§2. MHD and Alfvén Waves

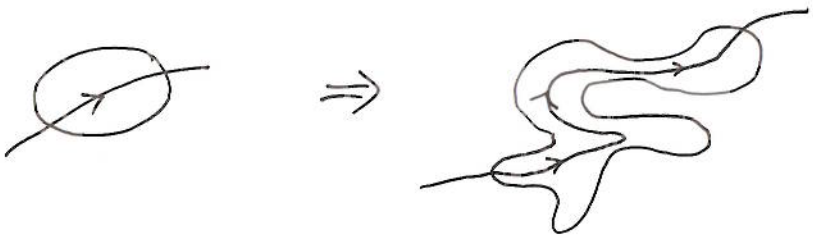
MHD Equations: from $\nabla\cdot\vec{u}=0$; Lorentz force $\vec{f}=\frac{c}{4\pi}\nabla\times\vec{B}$ (Ampère's law)

$$\begin{cases} \rho\left(\frac{\partial\vec{u}}{\partial t}+\vec{u}\cdot\nabla\vec{u}\right) = -\nabla p+\rho\nu\nabla^2\vec{u}+\frac{1}{c}\vec{f}\times\vec{B} \\ \frac{\partial\vec{B}}{\partial t} = \nabla\times(\vec{u}\times\vec{B})+\eta\nabla^2\vec{B} \end{cases}$$

$\nabla\cdot\vec{B}=0$, $\nabla\cdot\vec{u}=0$ $\Rightarrow$ $\frac{\partial\vec{B}}{\partial t}+\underbrace{\vec{u}\cdot\nabla\vec{B}}_{\text{advection}} = \vec{B}\cdot\nabla\vec{u}+\eta\nabla^2\vec{B}$ (diffusion)

Magnetic Reynolds # $Rm=\frac{UL}{\eta}$.

When $Rm\gg 1$, magnetic field is "frozen" into the fluid



Lorentz force: $\frac{1}{c}\vec{f}\times\vec{B} = -\nabla_\perp\frac{B^2}{8\pi}+\frac{\hat{b}\cdot\nabla\hat{b}\,B^2}{4\pi}$

mag. pressure; curvature force

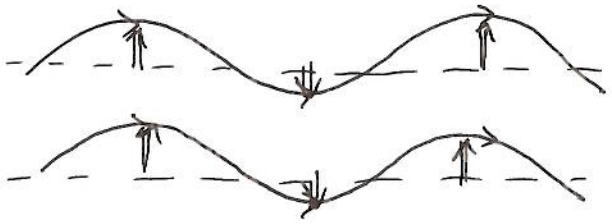
field does not like being compressed

field does not like being bent.

The magnetic field imparts certain elasticity to the fluid (this analogy can be made precise - polymers)

The forces give rise to waves:

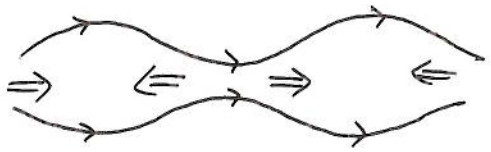
Alfvén waves:



$\vec{B} = \vec{B}_0 + \delta\vec{B}_\perp$

$\vec{u} = \cancel{\vec{u}_0}^{\,0} + \delta\vec{u}_\perp$

Slow waves:
("pseudo-Alfvén waves")



$\vec{B} = \vec{B}_0 + \delta B_\parallel \hat{b}_0$

$\vec{u} = \cancel{\vec{u}_0}^{\,0} + \delta u_\parallel \hat{b}_0$

Let me make this more precise.

$\vec{B} = \vec{B}_0 + \delta\vec{B}$, $\vec{u} = \cancel{\vec{u}_0}^{\,0} + \delta\vec{u} \equiv \tilde{\rho}$

$$\rho_0\left(\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = -\nabla\left(p + \frac{B^2}{8\pi}\right) + \frac{\vec{B}\cdot\nabla\vec{B}}{4\pi} + \nu\nabla^2\vec{u}$$

$$\frac{\partial\delta\vec{u}}{\partial t} + \delta\vec{u}\cdot\nabla\delta\vec{u} = -\nabla\frac{\tilde{p}}{\rho_0} + \underbrace{\frac{B_0^2}{4\pi\rho_0}}_{\equiv V_A^2}\nabla_\parallel\frac{\delta\vec{B}}{B_0} + \frac{\delta\vec{B}\cdot\nabla\delta\vec{B}}{4\pi\rho_0} + \nu\nabla^2\delta\vec{u}$$

$$\frac{\partial\delta\vec{B}}{\partial t} + \delta\vec{u}\cdot\nabla\delta\vec{B} = B_0\nabla_\parallel\delta\vec{u} + \delta\vec{B}\cdot\nabla\delta\vec{u} + \eta\nabla^2\delta\vec{B}$$

