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Intro to Linear Kinetics: Landau Damping, Waves, Instabilities

We first consider the simplest possible case:

- no magnetic field
  electrostatic perturbations $\vec{E} = -\nabla\phi$
- unperturbed ions in equilibrium
- electron distribution

ELECTRON KINETICS

$$f_e = f_{0e} + \delta f_e$$

satisfies Vlasov's kinetic equation

$$\frac{\partial f_e}{\partial t} + \vec{v}\cdot\nabla f_e - \frac{e}{m_e}\vec{E}\cdot\frac{\partial f_e}{\partial \vec{v}} = 0 \qquad (1)$$

$$\frac{\partial \delta f_e}{\partial t} + \vec{v}\cdot\nabla\delta f_e = -\frac{e}{m_e}(\nabla\phi)\cdot\frac{\partial f_{0e}}{\partial \vec{v}} \qquad (2) \quad \text{linearised}$$

Poisson's eq.: $$-\nabla^2\phi = 4\pi(\underbrace{Ze\delta n_i}_{0} - e\delta n_e) = -4\pi e\int d^3v\,\delta f_e \qquad (3)$$

Equations are linear, so let's decompose in plane waves:

$$\delta f_e(\vec{r},\vec{v},t) = \sum_{\vec{k}} \delta f_k(\vec{v},t)\, e^{i\vec{k}\cdot\vec{r}}$$

$$\phi(\vec{r},t) = \sum_{\vec{k}} \phi_k(t)\, e^{i\vec{k}\cdot\vec{r}}$$

Consider an initial value problem with

$$\delta f_e(\vec{r},\vec{v},t=0) = g(\vec{r},\vec{v}) = \sum_k g_k(\vec{v})\, e^{i\vec{k}\cdot\vec{r}}$$

$$\begin{cases} \frac{\partial}{\partial t}\delta f + i\vec{k}\cdot\vec{v}\,\delta f = -\frac{ie}{m_e}\phi\,\vec{k}\cdot\frac{\partial f_0}{\partial \vec{v}} & (4) \\ k^2\phi = -4\pi e\int d^3v\,\delta f & (5) \\ \delta f(t=0) = g \end{cases}$$

Omit the indices, and e indices as well

~~Solve~~ Solve by Laplace transform:

$$\delta f_p(\vec{v}) = \int_0^\infty dt\, e^{-pt}\,\delta f(\vec{v},t), \quad \text{same for } \phi_p.$$

