Firehose and Mirror Instabilities in High-β Plasmas

Plan

1) Kinetic MHD
2) Formal linear theory of firehose and mirror } these notes
3) Qualitative introduction to the physics of high-β turbulent magnetised weakly collisional plasma } attached reprint
   - exemplified by galaxy clusters
   - Origin of pressure anisotropies
   - Nonlinear evolution of firehose fluctuations
   - Possible implications for transport
4) Some more rigorous calculations of the above
   - see the attached reprint

① Kinetic MHD

(A quick and dirty introduction)

How do we describe electromagnetic fluctuations in plasmas on time scales s.t. $\omega \ll \Omega_i$ and spatial scales s.t. $k\rho_i \ll 1$? (Magnetised plasmas)

Our intuition likes fluid approaches, but we know we must be careful because when plasma is magnetised and weakly collisional or collisionless kinetic effects will be important. So how much can be rescued of the fluid description?

The kinetic equation for species s is

$$\frac{\partial f_s}{\partial t} + \vec{v}\cdot\nabla f_s + \frac{e_s}{m_s}\left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c}\right)\cdot\frac{\partial f_s}{\partial \vec{v}} = \cancel{\left(\frac{\partial f_s}{\partial t}\right)_c} \qquad (1)$$

$\left(\frac{\partial f_s}{\partial t}\right)_c$: collision integral. ignore it for now

Continuity: $$\frac{\partial n_s}{\partial t} + \nabla\cdot(n_s\vec{u}_s) = 0 \qquad (2)$$

Momentum: $$\underbrace{\frac{\partial}{\partial t}(m_s n_s \vec{u}_s)}_{=} + \underbrace{\nabla\cdot\int d^3\vec{v}\, m_s \vec{v}\vec{v}\vec{v} f_s}_{=} - e_s\left(\vec{E} + \frac{\vec{u}_s\times\vec{B}}{c}\right)n_s = 0 \qquad (3)$$

$$m_s\vec{u}_s\frac{\partial n_s}{\partial t} + m_s n_s\frac{\partial \vec{u}_s}{\partial t}$$

use (1)

$$m_s\nabla\cdot\left(n_s\vec{u}_s\vec{u}_s + \int d^3\vec{v}'\vec{v}'\vec{v}' f_s\right)$$
$$= \cancel{m_s\vec{u}_s\nabla\cdot(n_s\vec{u}_s)} + m_s n_s\vec{u}_s\cdot\nabla\vec{u}_s + \nabla\cdot\hat{P}_s$$

where $\vec{v}' = \vec{v} - \vec{u}_s$ peculiar velocity

and $\hat{P}_s = m_s\int d^3\vec{v}'\,\vec{v}'\vec{v}' f_s$ pressure tensor

(NB: $\sum_s F_s = 0$ → coll. friction (interspecies))

So we have

$$m_s n_s \left(\frac{\partial \vec{u}_s}{\partial t} + \vec{u}_s \cdot \nabla \vec{u}_s \right) = - \nabla \cdot \hat{P}_s + e_s n_s \left(\vec{E} + \frac{\vec{u}_s \times \vec{B}}{c} \right) + \vec{F}_s \qquad (4)$$

Add electron and ion equations, use $m_e \ll m_i$, assume that $m_e \vec{u}_e \ll m_i \vec{u}_i$ (we shall see later that this is OK), use $\sum_s e_s n_s = \frac{\nabla \cdot \vec{E}}{4\pi}$ small (quasineutrality)

$$\sum_s e_s n_s \vec{u}_s = \vec{J} = \frac{c}{4\pi} \nabla \times \vec{B} - \underbrace{\frac{1}{4\pi} \frac{\partial \vec{E}}{\partial t}}_{\text{small}} = \frac{c}{4\pi} \nabla \times \vec{B}$$

Ampère

[convince yourselves that this is fine – just as it is done in standard MHD derivations]

Finally,

$$\boxed{m_i n_i \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) = - \nabla \cdot \hat{P} + \frac{(\nabla \times \vec{B}) \times \vec{B}}{4\pi}} \qquad (5)$$

(I dropped the subscript on the fluid velocity: $\vec{u}_i \equiv \vec{u}$).

The magnetic field is given by the Faraday law:

$$\frac{\partial \vec{B}}{\partial t} = - c \nabla \times \vec{E} \qquad (6)$$

and the electric field by the electron momentum equation: (4) with $s = e$ and neglecting electron inertia,

$$\vec{E} + \frac{\vec{u}_e \times \vec{B}}{c} = - \frac{\nabla \cdot \hat{P}_e}{e n_e} + \cancel{\frac{\vec{F}_e}{e n_e}} \qquad \text{Ohm's law} \qquad (7)$$

$\vec{u}_e = \vec{u}_i - \frac{\vec{J}}{e n_e}$

↑ resistivity, small for $\sqrt{\frac{m_e}{m_i}} \ll 1$

whence $$\vec{E} = -\frac{\vec{u}\times\vec{B}}{c} + \underbrace{\frac{\vec{J}\times\vec{B}}{en_e} - \frac{\nabla\cdot\hat{P}_e}{en_e}}_{\text{Hall term}\quad\text{thermoelectric and other terms...}} \tag{8}$$

It is possible to prove (try!) that these terms are smaller than $\frac{\vec{u}\times\vec{B}}{c}$ when $k\rho_i \ll 1$.

So, for the perpendicular electric field, we have

$$\vec{E}_\perp = -\frac{\vec{u}\times\vec{B}}{c} \tag{9}$$

while $$E_\parallel = -\left(\frac{\nabla\cdot\hat{P}_e}{en_e}\right)\cdot\hat{b} \ll \vec{E}_\perp . \tag{10}$$

Inserting this into (6), we get

$$\boxed{\frac{\partial\vec{B}}{\partial t} = +\nabla\times(\vec{u}\times\vec{B})} \quad \underline{\text{induction equation}} \tag{11}$$

So we now have a set of equations that look like fluid equations:

(2): $$\frac{\partial n_i}{\partial t} + \nabla\cdot(n_i\vec{u}) = 0$$

(5): $$m_i n_i\left(\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = -\nabla\cdot\hat{P} + \frac{(\nabla\times\vec{B})\times\vec{B}}{4\pi}$$

(11): $$\frac{\partial\vec{B}}{\partial t} = \nabla\times(\vec{u}\times\vec{B})$$

The only kinetic input in these equations is via the pressure tensor $\hat{\mathbb{P}}$.

