

Critical Balance in MHD and Rotating Turbulence (Towards a Universal Scaling Conjecture?)

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Waves + turbulence is a common mix.

often there is a mean field of some kind that set a direction of anisotropy:

- mean magnetic field in MHD
- direction of rotation axis in rotating fluids
- direction of background gradient in strat'd fluids etc.

I will consider these 2 examples and explore how far the analogy between magnetic and hydro can go.

What I really want to discuss is the rotating case - to get feedback from experts.

Anisotropy is an empirically confirmed fact for such systems. Not only in DNS but also in measurements

[For MHD, most spectacularly, for the SW - Hobrey group @ IC]. So $k_{\parallel} \ll k_{\perp}$ (wrt mean field)

System supports waves that propagate mostly along the mean field:

$$\omega = k_{\parallel} v(k_{\perp}) \leftarrow$$

MHD: $v(k_{\perp}) = v_A$ [Note: general GK disp. rln for plasma is like this too]

Rot.: $v(k_{\perp}) = \frac{2\Omega}{k_{\perp}}$

Assume fluid-type nonlinearity:

$$\vec{u} \cdot \nabla \vec{u} \approx \vec{u}_\perp \cdot \nabla_\perp \vec{u} \quad (\text{because } k_\parallel \ll k_\perp)$$

↑ perpendicular!

[NB: assume $u_\parallel \sim u_\perp$]

$$\nabla \cdot \vec{u} = 0$$

means $\nabla_\perp \cdot \vec{u}_\perp = 0$

So the nonlinear time is

$$\frac{1}{\tau_{NL}} \sim k_\perp u_\perp$$

Kolmogorov theory (dimensional K41 argument à la Osukhov):

$$k_\perp E(k_\perp) \sim u_\perp^2(k_\perp) \sim \varepsilon \tau(k_\perp)$$

↑ const. flux ↑ "cascade time"

If we had no waves, it is dimensionally inevitable that

$$\tau \sim \tau_{NL} \Rightarrow u_\perp^2 \sim \varepsilon \frac{1}{k_\perp u_\perp} \Rightarrow u_\perp \sim \left(\frac{\varepsilon}{k_\perp} \right)^{1/3}$$

$$E(k_\perp) \sim \varepsilon^{2/3} k_\perp^{-5/3}$$

If there are waves in the system, there are 2 interesting timescales, ω and τ_{NL}^{-1} .

$\omega \gg \tau_{NL}^{-1}$ weak turbulence

$\omega \ll \tau_{NL}^{-1}$ 2D turbulence

Conjecture (GS95): $\omega \sim \tau_{NL}^{-1}$ Critical Balance

Discuss alternatives in a moment, but first let's see what it implies.

$$\tau \sim \omega \sim \tau_{NL}^{-1} \Rightarrow \text{K. spectrum still holds.}$$

From CB, get a relationship between k_\parallel and k_\perp

$$\omega = k_{\parallel} v(k_{\perp}) \sim \tau \omega' \sim k_{\perp} u_{\perp} \sim \epsilon^{1/3} k_{\perp}^{2/3}$$

$$k_{\parallel} \sim \frac{\epsilon^{1/3} k_{\perp}^{2/3}}{v(k_{\perp})}$$

$\epsilon^{1/3} v_A^{-1} k_{\perp}^{2/3}$

MHD

$\epsilon^{1/3} \Omega^{-1} k_{\perp}^{5/3}$

Rot.

Anisotropy scale-dependent!

Another important implication:

$$u_{\perp} \sim \frac{\epsilon^{1/3}}{k_{\perp}^{1/3}}$$

k_{\perp} is a fn of k_{\parallel} , so we can find "parallel spectra"

MHD: $k_{\parallel} \sim \epsilon^{1/3} v_A^{-1} k_{\perp}^{2/3}$

$$\Rightarrow k_{\perp} \sim \epsilon^{-1/2} v_A^{3/2} k_{\parallel}^{3/2}$$

$$u_{\perp} \sim \frac{\epsilon^{1/3}}{\epsilon^{-1/6} v_A^{1/2} k_{\parallel}^{1/2}}$$