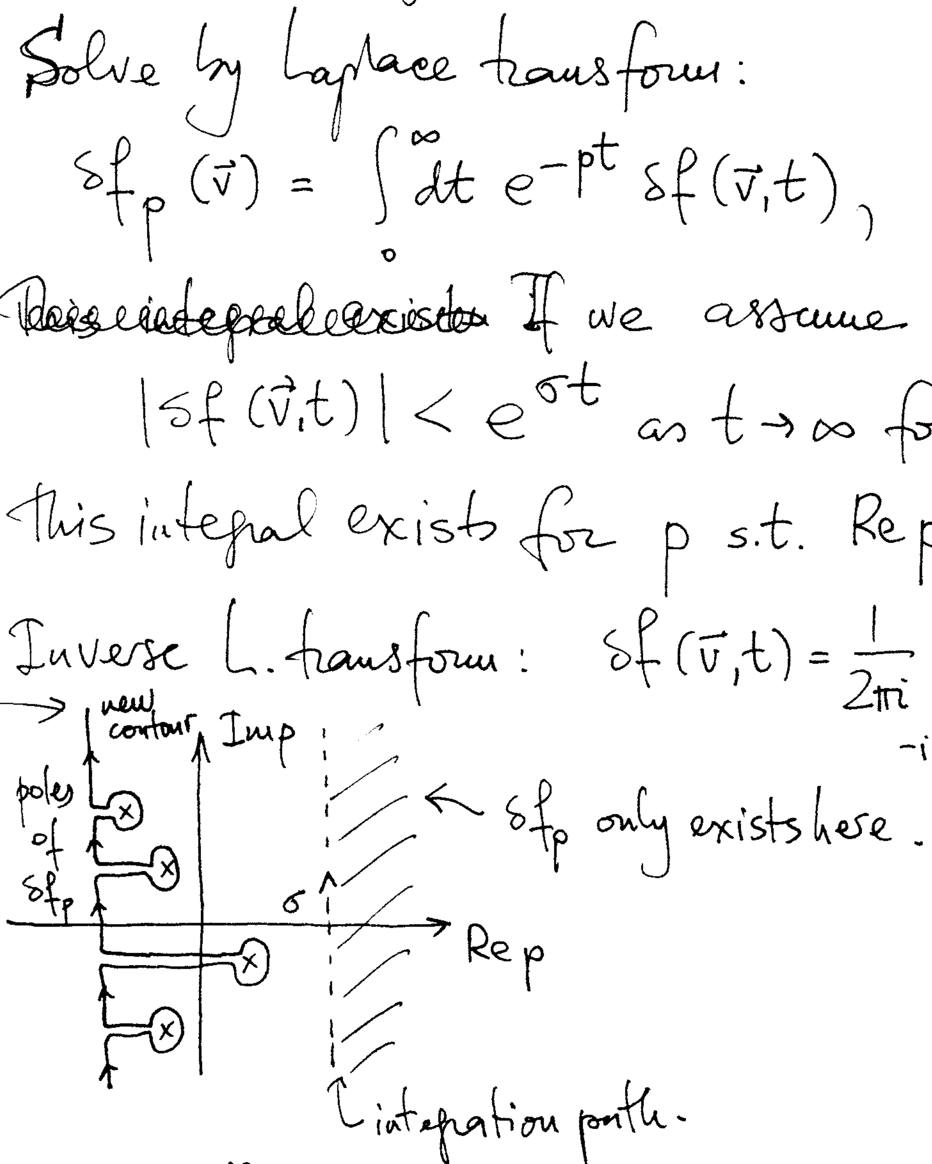
~~This integral exists~~ If we assume that

$|\delta f(\vec{v},t)| < e^{\sigma t}$ as $t\to\infty$ for some $\sigma>0$;

this integral exists for p s.t. $\mathrm{Re}\,p \geq \sigma$.

Inverse L. transform: $\delta f(\vec{v},t) = \frac{1}{2\pi i}\int_{-i\infty+\sigma}^{i\infty+\sigma} dp\, e^{pt}\,\delta f_p(\vec{v})$

see p. 4



Now $\int_0^\infty dt\, e^{-pt}$ (4) gives

$$\int_0^\infty dt\, e^{-pt}\frac{\partial \delta f}{\partial t} = e^{-pt}\delta f\Big|_0^\infty + p\int_0^\infty dt\, e^{-pt}\delta f =$$

$$= -g + p\,\delta f_p = -i\vec{k}\cdot\vec{v}\,\delta f_p - \frac{ie}{m_e}\phi_p\,\vec{k}\cdot\frac{\partial f_0}{\partial \vec{v}}.$$

Solution:

$$\boxed{\delta f_p = \underbrace{\frac{1}{p+i\vec{k}\cdot\vec{v}}}_{\text{this is a pole}} \Big[-\frac{ie}{m_e} \underbrace{\phi_p}_{\substack{\text{this has poles}\\ p_1,\, p_2,\, p_3 \dots}} \vec{k}\cdot\frac{\partial f_0}{\partial \vec{v}} + g \Big]} \quad \text{has poles.} \qquad (6)$$