When $\omega \ll \Omega_i$, $k\rho_i \ll 1$ the distribution functions can be written as a function only of $v_\perp'$ and $v_\parallel'$ but not of the gyroangle ϑ. Indeed, to lowest order in $\frac{\omega}{\Omega_i}$, $k\rho_i$, eq. (1) [using eq. (9)] is simply

$$\frac{\vec{v}' \times \vec{B}}{c} \cdot \frac{\partial f_s}{\partial \vec{v}'} = 0 \quad \Rightarrow \quad \frac{\partial f_s}{\partial \vartheta} = 0 \qquad \text{(check this!)} \quad (12)$$

[illegible]

Then

$$\hat{\mathbb{P}}_s = m_s \int d^3\vec{v}' \, \vec{v}' \vec{v}' \, f_s(t, \vec{r}, v_\perp', v_\parallel') =$$

$$= m_s 2\pi \int_0^\infty dv_\perp' \, v_\perp' \int_{-\infty}^{+\infty} dv_\parallel' \, f_s \underbrace{\langle \vec{v}' \vec{v}' \rangle}_{\text{avg. over gyroangles}}$$

$$= \frac{v_\perp'^2}{2}(\mathbb{1} - \hat{b}\hat{b}) + v_\parallel'^2 \hat{b}\hat{b}$$

$$= p_{\perp s}(\mathbb{1} - \hat{b}\hat{b}) + p_{\parallel s}\hat{b}\hat{b} = \begin{pmatrix} p_\perp & & \\ & p_\perp & \\ & & p_\parallel \end{pmatrix} \qquad (13)$$

where $p_{\perp s} = m_s \int d^3\vec{v}' \, \frac{v_\perp'^2}{2} f_s \qquad p_{\parallel s} = m_s \int d^3\vec{v}' \, v_\parallel'^2 f_s$

We can now rewrite the momentum equation so:

$$m_i n_i \left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = -\nabla\left(p_\perp + \frac{B^2}{8\pi}\right) + \nabla\cdot\left[\hat{b}\hat{b}\left(p_\perp - p_\parallel + \frac{B^2}{4\pi}\right)\right] \quad (14)$$

(scalar pressure) — (pressure anisotropy) — (tension force $\frac{\vec{B}\cdot\nabla\vec{B}}{4\pi}$)

$$\left.\begin{aligned} p_\perp &= p_{\perp i} + p_{\perp e} \\ p_\parallel &= p_{\parallel i} + p_{\parallel e} \end{aligned}\right\}$$

In order to calculate $p_\perp$ and $p_\parallel$, we need to know f_s — so we still need a kinetic equation. We could derive it by systematically expanding (1) in small $\frac{\omega}{\Omega_i}$ and $k\rho_i$ - as is done in the standard kinetic MHD derivation by Kulsrud (see his chapter in the Handbook of Plasma Physics or his Varenna lectures in 1960's).

Here I want to take a slightly more physically transparent short cut.

If we expect that $f_s = f_s(t, q)$, ← phase-space variable,

the kinetic equation is the statement of the phase volume conservation:

$$\frac{\partial f_s}{\partial t} + \frac{dq}{dt}\cdot\frac{\partial f_s}{\partial q} = 0$$

↑ this comes from the equations of motion of ind. particle.

Let us represent the distribution function in the limit $\frac{\omega}{\Omega_i} \ll 1$, $k\rho_i \ll 1$ as $f_s = f_s(t, \vec{r}, \mu_s, v_\parallel')$,

where we use the first adiabatic invariant $\mu_s = \frac{m_s v_\perp'^2}{2B}$ as the phase-space variable instead of $v_\perp'$.

We can now write the kinetic equation averaged over gyroangles:

$$\frac{\partial f_s}{\partial t} + \left\langle \frac{d\vec{r}}{dt} \right\rangle \cdot \frac{\partial f_s}{\partial \vec{r}} + \left\langle \frac{d\mu_s}{dt} \right\rangle \cdot \frac{\partial f_s}{\partial \mu_s} + \left\langle \frac{dv_\parallel'}{dt} \right\rangle \cdot \frac{\partial f_s}{\partial v_\parallel'} = 0 \qquad (15)$$

$= 0$ for $k\rho_i \ll 1$, $\frac{\omega}{\Omega_i} \ll 1$ (check this!) – conservation of μ

$\left\langle \frac{d\vec{r}}{dt} \right\rangle = \langle \vec{v} \rangle = \vec{u} + \langle \vec{v}' \rangle = \vec{u} + v_\parallel' \hat{b}$

$= \frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} + \underbrace{\langle \vec{v}' \rangle}_{v_\parallel' \hat{b}} \cdot \nabla \vec{u}$

Now $\left\langle \frac{dv_\parallel'}{dt} \right\rangle = \left\langle \vec{v}' \cdot \frac{D\hat{b}}{Dt} \right\rangle + \hat{b} \cdot \left\langle \frac{d\vec{v}'}{dt} \right\rangle - \hat{b} \cdot \left\langle \frac{D\vec{u}}{Dt} \right\rangle =$

along particle trajectory: $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u}\cdot\nabla + \vec{v}'\cdot\nabla$

$= v_\parallel' \hat{b} \cdot \left(\frac{\partial \hat{b}}{\partial t} + \vec{u}\cdot\nabla\hat{b} \right) + \langle \vec{v}'\vec{v}' \rangle : \nabla \hat{b} =$

$\frac{\partial \hat{b}}{\partial t} + \vec{u}\cdot\nabla\hat{b} = 0$ because $|\hat{b}|^2 = 1$

$= \left[\frac{v_\perp'^2}{2} (\mathbb{1} - \hat{b}\hat{b}) + v_\parallel'^2 \hat{b}\hat{b} \right] : \nabla \hat{b} = \frac{v_\perp'^2}{2} \nabla\cdot\hat{b} + \left(v_\parallel'^2 - \frac{v_\perp'^2}{2} \right) \hat{b}\cdot\nabla \frac{|\hat{b}|^2}{2} = -\frac{\mu_s}{m_s} \nabla_\parallel B$

