \phi_i^e t \) [eq. (12)]

where \( \phi_k(E) = -iE_k \phi_k^{(i)} + \phi_k^{(i)*} \) are solutions of \( E(\phi_k, E) = 0 \)

and \( \phi_k(E) = \phi_k^{(i)}(0) \) is the initial amplitude for mode \( i \).

Then

\[
\frac{|E_k|^2}{8\pi} = \sum_{ij} \frac{k^2 |c_i|^2}{8\pi} \int d^3\mathbf{r} \frac{(\phi_k^{(i)}(\mathbf{r}, 0) \phi_k^{(j)}(\mathbf{r})^* - i(w_k, w_k^{(i)}) t + (w_k^{(i)}, w_k^{(j)}) t) \phi_k^{(j)}(\mathbf{r}, 0) \phi_k^{(i)}(\mathbf{r})^*}}{\text{Oscillatory}}
\]

Now, if we time average this, we'll get

\[
0 \text{ for all terms except } i = j, \text{ so}
\]

\[
\frac{|E_k|^2}{8\pi} = \sum_i \frac{k^2 |c_i|^2}{8\pi} |\phi_k^{(i)}(0)|^2 e^{-2 \gamma_k t}
\]
Then \[
\frac{d}{dt} \frac{\mathcal{E}_k}{8\pi} = \sum_i 2\gamma_k^{(i)} \frac{|\mathcal{E}^{(i)}_k|^2}{8\pi}
\]

From Parcell's theorem, \[
\int d^3\mathbf{r} \frac{|\mathcal{E}|^2}{8\pi} = V \sum_k \frac{|\mathcal{E}_k|^2}{8\pi}
\]

We can now average the energy conservation law and use Eq. (26) for the avg. change in el. energy:

\[
\frac{d}{dt} \mathcal{E} = -\frac{d}{dt} \int d^3\mathbf{r} \frac{|\mathcal{E}|^2}{8\pi} = -V \sum_k \sum_i 2\gamma_k^{(i)} \frac{|\mathcal{E}^{(i)}_k|^2}{8\pi} > 0 \quad \text{if damping } (\gamma_k^{(i)} > 0)
\]

So, if waves are damped, the particle energy increases ("heating"). If there is an instability, particles can lose energy.

Let us now consider the simplest specific example:
- electrons in homogeneous Maxwellian equilibrium

\[
\mathcal{E}_{\text{e}} = \frac{n_{\text{e}}v^2}{(2\pi m_{\text{e}})^{3/2}} e^{-\frac{v^2}{2m_{\text{e}}}} = n_{\text{e}} \left(\frac{m_{\text{e}}}{2\pi T_{\text{e}}}ight)^{3/2} e^{-\frac{m_{\text{e}}v^2}{2T_{\text{e}}}}
\]

- ions motionless \((T_i = 0, \mathbf{v}_i = 0)\)

In such a system, we just have Landau-damped Langmuir waves. The electron energy is

\[
\mathcal{E}_{\text{e}} = V \int d^3\mathbf{r} \frac{n_{\text{e}}v^2}{2} \mathcal{E}_{\text{e}} = \frac{3}{2} V n_{\text{e}} T_{\text{e}}
\]

Since, in a homogeneous system, \(n_{\text{e}} = \text{const}\), we have

\[
\frac{d\mathcal{E}_{\text{e}}}{dt} = \frac{3}{2} V n_{\text{e}} \frac{dT_{\text{e}}}{dt} = -V \sum_k 2\gamma_k^{(i)} \frac{|\mathcal{E}_k|^2}{8\pi}
\]

So damping \(\Rightarrow\) heating of the equilibrium
There is another very important conservation law that all kinetic systems must respect. Consider the entropy of the plasma

\[ S = \frac{1}{\mathcal{Z}} \int d^3 \mathbf{v} \int d^3 \mathbf{r} f_s \ln f_s \]

The evolution eqn for \( S \) is derived by multiplying the kinetic equation by \(-(1 + \ln f_s)\) and integrating over the entire phase space.

Since \((1 + \ln f_s) \nabla f_s = 0\) (\(f_s \ln f_s\)), [derivative w.r.t. \(t\)]

we have

\[ \frac{dS}{dt} = - \int \int \int \left( \frac{\partial f_s}{\partial t} \right) \ln f_s \]

(29)

It is possible to prove that

1) Rhs of (29) \(>0\) \(\Rightarrow\) entropy never decreased by collisions (and, in eq. (29), is only changed by these)

- Boltzmann's H theorem

2) Rhs of (29) = 0 \(\Leftrightarrow\) \(f_s\) is a Maxwellian.