Since δf_p is finite for $\operatorname{Re} p \geq \sigma$, all these poles must have $\operatorname{Re} p_i < \sigma$. Find them: from eq. (5),

$$k^2 \phi_p = -4\pi e \int d^3v\, \delta f_p =$$

$$= \left[\frac{4\pi e^2}{m_e} i \int d^3v \frac{1}{p+i\vec{k}\cdot\vec{v}} \vec{k}\cdot\frac{\partial f_0}{\partial \vec{v}} \right] \phi_p -$$

$$- 4\pi e \int d^3v \frac{g(\vec{v})}{p+i\vec{k}\cdot\vec{v}}, \quad \text{so}$$

$$\boxed{\phi_p = -\frac{4\pi e}{k^2 \epsilon(p,\vec{k})} \int d^3v \frac{g(\vec{v})}{p+i\vec{k}\cdot\vec{v}}} \qquad (7)$$

$$\boxed{\epsilon(p,\vec{k}) = 1 - \frac{4\pi e^2}{m_e k^2} i \int d^3v \frac{1}{p+i\vec{k}\cdot\vec{v}} \vec{k}\cdot\frac{\partial f_0}{\partial \vec{v}}} \qquad (8)$$

Poles of ϕ_p are solutions of

$$\boxed{\epsilon(p_i, \vec{k}) = 0}, \quad i = 1, 2, 3 \dots \qquad (9)$$

Now, since ϕ_p is finite except for $p = p_i$, we can analytically continue it to the half-plane $\mathrm{Re}\, p < \sigma$ (except to poles). This give the analytic continuation of δf_p via eq. (6) and we can do the inverse Laplace transform by shifting the contour $\int_{-i\infty+\sigma}^{i\infty+\sigma}$ to $\mathrm{Re}\, p \leq 0$ (see figure on p. 2), but <u>without crossing the poles</u>. Now only the poles will contribute to the integral, the rest is exponentially small:

we have $\displaystyle \phi_p = \sum_i \frac{c_i}{p - p_i} + \text{analytic part}$

$$(6) \Rightarrow \delta f(\vec{v}, t) = \frac{1}{2\pi i} \underbrace{\int dp\, e^{pt} \delta f_p(\vec{v})}_{\text{contour as explained}} =$$

$$= -\frac{ie}{m_e} \sum_i \frac{c_i}{p_i + i\vec{k}\cdot\vec{v}}\, \vec{k} \cdot \frac{\partial f_0}{\partial \vec{v}}\, e^{p_i t} +$$

$$+ \underbrace{\left[-\frac{ie}{m_e} \frac{\phi_{-i\vec{k}\cdot\vec{v}}}{}\, \vec{k} \cdot \frac{\partial f_0}{\partial \vec{v}} + g(\vec{v}) \right] e^{-i\vec{k}\cdot\vec{v}t}}_{\text{ballistic response}}$$

NB: This bit of δf does not decay!

$$(7) \Rightarrow \phi(t) = -\frac{4\pi e}{k^2} \sum_i c_i e^{p_i t}$$

Thus, $p_i = p_i(\vec{k})$ are fundamental "modes" of plasma pulsations

What remains to be done is find ϕ_p according to (7) and calculate p_i, C_i and $\phi_{-i\vec{k}\cdot\vec{v}}$.

solve "dispersion relation" (9)

Consider (7) & (8): how do we do the $\int d^3v$ integrals?