$\hat{b}\cdot\nabla|\hat{b}|^2 = 0$ because $\nabla\cdot\hat{b} = -\frac{\nabla_\parallel B}{B}$

$$= -\hat{b}\cdot\frac{d\vec{u}}{dt} - v_\parallel' \hat{b}\hat{b} : \nabla\vec{u} - \frac{\mu_s}{m_s} \nabla_\parallel B + \hat{b}\cdot\left\langle \frac{e_s}{m_s} \left(\vec{E} + \frac{\vec{v}\times\vec{B}}{c} \right) \right\rangle \qquad (16)$$

non-inertial force because we are using peculiar velocity

mirror force

$\frac{e_s}{m_s} E_\parallel$

par. electric acceleration

Proof of μ conservation:

$$\left\langle \frac{d\mu_s}{dt} \right\rangle = \left\langle \frac{d}{dt}\frac{m_s v_\perp'^2}{2B} \right\rangle = -\mu_s \left\langle \frac{D\ln B}{Dt} \right\rangle + \frac{m_s}{2B}\left\langle \frac{d}{dt}\left[\vec{v}'\cdot(\mathbb{1}-\hat{b}\hat{b})\cdot\vec{v}'\right]\right\rangle$$

$$= -\mu_s\left(\frac{d\ln B}{dt} + v_\parallel'\hat{b}\cdot\nabla\ln B\right) + \frac{m_s}{B}(\mathbb{1}-\hat{b}\hat{b}):\left\langle \vec{v}'\frac{d\vec{v}'}{dt}\right\rangle - \frac{m_s}{2B}\left\langle \vec{v}'\cdot\frac{D\hat{b}}{Dt}\right\rangle v_\parallel'$$

$$= -\mu_s\left(\frac{d\ln B}{dt} + v_\parallel'\hat{b}\cdot\nabla\ln B\right) + \frac{m_s}{B}(\mathbb{1}-\hat{b}\hat{b}):\left\langle \vec{v}'\left[\frac{e_s}{m_s}\left(\vec{E}+\frac{\vec{u}\times\vec{B}}{c}+\frac{\vec{v}'\times\vec{B}}{c}\right) - \frac{D\vec{u}}{Dt}\right]\right\rangle$$

$$-\frac{m_s v_\parallel'}{B}\left\langle \vec{v}'\vec{v}'\right\rangle:\nabla\hat{b} =$$

$$\frac{m_s}{B}(\mathbb{1}-\hat{b}\hat{b}):\left\{v_\parallel'\hat{b}\left[\frac{e_s}{m_s}\left(\vec{E}+\frac{\vec{u}\times\vec{B}}{c}\right) - \frac{d\vec{u}}{dt}\right]\right\}$$

$$+\frac{m_s}{B}\left\langle \vec{v}_\perp'\cdot\frac{\vec{v}_\perp'\times\vec{B}}{c}\right\rangle - \frac{m_s}{B}(\mathbb{1}-\hat{b}\hat{b}):\left\langle \vec{v}'\vec{v}'\cdot\nabla\vec{u}\right\rangle$$

$$= -\frac{m_s}{B}(\mathbb{1}-\hat{b}\hat{b}):\left[\frac{v_\perp'^2}{2}(\mathbb{1}-\hat{b}\hat{b})\cdot\nabla\vec{u} + v_\parallel'^2\hat{b}\hat{b}\cdot\nabla\vec{u}\right]$$

$$= -\frac{m_s v_\perp'^2}{2B}(\mathbb{1}-\hat{b}\hat{b}):\nabla\vec{u} = -\mu_s\left(\nabla\cdot\vec{u} - \hat{b}\hat{b}:\nabla\vec{u}\right)$$

$$\frac{m_s v_\perp'^2}{2B}v_\parallel'\hat{b}\cdot\nabla\ln B \quad\text{and}\quad \frac{v_\perp'^2}{2}\nabla\cdot\hat{b}$$

$$= -\mu_s\left(\frac{d\ln B}{dt} - \hat{b}\hat{b}:\nabla\vec{u} + \nabla\cdot\vec{u}\right) = 0$$ from induction eqn

because $$\frac{dB}{dt} = B\left(\hat{b}\hat{b}:\nabla\vec{u} - \nabla\cdot\vec{u}\right)$$

So, the kinetic equation is

$$\boxed{\frac{df_s}{dt} + v_{\parallel}'\hat{b}\cdot\nabla f_s + \left(-\hat{b}\cdot\frac{d\vec{u}}{dt} - v_{\parallel}'\hat{b}\hat{b}:\nabla\vec{u} - \frac{\mu_s\hat{b}\cdot\nabla B}{m_s} + \frac{e_s}{m_s}E_{\parallel}\right)\frac{\partial f_s}{\partial v_{\parallel}'} = 0} \quad (17)$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{u}\cdot\nabla$.

From this equation, we calculate pressures:

note that $v_{\perp}'^2 = \frac{2B\mu_s}{m_s}$, so $2\pi v_{\perp}' dv_{\perp}' = \pi dv_{\perp}'^2 = \frac{2\pi B}{m_s}d\mu_s$

and

$$\left.\begin{aligned} p_{\perp s} &= \frac{2\pi B^2}{m_s}\int_0^{\infty} d\mu_s \int_{-\infty}^{+\infty} dv_{\parallel}'\ \mu_s f_s \\ p_{\parallel s} &= 2\pi B \int_0^{\infty} d\mu_s \int_{-\infty}^{+\infty} dv_{\parallel}'\ v_{\parallel}'^2 f_s \end{aligned}\right\} \rightarrow \text{into eq. (14)}$$

The full system is eq. (17) and

(2): $\frac{\partial n_i}{\partial t} + \nabla\cdot(n_i\vec{u}) = 0$ ← superfluous - density info already contained in eq. (17)

(14): $m_i n_i \frac{d\vec{u}}{dt} = -\nabla\left(p_{\perp} + \frac{B^2}{8\pi}\right) + \nabla\cdot\left[\hat{b}\hat{b}\left(p_{\perp} - p_{\parallel} + \frac{B^2}{4\pi}\right)\right]$

This is kinetic MHD

(11): $\frac{\partial\vec{B}}{\partial t} = \nabla\times(\vec{u}\times\vec{B}) \Rightarrow \frac{d\vec{B}}{dt} = \vec{B}\cdot\nabla\vec{u} - \vec{B}\nabla\cdot\vec{u}$ (18)