Introduce Elsasser variables: $\vec{z}^{\pm} = \delta\vec{u} \pm \frac{\delta\vec{B}}{\sqrt{4\pi\rho_0}}$

Then $\frac{\partial\vec{z}^{\pm}}{\partial t} = \frac{\partial\delta\vec{u}}{\partial t} \pm \frac{1}{\sqrt{4\pi\rho_0}}\frac{\partial\delta\vec{B}}{\partial t} =$

$$= -\delta\vec{u}\cdot\nabla\delta\vec{u} - \frac{\nabla\tilde{p}}{\rho_0} + \frac{B_0}{4\pi\rho_0}\nabla_\parallel\delta\vec{B} + \frac{\delta\vec{B}\cdot\nabla\delta\vec{B}}{4\pi\rho_0} + \nu\nabla^2\delta\vec{u}$$

$$\mp\delta\vec{u}\cdot\nabla\frac{\delta\vec{B}}{\sqrt{4\pi\rho_0}} \pm \frac{B_0}{\sqrt{4\pi\rho_0}}\nabla_\parallel\delta\vec{u} \pm \frac{\delta\vec{B}}{\sqrt{4\pi\rho_0}}\cdot\nabla\delta\vec{u} + \eta\nabla^2\frac{\delta\vec{B}}{\sqrt{4\pi\rho_0}} =$$

$$= \pm V_A \nabla_\parallel \vec{z}^{\pm} - \vec{z}^{\mp} \cdot \nabla \vec{z}^{\pm} - \frac{\nabla \tilde{p}}{\rho_0} + \frac{\nu \pm \eta}{2} \nabla^2 \vec{z}^{+} + \frac{\nu \mp \eta}{2} \nabla^2 \vec{z}^{-}$$

Then, (from $\nabla \cdot \vec{z}^{\pm} = 0$)

$$\boxed{\frac{\partial \vec{z}^{\pm}}{\partial t} \mp V_A \nabla_\parallel \vec{z}^{\pm} + \vec{z}^{\mp} \cdot \nabla \vec{z}^{\pm} = -\frac{\nabla \tilde{p}}{\rho_0} + \frac{\nu+\eta}{2} \nabla^2 \vec{z}^{\pm} + \frac{\nu-\eta}{2} \nabla^2 \vec{z}^{\mp}}$$

NB: if, say, $\vec{z}^{-} = \delta \vec{u} - \frac{\delta \vec{B}}{\sqrt{4\pi \rho_0}} = 0$, then $\vec{z}^{+}$ satisfies

$$\frac{\partial \vec{z}^{+}}{\partial t} - V_A \nabla_\parallel \vec{z}^{+} = -\frac{\nabla \tilde{p}}{\rho_0} + \frac{\nu+\eta}{2} \nabla^2 \vec{z}^{+}$$

$\nabla \tilde{p} / \rho_0 = 0$ (because $\nabla \cdot \vec{z}^{+}$ is satisfied)

So no nonlinearity, $\vec{z}^{+}$ is a wave + diffusion. Similar for $\vec{z}^{+} = 0$.

Waves: $\boxed{\omega = k_\parallel V_A}$ Alfvén waves - there are not just linear waves but also finite-amplitude solutions:

$\vec{z}_- = 0 \quad \vec{z}_+ = \vec{f}_+ (x, y, z + V_A t)$ Elsasser states

$\vec{z}_+ = 0 \quad \vec{z}_- = \vec{f}_- (x, y, z - V_A t)$

These are isolated wave packets that propagate along z with velocity V_A — they never catch up with each other (everyone has the same velocity) and never interact.

Interaction (worked by nonlinearity!) only occurs between counterpropagating wave packets.

Energetics of Elsasser variables:

$$\int d^3x \frac{\rho_0}{2}\left(|\vec{z}^+|^2 + |\vec{z}^-|^2\right) = \int d^3x \left(\frac{1}{2}\rho_0 \delta u^2 + \frac{\delta B^2}{8\pi}\right) \quad \text{energy}$$

$$\int d^3x \frac{\rho_0}{2}\left(|\vec{z}^+|^2 - |\vec{z}^-|^2\right) = \int d^3x \frac{2\rho_0}{\sqrt{4\pi\rho_0}} \delta\vec{u}\cdot\delta\vec{B} \quad \text{cross-helicity}$$

Both of these quantities are conserved in MHD, so the "energy" of each Elsasser variable is conserved (Ex.: check this!)

Thus, interactions do not change energy of the Elsasser fields — they are more like scattering of wave packets off each other or "passive" advection of $\vec{z}^{\pm}$ by $\vec{z}^{\mp}$.

NB: Cross-helicity is a measure of imbalance between the + and − fields.