Let z axis be along $\vec{k}$. Then let

$$F_0(v_z) = \int dv_x\, dv_y\, f_0(v_x, v_y, v_z)$$

$$G(v_z) = \int dv_x\, dv_y\, g(v_x, v_y, v_z)$$

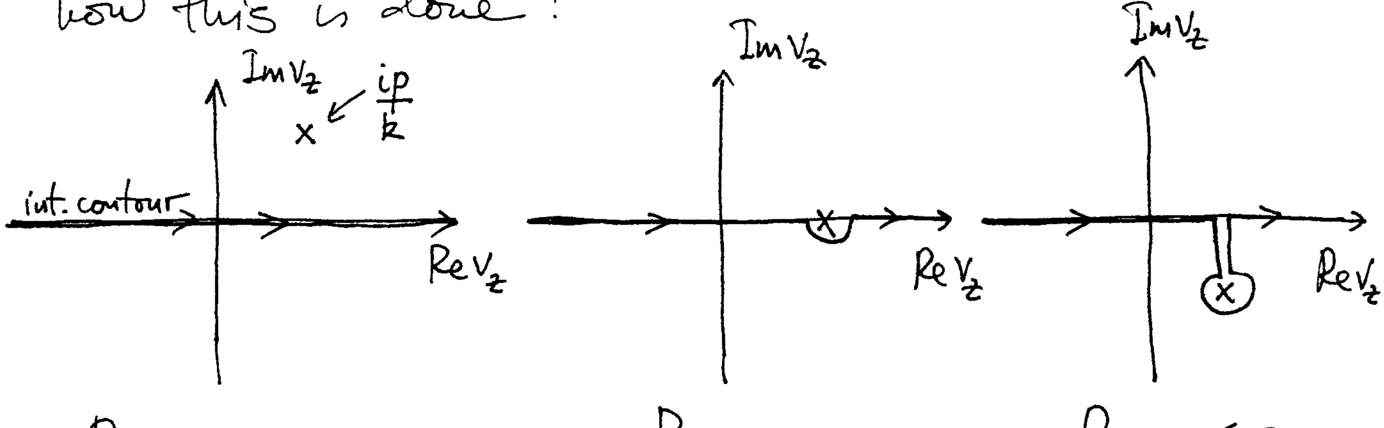
$$(7) \Rightarrow \quad \phi_p = -\frac{4\pi e}{k^2 \epsilon(p,\vec{k})} \int_{-\infty}^{+\infty} dv_z \frac{G(v_z)}{p + ikv_z}$$

← assume entire (no poles)

$$\epsilon(p,\vec{k}) = 1 - \frac{4\pi e^2}{m_e k}\, i \int_{-\infty}^{+\infty} dv_z \frac{1}{p+ikv_z} \frac{\partial F_0}{\partial v_z}$$

↑ pole at $v_z = \frac{ip}{k}$

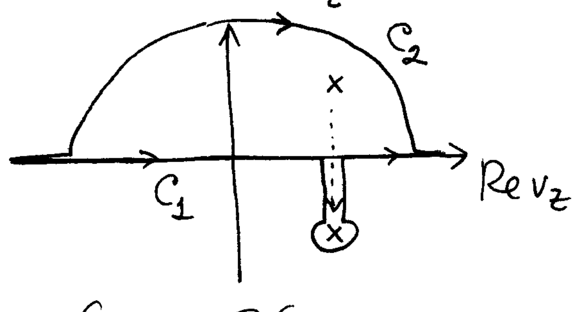
I said we analytically continue ϕ_p from $\mathrm{Re}\, p \geq \sigma$ to $\mathrm{Re}\, p < \sigma$, but did not say how this is done:



Re $p > 0$ Re $p = 0$ Re $p < 0$

Deform int. contour, so pole remains above it.

Here is why:



$$\int_{C_1} dv_z \, \frac{G(v_z)}{p+ikv_z} = \int_{C_2} dv_z \, \frac{G(v_z)}{p+ikv_z} + 2\pi i \, G\left(\frac{ip}{k}\right) \qquad (10)$$

remains analytic wherever the pole is

Keeping this contour below the pole is a way to keep the lhs of (10) equal to rhs, so it remains analytic.

Similar argument for $\frac{\partial F_0}{\partial z}$ ...

C_1 is called the Landau contour.