(10): $E_{\parallel} = -\frac{(\nabla\cdot\hat{P}_e)\cdot\hat{b}}{en_e} = -\frac{\hat{b}\cdot\nabla p_{\perp e} - \{\nabla\cdot[\hat{b}\hat{b}(p_{\perp e} - p_{\parallel e})]\}\cdot\hat{b}}{en_e} =$

$$= -\frac{\cancel{\hat{b}\cdot\nabla p_{\perp e}} - (\nabla\cdot\hat{b})(p_{\perp e} - p_{\parallel e}) - \cancel{\hat{b}\cdot\nabla(p_{\perp e} - p_{\parallel e})}}{en_e} =$$

$$= -\frac{\hat{b}\cdot\nabla p_{\parallel e}}{en_e} + (p_{\perp e} - p_{\parallel e})\frac{\nabla\cdot\hat{b}}{en_e} = \frac{-\hat{b}_0\cdot\nabla B}{B} \quad (19)$$

(2) Linear Theory

Let us linearise the kinetic MHD equations around a uniform equilibrium, so

$$\vec{u} = \vec{u}_0 + \delta\vec{u} \qquad \vec{u}_0 = \text{const} \ (=0)$$
$$\vec{B} = \vec{B}_0 + \delta\vec{B} \qquad \vec{B}_0 = \text{const (straight, uniform)}$$

and an eq. particle distribution function that I will specify later, but that has

$$n_{0i} = Z n_{0e} = \text{const}$$

$$p_{\perp 0} = \text{const}, \quad p_{\parallel 0} = \text{const}, \quad p_{\perp 0} \neq p_{\parallel 0}$$

Linearising (14),

$$m_i n_{0i}(-i\omega)\delta\vec{u} = -i\vec{k}\left(\delta p_\perp + \frac{B_0 \delta B_\parallel}{4\pi}\right) +$$

($\vec{k} = \vec{k}_\perp + k_\parallel \hat{b}_0$; because $\delta\hat{b} = \frac{\delta\vec{B}_\perp}{B_0}$ and $\vec{k}\cdot\delta\vec{B} = 0$, $-k_\parallel \frac{\delta B_\parallel}{B_0}$)

$$+ \hat{b}_0\left[\left(p_{\perp 0} - p_{\parallel 0} + \frac{B_0^2}{4\pi}\right) i\vec{k}\cdot\delta\hat{b} + i k_\parallel \left(\delta p_\perp - \delta p_\parallel + \frac{B_0 \delta B_\parallel}{2\pi}\right)\right]$$

$$+ \delta\hat{b}\, i k_\parallel \left(p_{\perp 0} - p_{\parallel 0} + \frac{B_0^2}{4\pi}\right) \tag{20}$$

$$= -i\vec{k}_\perp\left(\delta p_\perp + \frac{B_0^2}{4\pi}\frac{\delta B_\parallel}{B_0}\right) -$$

(from (18), $-i\omega\frac{\delta B_\parallel}{B_0} = -i\vec{k}_\perp\cdot\delta\vec{u}_\perp$)

$$- i\hat{b}_0 k_\parallel\left[\left(p_{\perp 0} - p_{\parallel 0}\right)\frac{\delta B_\parallel}{B_0} + \delta p_\parallel\right]$$

"magnetosonic perturbations"

$$+ i\delta\hat{b}\, k_\parallel \left(p_{\perp 0} - p_{\parallel 0} + \frac{B_0^2}{4\pi}\right)$$

(from (18) $-i\omega\delta\hat{b} = -i\omega\frac{\delta\vec{B}_\perp}{B_0} = i k_\parallel \delta\vec{u}_\perp$ (curvature perturbation))

"Alfvénic perturbations"

We see that we can separate the Alfvénic perturbations:

formally, take $\vec{k}_\perp \times (20)_\perp$:

$$-m_i n_{oi}\,\omega\left(\vec{k}_\perp \times \delta\vec{u}_\perp\right) = -\frac{k_\parallel}{\omega}\left(\vec{k}_\perp \times \delta\vec{u}_\perp\right)\cdot k_\parallel\left(p_{\perp o} - p_{\parallel o} + \frac{B_o^2}{4\pi}\right)$$

$$\boxed{\omega^2 = k_\parallel^2\left(\frac{p_{\perp o} - p_{\parallel o}}{m_i n_{oi}} + V_A^2\right)} \quad \text{where} \quad V_A = \frac{B_o}{\sqrt{4\pi n_{oi} m_i}} \tag{21}$$

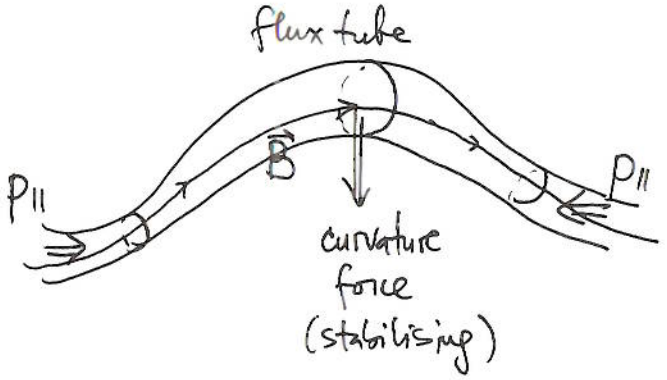
If $p_{\perp o} > p_{\parallel o}$, these are just Alfvén waves (modified by the pressure anisotropy).

If $p_{\perp o} < p_{\parallel o}$, we may have $\omega^2 < 0$ and an instability.

Criterion:
$$\frac{p_{\parallel o} - p_{\perp o}}{m_i n_{oi}} > V_A^2 = \frac{B_o^2}{4\pi m_i n_{oi}}$$

or
$$\boxed{\frac{p_{\parallel o} - p_{\perp o}}{p_{\perp o}} > \frac{B_o^2}{4\pi p_{\perp o}} \equiv \frac{2}{\beta_\perp}} \tag{22}$$

This is called the firehose instability



If there is not enough tension in the magnetic field, it cannot spring back.

Refs: Rosenbluth, LANL Report 2030 (1956)
Chandrasekhar, Kaufman, Watson, Proc. Roy. Soc. A245, 435 (1958)
Parker, Phys. Rev. 109, 1874 (1958)
Vedenov & Sagdeev, Doklady 3, 278 (1958)

Note that in order to get the firehose instability, we did not need to do any kinetics (no pressure perturbations were present in the Alfvénic/firehose fluctuations).

In order to find out what happens to magnetosonic perturbations, we will need to calculate $\delta p_\perp$ and $\delta p_\parallel$ from the kinetic equation (17).