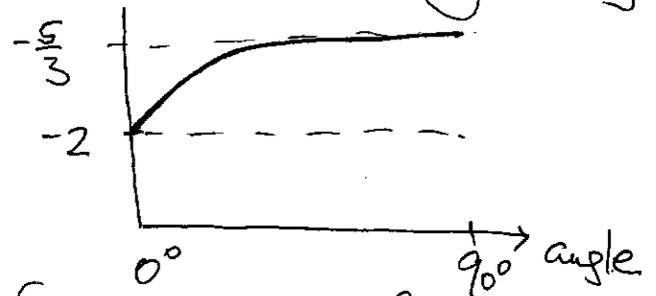
$$\Rightarrow k_{\parallel} E(k_{\parallel}) \sim u_{\perp}^2 \sim \epsilon v_A^{-1} k_{\parallel}^{-1}$$

Rotating: $k_{\parallel} \sim \epsilon^{1/3} \Omega^{-1} k_{\perp}^{5/3}$

$E(k_{\parallel}) \sim \epsilon v_A^{-1} k_{\parallel}^{-2}$

$$k_{\perp} \sim \epsilon^{-1/5} \Omega^{3/5} k_{\parallel}^{3/5}$$

Confirmed in the SW by Morfing



~~scribbled out text~~

$$u_{\perp} \sim \frac{\epsilon^{1/3}}{(\epsilon^{-1/5} \Omega^{3/5} k_{\parallel}^{3/5})}$$

$E(k_{\parallel}) \sim \epsilon^{4/5} \Omega^{-2/5} k_{\parallel}^{-7/5}$

[see also papers by R. Wicks and C. Chen]

This has never been seen - would be interesting to measure.

NB: In MHD mean field has to be computed locally
 So perhaps in rot. turbulence, one has to measure this wrt local vorticity direction.

Before I discuss further the implications of critical balance for rotating turbulence, let me exclude the alternatives.

1) Weak turbulence. $\omega\tau_{NL} \gg 1$ } Very detailed math. theory is possible, but it boils down to this:

We had $k_{\perp} E(k_{\perp}) \sim \varepsilon \tau(k_{\perp})$

\uparrow calculate the cascade time.

"One interaction" between wave packets produces a small perturbation of the amplitude:

$$\delta u_{\perp} \sim \frac{u_{\perp}}{\omega\tau_{NL}}$$

If these accumulate as a random walk, after n interactions, we have

$$n^{1/2} \delta u_{\perp} \sim u_{\perp}$$

\uparrow this defines τ , $n = \tau\omega$

$$\text{So, } (\tau\omega)^{1/2} \frac{u_{\perp}}{\omega\tau_{NL}} \sim u_{\perp} \Rightarrow \tau \sim \omega\tau_{NL}^2$$



$$u_{\perp}^2 \sim \varepsilon \omega\tau_{NL}^2 \sim \frac{\varepsilon k_{\parallel} v(k_{\perp})}{k_{\perp}^2 u_{\perp}^2}$$

$$u_{\perp} \sim \frac{\varepsilon^{1/4} k_{\parallel}^{1/4} [v(k_{\perp})]^{1/4}}{k_{\perp}^{1/2}}$$

(GS97...)

$$E(k_{\perp}) \sim \varepsilon^{1/2} k_{\parallel}^{1/2} [v(k_{\perp})]^{1/2} k_{\perp}^{-2}$$

$\rightarrow (\varepsilon k_{\parallel} v_A)^{1/2} k_{\perp}^{-2}$ MHD

$\rightarrow (\varepsilon k_{\parallel} \Omega)^{1/2} k_{\perp}^{-9/2}$ rot.

(Galtier 2003)

\rightarrow NB: One ~~usually~~ finds that there is no cascade in k_{\parallel} , so k_{\parallel} is a parameter here.

When is this valid?

$$\omega T_{NL} \sim \frac{k_{\perp} V(k_{\perp})}{k_{\perp} u_{\perp}} \gg 1$$

↑ shear rate goes up with k_{\perp}

Eventually, nonlinearity becomes comparable to ω !