With this prescription, we have, in terms of integrals along the real axis:

$$\int_{C_1} dv_z \, \frac{1}{p+ikv_z} \, G(v_z) = \begin{cases} \int_{-\infty}^{+\infty} (\ldots) & \mathrm{Re}\, p > 0 \\ P\int_{-\infty}^{+\infty} (\ldots) + i\pi \, G\left(\frac{ip}{k}\right) & \mathrm{Re}\, p = 0 \\ \int_{-\infty}^{+\infty} (\ldots) + 2i\pi \, G\left(\frac{ip}{k}\right) & \mathrm{Re}\, p < 0 \end{cases}$$

or $\frac{\partial F_0}{\partial v_z}$

principal value of the formally divergent integral

OK, now let us do some practical calculations: find p_i by solving

$$\epsilon(p,k) = 1 - \frac{4\pi e^2}{m_e k}\, i \int_C dv_z \frac{1}{p + ikv_z} \frac{\partial F_0}{\partial v_z} = 0 \qquad (11)$$

Look for weakly damped waves:

$$p = -i(\omega + i\gamma) = -i\omega + \underset{\text{negative if damping}}{\gamma}, \qquad \gamma \ll \omega$$

$$\epsilon(p,k) = 1 - \frac{4\pi e^2}{m_e k^2} \int_C dv_z \frac{1}{v_z - \frac{ip}{k}} \frac{\partial F_0}{\partial v_z} =$$

$$= 1 - \frac{4\pi e^2}{m_e k^2} \int_C dv_z \frac{1}{v_z - \frac{\omega}{k} - \frac{i\gamma}{k}} \frac{\partial F_0}{\partial v_z} =$$

$$= 1 - \frac{4\pi e^2}{m_e k^2} \left[\int_C dv_z \frac{1}{v_z - \frac{\omega}{k}} \frac{\partial F_0}{\partial v_z} + i\gamma \frac{\partial}{\partial \omega} \int_C dv_z \frac{1}{v_z - \frac{\omega}{k}} \frac{\partial F_0}{\partial v_z} \right]$$

Now

$$\int_C dv_z \frac{1}{v_z - \frac{\omega}{k}} \frac{\partial F_0}{\partial v_z} \underset{\substack{\text{pole on} \\ \text{real axis}}}{=} \underbrace{\mathcal{P} \int_{-\infty}^{+\infty} dv_z \frac{1}{v_z - \frac{\omega}{k}} \frac{\partial F_0}{\partial v_z}} + i\pi F_0'\left(\frac{\omega}{k}\right)$$

Taylor expand assuming $\frac{\omega}{k} \gg v_{the}$ ("coldish electrons")

$$\mathcal{P} \int_{-\infty}^{+\infty} dv_z \frac{\partial F_0}{\partial v_z} \left(-\frac{\omega}{k}\right)^{-1} \left(\underset{0}{\cancel{1}} + \frac{kv_z}{\omega} + \ldots\right) = \frac{k^2}{\omega^2} \underset{\int_{-\infty}^{+\infty} dv_z F_0(v_z)}{n_{0e}} + \ldots$$

Substitute back into $\epsilon(p,k)$:

$$\epsilon(p,k) = 1 - \frac{4\pi e^2}{m_e k^2}\left(1 + i\gamma\frac{\partial}{\partial\omega}\right)\left[\frac{k^2}{\omega^2}n_{oe} + \ldots + i\pi F_0'\left(\frac{\omega}{k}\right)\right] = 0$$

Real part (to lowest order):

$$1 - \frac{4\pi e^2 n_{oe}}{m_e\omega^2} = 0 \Rightarrow \omega^2 = \frac{4\pi e^2 n_{oe}}{m_e} \equiv \omega_{pe}^2$$

plasma frequency

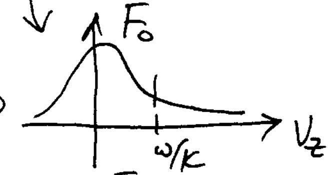
Im. part:

$$-\frac{\cancel{4}\pi^{\cancel{2}}\cancel{e^2}}{\cancel{m_e k^2}}F_0'\left(\frac{\omega}{k}\right) + 2\frac{4\pi \cancel{e^2}}{\cancel{m_e k^2}}\gamma\frac{k^2}{\omega^3}n_{oe} = 0$$