Let us linearise this equation around some eq. distribution about which we will only assume for now that it does not depend on space and time (or depends on them slowly). Later on we will see how an equilibrium distribution with pressure anisotropy can emerge.

So, eq. (17) is linearised as follows:

$$-i\omega\delta f_s + ik_\parallel v_\parallel' \delta f_s + \left(i\omega \delta u_\parallel - ik_\parallel v_\parallel' \delta u_\parallel - \frac{\mu_s k_\parallel \delta B_\parallel}{m_s} + \frac{e_s}{m_s}\delta E_\parallel \right) \frac{\partial f_{0s}}{\partial v_\parallel'} = 0$$

$$\delta f_s = \delta u_\parallel \frac{\partial f_{0s}}{\partial v_\parallel'} - \underbrace{\frac{\mu_s k_\parallel \delta B_\parallel + i e_s \delta E_\parallel}{\omega - k_\parallel v_\parallel'}}_{\text{Barnes/Landau resonance}} \frac{\perp}{m_s} \frac{\partial f_{0s}}{\partial v_\parallel'} \qquad (23)$$

Barnes/Landau resonance
-so kinetics are essential!

Now we calculate the pressure perturbations:

$$\delta p_{\perp s} = \delta\left[\frac{2\pi B_0^2}{m_s}\int_0^\infty d\mu_s \mu_s \int_{-\infty}^{+\infty} dv_\parallel' f_s\right] =$$

(integrate along Landau contour; $\delta v_\parallel \frac{\partial f_{0s}}{\partial v_\parallel'}$ vanishes)

$$= 2\frac{\delta B_\parallel}{B_0} p_{\perp 0s} + \frac{2\pi B_0^2}{m_s}\int_0^\infty d\mu_s \mu_s \int_{-\infty}^{+\infty} dv_\parallel' \left[\delta v_\parallel \frac{\partial f_{0s}}{\partial v_\parallel'} - \frac{\mu_s k_\parallel \delta B_\parallel + i e_s \delta E_\parallel}{\omega - k_\parallel v_\parallel'} \frac{1}{m_s}\frac{\partial f_{0s}}{\partial v_\parallel'}\right]$$

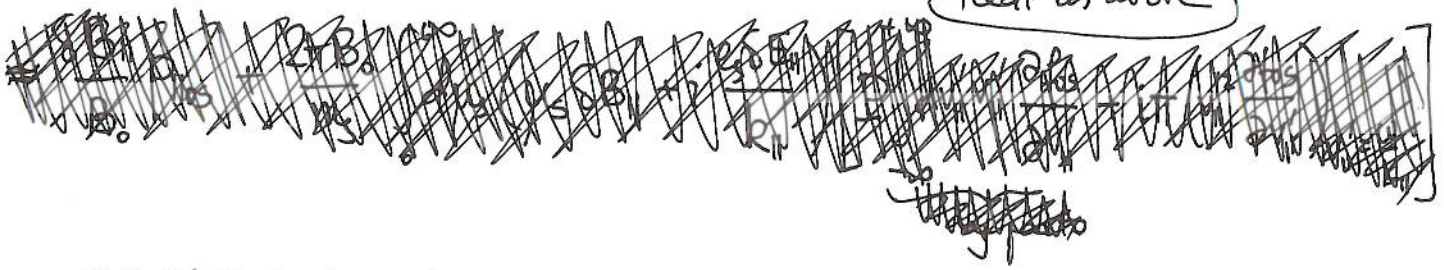
$$= 2\frac{\delta B_\parallel}{B_0} p_{\perp 0s} + \frac{2\pi B_0^2}{m_s^2}\int_0^\infty d\mu_s \mu_s \left(\mu_s \delta B_\parallel + i\frac{e_s \delta E_\parallel}{k_\parallel}\right)\left[\underbrace{P\int_{-\infty}^{+\infty} dv_\parallel' \frac{1}{v_\parallel' - \frac{\omega}{k_\parallel}}\frac{\partial f_{0s}}{\partial v_\parallel'}}_{\text{principal value integral - this can be computed in the low-freq. limit: } \omega \ll k_\parallel v_\parallel'} + i\pi \frac{\partial f_{0s}}{\partial v_\parallel'}\bigg|_{v_\parallel' = \omega/k_\parallel}\right]$$

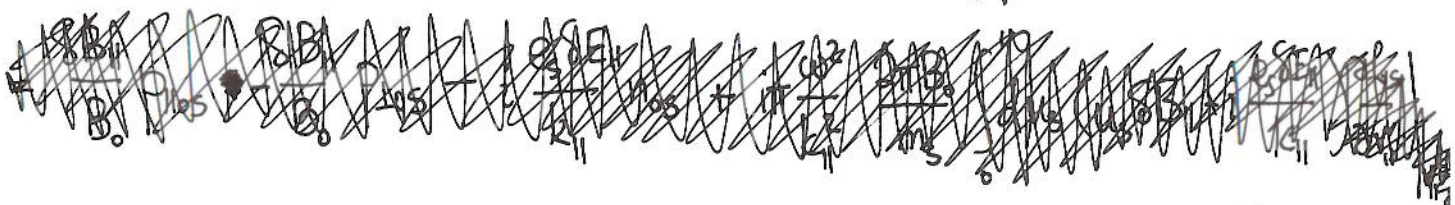
$$\approx 2\frac{\delta B_\parallel}{B_0} p_{\perp 0s} + \frac{2\pi B_0^2}{m_s^2}\int_0^\infty d\mu_s \mu_s \left(\mu_s \delta B_\parallel + i\frac{e_s \delta E_\parallel}{k_\parallel}\right)\left[P\int_{-\infty}^{+\infty} \frac{dv_\parallel'}{v_\parallel'}\frac{\partial f_{0s}}{\partial v_\parallel'} + i\pi \frac{\partial f_{0s}}{\partial v_\parallel'}\bigg|_{v_\parallel' = \omega/k_\parallel}\right]$$

$$\delta p_{\parallel s} = \delta\left[2\pi B \int_0^\infty d\mu_s \int_{-\infty}^{+\infty} dv_\parallel' v_\parallel'^2 f_s\right] = \quad (24)$$