The law of entropy increase is related to irreversibility. In an exactly collisionless system, entropy is conserved and everything is reversible — so how can we have collisionless damping?
Let us go back to our simple example:

\[ f_e = f_{oe} + \delta f_e \]

\[ \text{homogeneous Maxwellian} \]

*small perturbation*

\[ S = - \int d^3 \vec{r} \int d^3 \vec{v} \left( f_e + \delta f_e \right) \ln \left( f_{oe} + \delta f_e \right) = \]

\[ = - \int d^3 \vec{r} \int d^3 \vec{v} \left( f_{oe} \ln f_{oe} + \frac{\ln f_{oe} + \ln \left( 1 + \frac{\delta f_e}{f_{oe}} \right)}{\delta f_e} \right) = \]

\[ = - \frac{\delta f_e}{f_{oe}} \int d^3 \vec{r} \int d^3 \vec{v} \left[ f_{oe} \ln f_{oe} + (1 + \ln f_{oe}) \delta f_e + \frac{\ln f_{oe}}{2} + \frac{\delta f_e^2}{f_{oe}} + \cdots \right] \]

\[ \downarrow \]

\[ S_0 \] vanishes upon time averaging

\[ \overline{S} = S_0 + \overline{dS} \]

Where

\[ S_0 = - \int d^3 \vec{r} \int d^3 \vec{v} f_{oe} \left[ \ln n_e \left( \frac{m_e}{2 \pi T_e} \right)^{3/2} - \frac{3}{2} \frac{m_e V^2}{2 T_e} \right] = \]

\[ = - V n_e \ln \left( n_e \left( \frac{m_e}{2 \pi T_e} \right)^{3/2} \right) + \frac{3}{2} V n_e \ln T_e + \frac{V}{T_e} \frac{3}{2} n_e T_e \]

and

\[ \overline{dS} = - \int d^3 \vec{r} \int d^3 \vec{v} \frac{\delta f_e^2}{2 f_{oe}} \]

Now time average eq. (2a) and substitute the above:

\[ \frac{d^3 \vec{r}}{dt} = \frac{dS_0}{dt} + \frac{d\overline{dS}}{dt} = - \int d^3 \vec{r} \int d^3 \vec{v} \left[ \left( \frac{\partial f_e}{\partial t} \right)_c + \left( \frac{\partial \delta f_e}{\partial t} \right)_c + \cdots \right] \left( \ln f_{oe} + \frac{\delta f_e}{f_{oe}} + \cdots \right) \]

\[ \frac{3}{2} V n_e \frac{dT_e}{T_e} \frac{dt}{dt} \quad (30) \]

(heating = eq. entropy increase)

\[ = - \int d^3 \vec{r} \int d^3 \vec{v} \frac{\delta f_e}{f_{oe}} \left( \frac{\partial \delta f_e}{\partial t} \right)_c \]

\[ \text{because Maxwellian (for e-e collisions)} \]
So we have derived
\[
\frac{3}{2} V\mathcal{Q}_{\text{oe}} \frac{d}{dt} \left( \frac{1}{T_{\text{oe}}} \right) - \frac{d}{dt} \left( \frac{1}{2} \int \int \int \int \frac{8\pi}{\text{foc}} \, d^3 \mathbf{v} \, \mathbf{v}^2 \right) = \int \int \int \frac{8\pi}{\text{foc}} \left( \frac{\mathcal{Q}_{\text{oe}}}{T_{\text{oe}}} \right) 
\]
and we have the full cons. law:
\[
\frac{d}{dt} \left( \int \int \int \int \frac{8\pi}{\text{foc}} \, d^3 \mathbf{v} \, \mathbf{v}^2 \right) = \int \int \int \frac{8\pi}{\text{foc}} \left( \frac{\mathcal{Q}_{\text{oe}}}{T_{\text{oe}}} \right) \leq 0 
\]
This is directly generalizable to multiple species.