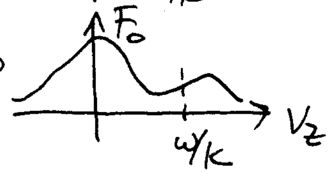
$$\boxed{\gamma = +\frac{\pi}{2}\frac{\omega^3}{k^2}\frac{1}{n_{oe}}F_0'\left(\frac{\omega}{k}\right)}$$

Landau damping (12)

This is damping if $\gamma < 0$ $\quad F_0'\left(\frac{\omega}{k}\right) < 0$



instability if $\gamma > 0$ $\quad F_0'\left(\frac{\omega}{k}\right) > 0$



bump-on-tail instability

NB: An important (particular) choice of the eq. distribution is a Maxwellian:

$$f_{oe} = \frac{n_{oe}}{(\pi v_{the}^2)^{3/2}} e^{-v^2/v_{the}^2} \qquad v_{the} = \sqrt{\frac{2T_{oe}}{m_e}}$$

NB definition

$$F_0(v_z) = \frac{n_{oe}}{\sqrt{\pi v_{the}^2}} e^{-v_z^2/v_{the}^2}$$

In this case, define $u = \frac{v_z}{v_{th}}$, $\zeta = \frac{ip}{k v_{th}}$ and absorb the Landau integration rule into a new special function ("plasma dispersion fn")

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_C du \, \frac{e^{-u^2}}{u-\zeta}$$

(look up its properties in the Plasma Formulary)

Then

$$\epsilon(p,k) = 1 + \frac{\omega_{pe}^2}{k^2 v_{the}^2} \underbrace{\int_C du \, \frac{1}{u-\zeta} \frac{2}{\sqrt{\pi}} u \, e^{-u^2}}_{= 2\left[1 + \zeta Z(\zeta)\right]}$$

$$= 1 + \underbrace{2 \frac{\omega_{pe}^2}{k^2 v_{the}^2}}_{= \frac{1}{k^2 \lambda_{De}^2}} \left[1 + \zeta Z(\zeta)\right] = 0$$

Note that in the limit we considered before,

$$\zeta = \frac{ip}{k v_{th}} \gg 1 \quad \text{(coldish electrons)},$$

$$Z(\zeta) \approx i\sqrt{\pi} e^{-\zeta^2} - \frac{1}{\zeta}\left(1 + \frac{1}{2\zeta^2} + \frac{3}{4\zeta^4} + \dots\right), \quad \zeta \to \infty$$

$$1 + \frac{2\omega_{pe}^2}{k^2 v_{the}^2}\left[\cancel{1} + i\zeta\sqrt{\pi} e^{-\zeta^2} - \cancel{1} - \frac{1}{2\zeta^2} - \frac{3}{4\zeta^4} - \dots\right] = 0$$

Real part:

$$1-\frac{\omega_{pe}^2}{k^2 v_{the}^2}\left(\frac{1}{\zeta_r^2}+\frac{3}{2\zeta_r^4}+\dots\right)=0 \quad , \zeta_r = \mathrm{Re}\,\zeta$$

$$\zeta_r^2 \approx \frac{\omega_{pe}^2}{k^2 v_{the}^2}\Big(1+\frac{3}{2}\underbrace{\frac{1}{\omega_{pe}^2}k^2 v_{the}^2}_{=k^2\lambda_{De}^2}\Big)$$

$$\zeta_r^2 = \frac{\omega^2}{k^2 v_{the}^2}$$

$$\omega^2 \approx \omega_{pe}^2\left(1+\frac{3}{2}\frac{k^2 v_{the}^2}{\omega_{pe}^2}\right) \qquad \text{Langmuir wave} \quad (13)$$

thermal correction to plasma oscillation. (repulsion of electrons by el. pressure force)