(vanishes after integration by parts and using $\int dv_\parallel' v_\parallel' f_{0s} = 0$)

$$= \frac{\delta B_\parallel}{B_0} p_{\parallel 0s} + 2\pi B_0 \int_0^\infty d\mu_s \int_{-\infty}^{+\infty} dv_\parallel' v_\parallel'^2 \left[\delta v_\parallel \frac{\partial f_{0s}}{\partial v_\parallel'} - \underbrace{\frac{\mu_s k_\parallel \delta B_\parallel + i e_s \delta E_\parallel}{\omega - k_\parallel v_\parallel'}}_{\text{treat as above}} \frac{1}{m_s}\frac{\partial f_{0s}}{\partial v_\parallel'}\right] =$$





$$= \frac{\delta B_\parallel}{B_0} p_{\parallel os} + \frac{2\pi B_0}{m_s} \int_0^\infty d\mu_s \left(\mu_s \delta B_\parallel + i \frac{e_s \delta E_\parallel}{k_\parallel}\right) \left[\mathcal{P} \int_{-\infty}^{+\infty} dv_\parallel' \frac{v_\parallel'^2}{v_\parallel' - \frac{\omega}{k_\parallel}} \frac{\partial f_{os}}{\partial v_\parallel'} + \right.$$

$$\left. + i\pi \frac{\omega^2}{k_\parallel^2} \frac{\partial f_{os}}{\partial v_\parallel'}\bigg|_{v_\parallel' = \frac{\omega}{|k_\parallel|}} \right]$$

$$= \frac{\delta B_\parallel}{B_0} p_{\parallel os} +$$

$$- \frac{\delta B_\parallel}{B_0} p_{\perp os} - i \frac{e_s n_{os}}{k_\parallel} \delta E_\parallel$$

$$+ \frac{\omega^2}{k_\parallel^2} \frac{2\pi B_0}{m_s} \int_0^\infty d\mu_s \left(\mu_s \delta B_\parallel + i \frac{e_s \delta E_\parallel}{k_\parallel}\right) \left[\mathcal{P} \int_{-\infty}^{+\infty} \frac{dv_\parallel'}{v_\parallel'} \frac{\partial f_{os}}{\partial v_\parallel'} + i\pi \frac{\partial f_{os}}{\partial v_\parallel'}\bigg|_{v_\parallel' = \frac{\omega}{|k_\parallel|}} \right] \quad (25)$$

$$\mathcal{P} \int_{-\infty}^{+\infty} dv_\parallel' \frac{v_\parallel'^2 - \frac{\omega^2}{k_\parallel^2} + \frac{\omega^2}{k_\parallel^2}}{v_\parallel' - \frac{\omega}{k_\parallel}} \frac{\partial f_{os}}{\partial v_\parallel'} =$$

(integral vanishes; expand $\frac{\omega}{k_\parallel v_\parallel'} \ll 1$)

$$= \mathcal{P} \int_{-\infty}^{+\infty} dv_\parallel' \left[v_\parallel' + \frac{\omega}{k_\parallel} + \frac{\omega^2}{k_\parallel^2} \frac{1}{v_\parallel' - \frac{\omega}{k_\parallel}} \right] \frac{\partial f_{os}}{\partial v_\parallel'} =$$

$$= -\int_{-\infty}^{+\infty} dv_\parallel' f_{os} + \frac{\omega^2}{k_\parallel^2} \mathcal{P} \int_{-\infty}^{+\infty} dv_\parallel' \frac{1}{v_\parallel'} \frac{\partial f_{os}}{\partial v_\parallel'}$$

This eq. will help us calculate $\delta E_\parallel$: the linearized (19) is

$$\delta E_\parallel = - \frac{i k_\parallel \delta p_{\parallel e}}{e n_{oe}} - \frac{p_{\perp oe} - p_{\parallel oe}}{e n_{oe}} i k_\parallel \frac{\delta B_\parallel}{B_0} \quad (26)$$

$$i \frac{e n_{oe}}{k_\parallel} \delta E_\parallel = \delta p_{\parallel e} + (p_{\perp oe} - p_{\parallel oe}) \frac{\delta B_\parallel}{B_0} =$$

↑ substitute (25)

NB $e_e = -e$

$$= \frac{\delta B_\parallel}{B_0} (p_{\parallel oe} - p_{\perp oe}) + i \frac{e n_{oe}}{k_\parallel} \delta E_\parallel + (p_{\perp oe} - p_{\parallel oe}) \frac{\delta B_\parallel}{B_0}$$

$$+ \frac{\omega^2}{k_\parallel^2} \frac{2\pi B_0}{m_e} \int_0^\infty d\mu_e \left(\mu_e \delta B_\parallel - i \frac{e \delta E_\parallel}{k_\parallel}\right) \left[\mathcal{P} \int_{-\infty}^{+\infty} \frac{dv_\parallel'}{v_\parallel'} \frac{\partial f_{oe}}{\partial v_\parallel'} + i\pi \frac{\partial f_{oe}}{\partial v_\parallel'}\bigg|_{v_\parallel' = \frac{\omega}{|k_\parallel|}} \right]$$

These are small as $\sqrt{m_e/m_i}$

So, setting the last term to 0, we have

$$i\frac{e\delta E_\parallel}{k_\parallel} = \frac{\dfrac{2\pi B_0^2}{m_e}\displaystyle\int_0^\infty d\mu_e \mu_e \left[P\int_{-\infty}^{+\infty} \frac{dv_\parallel'}{v_\parallel'}\frac{\partial f_{0e}}{\partial v_\parallel'} + i\pi \frac{\partial f_{0e}}{\partial v_\parallel'}\Big|_{v_\parallel'=\omega/k_\parallel}\right]}{\dfrac{2\pi B_0}{m_e}\displaystyle\int_0^\infty d\mu_e \left[P\int_{-\infty}^{+\infty} \frac{dv_\parallel'}{v_\parallel'}\frac{\partial f_{0e}}{\partial v_\parallel'} + i\pi \frac{\partial f_{0e}}{\partial v_\parallel'}\Big|_{v_\parallel'=\omega/k_\parallel}\right]} \frac{\delta B_\parallel}{B_0} \qquad (27)$$

$\equiv \tilde{p}_e/n_{0e}$ (short-hand)

We are now ready to derive the dispersion relation:

take eq. (20) $\cdot i\vec{k}_\perp$ and use $\vec{k}_\perp\cdot\delta\vec{u}_\perp = \omega\dfrac{\delta B_\parallel}{B_0}$ and