So what would happen if we had an exactly collisionless system?

\[
\text{RHS of (31)} \rightarrow 0 \Rightarrow \frac{d}{dt} \left[ \int \int \int \int \frac{8\pi}{\text{foc}} \, d^3 \mathbf{v} \, \mathbf{v}^2 + \frac{1}{8\pi} \right] = 0 
\]
\[
\text{This must be growing then} \Leftarrow \text{this is damping if there is damping} 
\]
Indeed, damping \Rightarrow heating [Eq.(28)] \Rightarrow eq. entropy grows [Eq.(30)]

But total entropy is conserved [Eq.(29)] \Rightarrow growth of eq. entropy must be compensated by decrease of the entropy of the perturbed distribution.

\[
-\int \int \int \int \frac{8\pi}{\text{foc}} \, d^3 \mathbf{v} \, \mathbf{v}^2 \text{ grows} \Rightarrow \mathcal{Q}_{\text{oe}} \text{ is growing} .
\]

So the perturbation does not decay away and stays forever ...

We also need some sort of special arrangement because \( \mathcal{Q}_{\text{e}} = -\frac{4\pi}{k^2} \int \int \int \frac{8\pi}{\text{foc}} \) grows.

One way to achieve this is by having a free oscillate rapidly in \( \nabla \Delta \) space - this is indeed what happens ...
In fact, what turns out to happen (as we are about to see) is that the flow tends to develop small scales in velocity space—"phase mixing". But then, sooner or later, the collision term will become important because it is a 2-order diffusion operator in velocity space:

\[ \frac{\partial \mathbf{S}}{\partial t} = -\nabla \cdot (\mathbf{S} \mathbf{V}) + \mathbf{V} \cdot \nabla \mathbf{S} \]

\[ \text{small coll. freq.} \quad \text{large velocity grad.} \]

so the of (31) < 0 and both the field and the perturbed distribution function can decay.

Where do small scales in velocity space come from?
Consider an ultra-simplified kinetic eqn: \( f_c = 0 \), 1D:

\[ \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} = 0, \quad f(0, x) = g(x, v) \]

\[ \text{some bump dependence} \]

Solution: \( f(t, x, v) = g(x - vt, v) \)

Typical scale in \( v \) space: \( \frac{\partial f}{\partial v} = -t \frac{\partial g}{\partial x} \), so \( \frac{\partial f}{\partial v} \sim kt \)

This is because particles with small velocity difference \( v - v' \) are widely separated at \( x \times \) at time \( t \), come from widely separated points \( x_0, x_0' \) at time 0 (a long time ago).

In \( k \) space this is explicit:

\[ \frac{\partial f_k}{\partial t} + ikv \frac{\partial f_k}{\partial x} = 0 \Rightarrow \mathbf{f}_k(x, v) = g_k(v) e^{-ikv} \]

ballistic response
Let us now go back to the linear theory and see if we can trace the ballistic response. The Laplace-transformed distribution function was:

$$
\hat{\delta f_{ks}}(p, \nu) = \frac{1}{p + iE \nu} \left[ i \frac{g_2}{m_s} \hat{\phi}_2(p) \Gamma \frac{\partial \phi_0}{\partial \nu} + g_{ks}(\nu) \right]
$$

Taking the inverse Laplace transform in the way described on p. 8, we get:

$$
\delta f_{ks}(t, \nu) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dp \, e^{pt} \frac{1}{p + iE \nu} \left[ i \frac{g_2}{m_s} \hat{\phi}_2(p) \Gamma \frac{\partial \phi_0}{\partial \nu} + g_{ks}(\nu) \right]
$$

$$
= e^{-iE \nu t} \left[ i \frac{g_2}{m_s} \hat{\phi}_2(-iE \nu) \Gamma \frac{\partial \phi_0}{\partial \nu} + g_{ks}(\nu) \right] + \frac{ig_2}{m_s} \sum \frac{c_i \Gamma}{p + iE \nu} e^{pt} \frac{\partial \phi_0}{\partial \nu}
$$

In the presence of collisions even with a small collision frequency, the ballistic part of the perturbed distribution will get smeared out by the collisional diffusion in the velocity space. Incorporating this effect exactly would require solving the kinetic equation with a collision operator --- quite difficult. Instead, we can take this effect into account by "coarse-graining" the distribution function in velocity space and thus average out the small scales, so only term (2) is left in (32):

$$
\sum \frac{c_i \Gamma}{p + iE \nu} e^{pt} \frac{\partial \phi_0}{\partial \nu}
$$
\[ \langle S_{\text{res}}(t, V) \rangle_V = \frac{e^i}{m_s} \sum_{i} \omega_i e^{p_i V} \int \frac{d^3x}{e^{\frac{\hbar}{kT}} + 1} \]  