Im. part:

$$2\frac{\omega_{pe}^2}{k^2 v_{the}^2}\left[\zeta_r \,\mathrm{Im}\, Z(\zeta_r) + \zeta_i \frac{\partial}{\partial \zeta_r}\mathrm{Re}\,\zeta_r Z(\zeta_r)\right]=0$$

$$\zeta_i = \mathrm{Im}\,\zeta$$

$$\zeta_i \approx -\frac{\zeta_r \,\mathrm{Im}\, Z(\zeta_r)}{\frac{\partial}{\partial \zeta_r}\mathrm{Re}\left[\zeta_r Z(\zeta_r)\right]} \approx +\frac{\zeta_r\sqrt{\pi}\, e^{-\zeta_r^2}}{\frac{\partial}{\partial\zeta_r}\left(1+\frac{1}{2\zeta_r^2}+\dots\right)} =$$

$$= -\sqrt{\pi}\,\zeta_r^4 e^{-\zeta_r^2}$$

$$\frac{\gamma}{\omega_{pe}} \approx \frac{\zeta_i}{\zeta_r} = -\sqrt{\pi}\left(\frac{\omega_{pe}}{k v_{the}}\right)^3 e^{-\left(\frac{\omega_{pe}^2}{k^2 v_{the}^2}+\frac{3}{2}\right)} \qquad (14)$$

Landau damping of Langmuir wave.

ION KINETICS

All of the above was done in neglect of ion dynamics. If this is restored, new waves appear.

Now the linearised kinetic system is

$$\frac{\partial}{\partial t}\delta f_s + i\vec{k}\cdot\vec{v}\,\delta f_s = \frac{iq_s}{m_s}\phi\,\vec{k}\cdot\frac{\partial f_{0s}}{\partial \vec{v}}$$

where $s = e, i$, $q_e = -e$, $q_i = Ze$, assume $Z=1$ (hydrogen)

$$k^2\phi = 4\pi\left(Ze\,\delta n_i - e\,\delta n_e\right) =$$

$$= \sum_s 4\pi q_s \int d^3v\,\delta f_s$$

Then $$\delta f_{sp} = \frac{1}{p+i\vec{k}\cdot\vec{v}}\left[\frac{iq_s}{m_s}\phi_p\,\vec{k}\cdot\frac{\partial f_{0s}}{\partial\vec{v}} + g_s\right]$$

$$k^2\phi_p = \sum_s \frac{4\pi q_s^2}{m_s}\, i\int d^3v\,\frac{1}{p+i\vec{k}\cdot\vec{v}}\,\vec{k}\cdot\frac{\partial f_{0s}}{\partial\vec{v}}\,\phi_p$$

$$+\sum_s 4\pi q_s\int d^3v\,\frac{g_s(\vec{v})}{p+i\vec{k}\cdot\vec{v}}$$

$$\phi_p = \frac{1}{k^2\epsilon(p,\vec{k})}\sum_s 4\pi q_s\int d^3v\,\frac{g_s(\vec{v})}{p+i\vec{k}\cdot\vec{v}}$$

$$\boxed{\begin{aligned}\epsilon(p,\vec{k}) &= 1 - \sum_s \frac{4\pi q_s^2}{m_s}\, i\int d^3v\,\frac{1}{p+i\vec{k}\cdot\vec{v}}\,\vec{k}\cdot\frac{\partial f_{0s}}{\partial\vec{v}} = \\ &= 1 - \sum_s \frac{4\pi q_s^2}{m_s k}\, i\int dv_z\,\frac{1}{p+ikv_z}\,\frac{\partial F_{0s}}{\partial v_z}\end{aligned}}$$