$\vec{k}_\perp\cdot\delta\hat{b} = -\dfrac{k_\parallel}{\omega}\vec{k}_\perp\cdot\delta\vec{u}_\perp = -k_\parallel\dfrac{\delta B_\parallel}{B_0}$. This gives

$$m_i n_{0i}\omega^2\frac{\delta B_\parallel}{B_0} = k_\perp^2\left(\delta p_\perp + \frac{B_0^2}{4\pi}\frac{\delta B_\parallel}{B_0}\right) + k_\parallel^2\left(p_{\perp 0} - p_{\parallel 0} + \frac{B_0^2}{4\pi}\right)\frac{\delta B_\parallel}{B_0} \qquad (28)$$

this term is small for $\frac{\omega}{k_\parallel v_\parallel'} \ll 1$

Substitute (24) and (27) $\propto \left(\frac{\delta B_\parallel}{B_0}\right)^2$

$e_i = Ze$

$$\delta p_{\perp i} = \frac{\delta B_\parallel}{B_0}\left\{ 2p_{\perp 0i} + \frac{2\pi B_0^2}{m_i^2}\int_0^\infty d\mu_i \mu_i\left(\mu_i B_0 + \frac{Z\tilde{p}_e}{n_{0e}}\right)\left[P\int_{-\infty}^{+\infty}\frac{dv_\parallel'}{v_\parallel'}\frac{\partial f_{0e}}{\partial v_\parallel'} + i\pi\frac{\partial f_{0i}}{\partial v_\parallel'}\Big|_{v_\parallel'=\frac{\omega}{k_\parallel}}\right]\right\}$$

$$\delta p_{\perp e} = \frac{\delta B_\parallel}{B_0}\left\{ p_{\perp 0e} + \frac{2\pi B_0^2}{m_e^2}\int_0^\infty d\mu_e \mu_e\left(\mu_e B_0 - \frac{\tilde{p}_e}{n_{0e}}\right)\left[P\int_{-\infty}^{+\infty}\frac{dv_\parallel'}{v_\parallel'}\frac{\partial f_{0e}}{\partial v_\parallel'} + i\pi\frac{\partial f_{0e}}{\partial v_\parallel'}\Big|_{v_\parallel'=\frac{\omega}{k_\parallel}}\right]\right\} \qquad (29)$$

This is smaller than the imaginary term in $\delta p_{\perp i}$ by $(m_e/m_i)^{1/2}$

To simplify things, I want to make 2 further assumptions:

1) $\boxed{f_{0i} = f_{0i}(\mu_i, v_\parallel'^2)}$ and $\boxed{\left.\frac{\partial f_{0i}}{\partial v_\parallel'^2}\right|_{v_\parallel'=0} \neq 0}$ (<0)

Then $\frac{\partial f_{0i}}{\partial v_\parallel'} = 2v_\parallel' \frac{\partial f_{0i}}{\partial v_\parallel'^2}$ and $\left.\frac{\partial f_{0i}}{\partial v_\parallel'}\right|_{v_\parallel'=\frac{\omega}{|k_\parallel|}} \approx \frac{2\omega}{|k_\parallel|} \frac{\partial f_{0i}}{\partial v_\parallel'^2}(\mu_i, 0)$

2) Electron pressure anisotropy is negligible — we will see later on that this is justified in certain physical contexts and that $\frac{p_{\perp e} - p_{\parallel e}}{p_{\perp i} - p_{\parallel i}} \sim \sqrt{\frac{m_e}{m_i}}$, so should be consistently ignored.

So, $\boxed{f_{0e} = f_{0e}(v'^2)}$, $v'^2 = v_\perp'^2 + v_\parallel'^2 = \frac{2B_0}{m_e}\mu_e + v_\parallel'^2$, $p_{\perp 0e} = p_{\parallel 0e} = p_{0e}$

$$\frac{1}{v_\parallel'}\frac{\partial f_{0e}}{\partial v_\parallel'} = 2\frac{\partial f_{0e}}{\partial v_\parallel'^2} = \frac{m_e}{B_0}\frac{\partial f_{0e}}{\partial \mu_e}$$

$$\frac{2\pi B_0^2}{m_e^2}\int_0^\infty d\mu_e\, \mu_e \left(\mu_e B_0 - \frac{\tilde{p}_e}{n_{0e}}\right) \mathcal{P}\int_{-\infty}^{+\infty} \frac{dv_\parallel'}{v_\parallel'}\frac{\partial f_{0e}}{\partial v_\parallel'} \overset{\text{by parts}}{=} -\frac{2\pi B_0}{m_e}\int_0^\infty d\mu_e \left(2B_0 - \frac{\tilde{p}_e}{n_{0e}}\right)\int_{-\infty}^{+\infty} dv_\parallel' f_{0e}$$

$$= -2p_{0e} + \tilde{p}_e \overset{\text{eq.(27)}}{=} -2p_{0e} + \frac{-2\pi B_0 \int_0^\infty d\mu_e \int_{-\infty}^{+\infty} dv_\parallel' f_{0e}}{+2\pi \int_0^\infty d\mu_e \int_{-\infty}^{+\infty} dv_\parallel' \frac{\partial f_{0e}}{\partial \mu_e}} n_{0e} =$$

$$= -2p_{0e} + \underbrace{\frac{m_e n_{0e}^2}{2\pi \int_{-\infty}^{+\infty} dv_\parallel' f_{0e}(0, v_\parallel')}}_{\tilde{p}_e} \quad \Rightarrow \delta p_{\perp e} = \tilde{p}_e \frac{\delta B_\parallel}{B_0} \qquad (30)$$

(see eq.(29))

NB: If f_{0e} is a Maxwellian, $\tilde{p}_e = p_{0e}$.