(33)

Let us calculate the el. field from this:

\[ \psi_k = \frac{4\pi i}{k^2} \sum_{i} \frac{1}{m_s} \int d^3V \langle S_{\text{res}}(t, V) \rangle_V = \]

\[ = \sum_{i} \omega_i e^{p_i t} \left[ \int \frac{d^3V}{m_s k^2} \left( \frac{1}{e^{\frac{\hbar}{kT}} + 1} - 1 + 1 \right) \right] = \]

\[ \sum_{i} \omega_i e^{p_i t} \]

(34)

as, indeed, we claimed in § 2.

This is why irreversible damping is possible: we deal with a smoothed-out distribution, where entropy does not need to be conserved.

What this procedure means is that we are first taking the limit \( t \to 0 \), keeping \( \rho \) and \( \eta \) finite and allowing collisions to get rid of the ballistic term (smooth the distr. function) and then take \( \rho \to 0 \) and \( \eta \to 0 \), allowing the damping due to wave-particle resonance to dominate.

Note: You might have spotted that in the entropy calculation on p. 18 I neglected the inter-species collision, which tend to equalize temperatures of different species — this is ok because those terms, unlike \( \frac{\rho}{t} \), are not singular in \( \rho \) and \( \eta \), i.e., their \( \rho \) and \( \eta \) do not make them small (they only involve \( \frac{\rho}{t} \) of the eq. distr., so \( \frac{\rho}{t} \) stay finite).
Let us estimate how long it takes collisions to eliminate the ballistic term [term $\text{\textcircled{1}}$ in (32)]:

$$\frac{\partial}{\partial t} \text{\textcircled{1}} \sim kv \text{\textcircled{1}} \implies v_{\text{coll}} u_{k} \frac{\partial}{\partial v} \text{\textcircled{1}} \sim v_{\text{coll}} u_{k} k^2 t^2 \text{\textcircled{1}}$$

This has to become big:

$$v_{\text{coll}} u_{k} k^2 t^2 \gg kv \sim w_k \implies t \gg \left( \frac{w_k}{v_{\text{coll}} u_{k} k^2} \right)^{1/2} \quad (35)$$

Compare this with what happens to term $\text{\textcircled{2}}$:

$$\frac{\partial}{\partial t} \text{\textcircled{2}} \sim \frac{1}{u_{k}} \text{\textcircled{2}} \implies v_{\text{coll}} u_{k} \frac{\partial}{\partial v} \text{\textcircled{2}} \sim v_{\text{coll}} \text{\textcircled{2}}$$

So we can neglect the effect of collisions on this term as long as $t \ll v_{\text{coll}}^{-1}$, which is consistent with (35) if

$$\frac{1}{v_{\text{coll}}} \gg \left( \frac{w_k}{v_{\text{coll}} k^2 u_{k}} \right)^{1/2} \quad \text{or} \quad v_{\text{coll}} \ll \frac{k^2 u_{k}^2}{w_k} \quad (\text{no problem})$$

Note that (35) is in fact a very conservative estimate. It is possible to argue that in nonlinear systems containing many waves, phase mixing can be accomplished in much faster way due to phase-space "turbulent diffusion" - the collisional scales in velocity space are then reached on $v_{\text{coll}}$-independent time scales of turbulent diffusion [Dupree 1966, Phys. Fluids 9, 1773].
Let me illustrate the effect of collisions and the non-interchangeability of limits using our ultra-simplified example: consider $v_e = 0$, 1D, so

$$\frac{\partial \delta f_e}{\partial t} + i k v \delta f_e = v_{\text{coll}} v_{th} \frac{\partial^2 \delta f_e}{\partial v^2}$$

[\text{magic!}]