Thus, ϵ picks up an extra term from the ions:

$$\epsilon(p,k) = 1 - \frac{4\pi e^2}{m_e k^2} \int_C dv_z \frac{1}{v_z - \frac{ip}{k}} \frac{\partial F_{0e}}{\partial v_z} - \underbrace{\frac{4\pi e^2}{m_i k^2} \int_C dv_z \frac{1}{v_z - \frac{ip}{k}} \frac{\partial F_{0i}}{\partial v_z}}_{\text{the new term.}} = 0 \qquad (15)$$

If both F_{0e} and F_{0i} are Maxwellian (in general, with $T_{0i} \neq T_{0e}$), this becomes

$$1 + 2\frac{\omega_{pe}^2}{k^2 v_{the}^2}\left[1 + \zeta_e Z(\zeta_e)\right] + 2\frac{\omega_{pi}^2}{k^2 v_{thi}^2}\left[1 + \zeta_i Z(\zeta_i)\right] = 0 \qquad (16)$$

If $\zeta_e \gg 1$ / $\frac{\omega}{k} \gg v_{the}$, ions give a small correction to Langmuir wave (ex.: check this, use $v_{thi} = \sqrt{\frac{2T_i}{m_i}} = \sqrt{\frac{m_e}{m_i}\frac{T_{0i}}{T_{0e}}}\, v_{the} \ll v_{the}$).

Consider now $\boxed{v_{thi} \ll \frac{\omega}{k} \ll v_{the}}$

$\zeta_i \gg 1$ cold ions $\qquad \zeta_e \ll 1$ hot electrons

$$Z(\zeta_i) \approx i\sqrt{\pi}\, e^{-\zeta_i^2} - \frac{1}{\zeta_i}\left(1 + \frac{1}{2\zeta_i^2} + \frac{3}{4\zeta_i^4} + \dots\right) \quad \zeta_i \to \infty$$

$$Z(\zeta_e) \approx i\sqrt{\pi} - 2\zeta_e + \dots \quad \zeta_e \to 0$$

Real part of (16):

$$1 + 2\frac{\omega_{pe}^2}{k^2 v_{the}^2}\left[1 + \dots\right] + 2\frac{\omega_{pi}^2}{k^2 v_{thi}^2}\left[-\frac{1}{2\zeta_i^2} - \frac{3}{4\zeta_i^4} + \dots\right] = 0$$

$$1 + 2\frac{\omega_{pe}^2}{k^2 v_{the}^2} - \frac{\omega_{pi}^2}{\omega^2}\Big(1 + \underbrace{\frac{3 v_{thi}^2 k^2}{2\omega^2}}_{\text{small}}\Big) = 0$$

$$\omega^2 = \frac{\omega_{pi}^2}{1 + 2\frac{\omega_{pe}^2}{k^2 v_{the}^2}} = \frac{1}{2}\underbrace{k^2 v_{the}^2 \frac{\omega_{pi}^2}{\omega_{pe}^2}}\,\frac{1}{1 + k^2\lambda_{De}^2/2}$$

$$\parallel$$

$$\frac{2T_{0e}}{m_e}\,\frac{4\pi e^2 n_{0i} m_e}{m_i 4\pi e^2 n_{0e}} = \frac{2T_{0e}}{m_i} \equiv 2c_s^2$$

$$\boxed{\omega^2 = \frac{k^2 c_s^2}{\underbrace{1 + k^2\lambda_{De}^2/2}_{\text{kinetic factor in the usual acoustic wave.}}}}$$

ion-acoustic wave (17)

To work out how they are damped, look at Im. part of (16):

$$2\frac{\omega_{pe}^2}{k^2 v_{the}^2}\sqrt{\pi}\,\zeta_e + 2\frac{\omega_{pi}^2}{k^2 v_{thi}^2}\Big[\zeta_i\sqrt{\pi}\, e^{-\zeta_i^2} + \underbrace{\mathrm{Im}\,\zeta_i \frac{1}{\zeta_i^3}}_{= \frac{\gamma}{\omega}\frac{1}{\zeta_i^2}}\Big] = 0$$