So, with these simplifications, eq. (29) is

$$\delta p_\perp = \underbrace{\delta p_{\perp i} + \delta p_{\perp e}}_{\delta p_{\perp}} = \frac{\delta B_\parallel}{B_0}\left\{ 2p_{\perp 0i} + \tilde{p}_e + \frac{4\pi B_0^2}{m_i^2}\int_0^\infty d\mu_i \mu_i \left(\mu_i B_0 + \frac{Z\tilde{p}_e}{n_{0e}}\right)\int_{-\infty}^{+\infty} dv_\parallel' \frac{\partial f_{0e}}{\partial v_\parallel'^2} + \right.$$

$$\left. + i\frac{\omega}{|k_\parallel|}\frac{4\pi^2 B_0^2}{m_i^2}\int_0^\infty d\mu_i \mu_i \left(\mu_i B_0 + \frac{Z\tilde{p}_e}{n_{0e}}\right)\frac{\partial f_{0i}}{\partial v_\parallel'^2}(\mu_i, 0)\right\} \quad (31)$$

Substitute this into (28):

$$i\frac{\omega}{|k_\parallel|}\frac{4\pi^2 B_0^2}{m_i^2}\int_0^\infty d\mu_i \mu_i \left(\mu_i B_0 + \frac{Z\tilde{p}_e}{n_{0e}}\right)\frac{\partial f_{0i}}{\partial v_\parallel'^2}(\mu_i, 0) =$$

$$= -\left[2p_{\perp 0i} + \tilde{p}_e + \frac{4\pi B_0^2}{m_i^2}\int_0^\infty d\mu_i \mu_i \left(\mu_i B_0 + \frac{Z\tilde{p}_e}{n_{0e}}\right)\int_{-\infty}^{+\infty} dv_\parallel' \frac{\partial f_{0i}}{\partial v_\parallel'^2}\right] - \frac{B_0^2}{4\pi}$$

$$- \frac{k_\parallel^2}{k_\perp^2}\left(p_{\perp 0i} - p_{\parallel 0i} + \frac{B_0^2}{4\pi}\right)$$

Using $\gamma = -i\omega$ (>0 means growth) and $\mu_i = \frac{m_i v_\perp'^2}{2B_0}$, we have

$$\gamma = \frac{|k_\parallel|}{\pi}\left[1 + \frac{\tilde{p}_e}{2p_{\perp 0i}} + \frac{1}{p_{\perp 0i}}\int d^3\vec{v}\,\frac{v_\perp'^2}{2}\left(\frac{m_i v_\perp'^2}{2} + \frac{Z\tilde{p}_e}{n_{0e}}\right)\frac{\partial f_{0i}}{\partial v_\parallel'^2} + \frac{1}{\beta_{\perp i}} + \underbrace{\frac{k_\parallel^2}{2k_\perp^2}\left(\frac{p_{\perp 0i} - p_{\parallel 0i}}{p_{\perp 0i}} + \frac{2}{\beta_{\perp i}}\right)}\right]\left[\frac{1}{p_{\perp 0i}}\int d^2\vec{v}_\perp \frac{v_\perp'^2}{2}\left(\frac{m_i v_\perp'^2}{2} + \frac{Z\tilde{p}_e}{n_{0e}}\right)\underbrace{\frac{\partial f_{0i}}{\partial v_\parallel'^2}(v_\perp', 0)}\right]^{-1} \quad (32)$$

Instability if this bracket is negative

This is negative (usually)

– the mirror instability

These terms are often dropped, assuming cold electrons.

Instability condition:

$$-\frac{1}{p_{\perp oi}}\int d^3\vec{v}\,\frac{v_\perp'^2}{2}\left(\frac{m_i v_\perp'^2}{2}+\frac{Z\tilde{p}_e}{n_{oe}}\right)\frac{\partial f_{oi}}{\partial v_\parallel'^2} > 1+\frac{\tilde{p}_e}{2p_{\perp oi}}+\frac{1}{\beta_{\perp i}} \qquad (33)$$

This is the general instability condition [Shapiro & Shevchenko Sov. Phys. JETP 18, 1109 (1964)] but we cannot really make physical sense of it without specifying the equilibrium distribution function, so we are back to the question of the origin of the anisotropy. We are about to address it, but first let us try the simplest conceivable anisotropic distribution: a bi-Maxwellian:

$$f_{oi}=\left(\frac{m_i}{2\pi}\right)^{3/2}\frac{n_{oi}}{T_{\perp oi}T_{\parallel i}^{1/2}}\,e^{-\frac{m_i v_\perp'^2}{2T_{\perp i}}-\frac{m_i v_\parallel'^2}{2T_{\parallel i}}} \qquad (34)$$

(and a Maxwellian $f_{oe} \Rightarrow \tilde{p}_e = p_{oe}$). Then

$$\text{lhs of (33)} = \frac{1}{n_{oi}T_{\perp i}}\frac{m_i}{2T_{\parallel i}}\int d^3\vec{v}\,\frac{v_\perp'^2}{2}\left(\frac{m_i v_\perp'^2}{2}+\frac{Z p_{eo}}{n_{oe}}\right)f_{oi} = \frac{p_{\perp oi}}{p_{\parallel oi}}+\frac{p_{oe}}{2p_{\parallel oi}}$$

so the instability condition becomes

$$\Delta \equiv \frac{p_{\perp i}-p_{\parallel oi}}{p_{\perp oi}} > \frac{2p_{\parallel oi}}{2p_{\perp oi}+p_{oe}}\,\frac{1}{\beta_{\perp i}} \qquad (35)$$

(So $p_\perp > p_\parallel$, opposite to firehose)

from (32),

$$\gamma = \frac{|k_\parallel|\,v_{th\parallel i}}{\sqrt{\pi}}\left[\Delta-\frac{2p_{\parallel oi}}{2p_{\perp oi}+p_{oe}}\,\frac{1}{\beta_{\perp i}}+\frac{k_\parallel^2}{2k_\perp^2}\left(\Delta+\frac{2}{\beta_{\perp i}}\right)\frac{2p_{\parallel oi}}{2p_{\perp oi}+p_{oe}}\right] \qquad (36)$$

Refs: Hasegawa, Phys. Fluids 12, 2642 (1969)

So we have completed the (linear) formal theory of firehose and mirror instabilities. We now want to address a number of physical questions:

1) Under what conditions do the instabilities occur?
   — the origin of pressure anisotropies
2) How fast do the perturbations grow?
   — we have derived the growth rates, but they all diverged as $\propto |k_\parallel|$. So, in fact, the kinetic MHD equations have a UV catastrophe and we have to include FLR effects to regularise everything. [Refs: Yoon, Wu, de Assis, Phys. Fluids B 5, 1971 (1993); Hellinger, Phys. Plasmas 14, 082105 (2007)]
3) How large are saturation amplitudes and at what scales are the saturated fluctuations sitting?
4) How do the instabilities affect transport properties?

I will attempt to answer these questions qualitatively for the rather interesting and representative astrophysical example: galaxy cluster plasmas.

Refs: ApJ 629, 139 (2005); Phys. Plasmas 13, 056501 (2006) } clusters
arXiv: 0709.3828 (PRL, accepted) ↳ reprint attached.