The solution of this equation can be sought in the form:

$$\delta f_e(t, v) = \Phi(t, u) e^{-i k v t - \frac{1}{3} v_{\text{coll}} v_{th}^2 k^2 t^3}$$

where $u = v - i k v_{\text{coll}} v_{th}^2 t^2$. Then

$$\frac{\partial \delta f_e}{\partial t} + i k v \delta f_e = \left[ \frac{\partial}{\partial t} - 2 i k v_{\text{coll}} v_{th}^2 \frac{\partial}{\partial u} \right] \Phi(t, u)$$

$$= -v_{\text{coll}} v_{th}^2 \frac{\partial}{\partial v} \left[ \frac{\partial}{\partial u} - i k t \delta \right] e^{-\frac{u}{2}}$$

$$= -v_{\text{coll}} v_{th}^2 \left[ \frac{\partial^2}{\partial u^2} - i k t \frac{\partial}{\partial u} - i k^2 t^2 \right] e^{-\frac{u}{2}}$$

So we get

$$\frac{\partial \delta f_e}{\partial t} + i k v \delta f_e = -v_{\text{coll}} v_{th}^2 \frac{\partial}{\partial u} e^{-\frac{u}{2}}$$

**Solution:**

$$\delta f_e(t, v) = e^{-i k v t - \frac{1}{3} v_{\text{coll}} v_{th}^2 k^2 t^3} \int_{-\infty}^{+\infty} dv' g(v') e^{\frac{(v - v')^2}{4 v_{\text{coll}} v_{th}^2 t^2}}$$

$$= e^{-i k v t - \frac{1}{12} v_{\text{coll}} v_{th}^2 k^2 t^3} \int_{-\infty}^{+\infty} dv' g(v') e^{\frac{(v - v')^2}{4 v_{\text{coll}} v_{th}^2 t^2}}$$

$$= \int_{-\infty}^{+\infty} dv' g(v') e^{\frac{(v - v')^2}{4 v_{\text{coll}} v_{th}^2 t^2}}$$

$$\Rightarrow g_e(v) e^{-i k v t} \quad \text{if} \quad v_{\text{coll}} \to 0$$

$$\Rightarrow 0 \quad \text{if} \quad t \to \infty$$

Collisions damp out the velocity structure after

$$v_{\text{coll}} v_{th}^2 k^2 t^3 \gg k v t$$

$$\alpha \equiv \left( \frac{v}{v_{\text{coll}} k v_{th}^2} \right)^{\frac{1}{2}} \sim \left( \frac{v_{\text{th}}^2}{v_{\text{coll}} k v_{th}^2} \right)^{\frac{1}{2}} \quad \text{cf. eq. (38)}$$
So what happens if our plasma is, after exactly collisionless or so weakly collisional that eq. (35) cannot be satisfied for $t \sim \gamma_k^{-1}$?

Let us go back to eq. (32) and substitute into it

$$
\Phi_k(-ieV) = \sum_i \frac{C_i}{-ieV - p_i} + \text{analytic part of } \Phi_k(p) @ p = -ieV
$$

$$
S_{\Phi_k}(t, V) = e^{-ieVt} \left[ i\frac{q_s}{m_s} \left( -\sum_i \frac{C_i}{p_i + ieV} + \text{analytic part} \right) + \left. \frac{\partial \Phi_k}{\partial V} \right|_{p = -ieV} + g_{ks}(V) \right] t + \frac{i q_s}{m_s} \sum_i \frac{C_i}{p_i + ieV} \cdot \frac{\partial \Phi_k}{\partial V}
$$

Then the electric field is given by

$$
\Phi_k(t) = \frac{4\pi}{k^2} \sum_{s} q_s \int d^3V e^{-ieVt} \left[ i\frac{q_s}{m_s} \sum_i \frac{C_i}{p_i + ieV} \cdot \frac{\partial \Phi_k}{\partial V} + g_{ks}(V) \right] + \sum_i C_i e^{ip_it}
$$

$$
= \sum_i C_i \left[ -i \frac{4\pi q_s^2}{k^2 m_s} \mathcal{P} \int d^3V \frac{e^{-ieVt}}{p_i + ieV} \cdot \frac{\partial \Phi_k}{\partial V} + \frac{4\pi q_s}{k^2} \int d^3V e^{-ieVt} g_{ks}(V) \right] + \sum_i C_i e^{ip_it}
$$

$$
\text{smaller than this: } C_i e^{ip_it} \quad \text{smaller still: } \sum_i C_i e^{ip_it} \quad \text{but the pasted distribution does not decay .}
$$

Vanishes because of $V$-space oscillation as long as width of initial distribution not too small:

$$
t \gg \frac{1}{k \gamma_k}
$$

$$
t \ll \frac{1}{k \gamma_k} \Rightarrow \Delta V \gg \frac{\gamma_k}{k}
$$