MHD: $k_{\perp c} \sim \epsilon^{-1/2} (k_{\parallel} V_A)^{3/2} \sim k_0 M_A^{-2}$

Rotating:

$$k_{\perp c} \sim \epsilon^{-1/5} (k_{\parallel} \Omega)^{3/5} \sim k_0 R_0^{-4/5}$$

$$u_{rms} \sim \left(\frac{\epsilon V_A}{k_0} \right)^{1/4}$$

$$\epsilon \sim M_A^4 V_A^3 k_0$$

$$M_A \sim \frac{u_{rms}}{V_A}$$

Alfvénic Mach

So, WT regime eventually breaks down.

$$u_{rms} \sim (\epsilon \Omega)^{1/4} k_0^{-1/2}$$

$$\epsilon \sim R_0^4 \Omega^3 k_0^{-2}$$

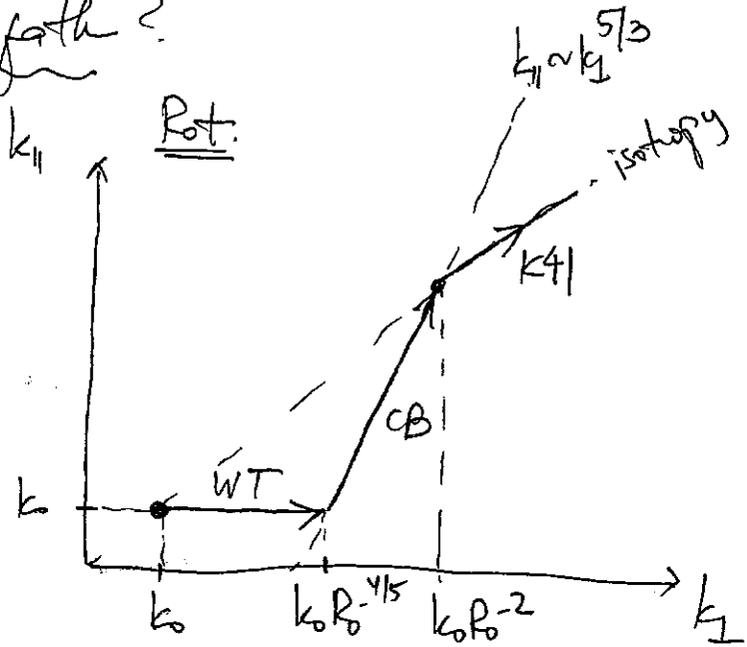
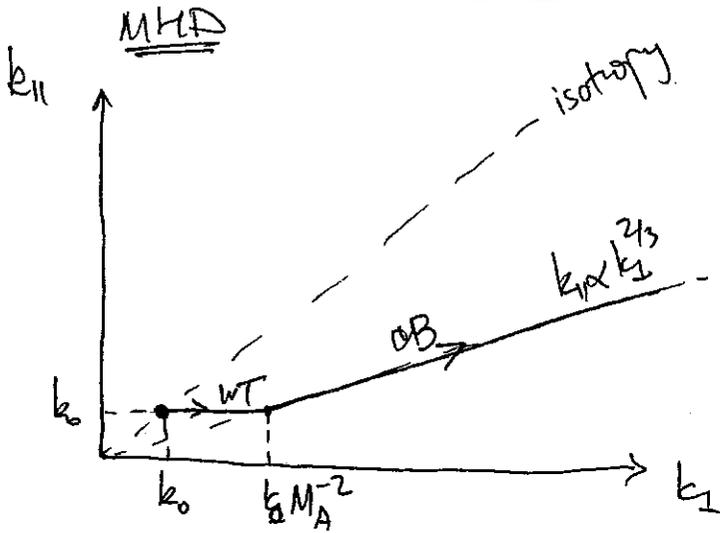
$$R_0 \sim \frac{u_{rms} k_0}{\Omega}$$

Rossby.

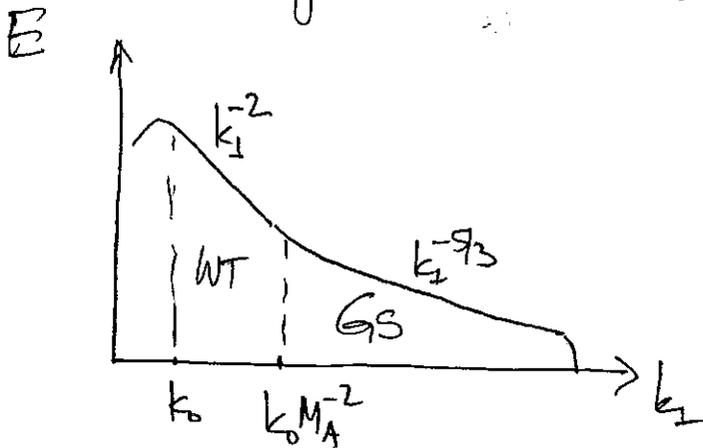
2) Why doesn't the turbulence become 2D?

Standard argument.

So, what is the cascade path?



Turbulence gets more anisotropic with scale



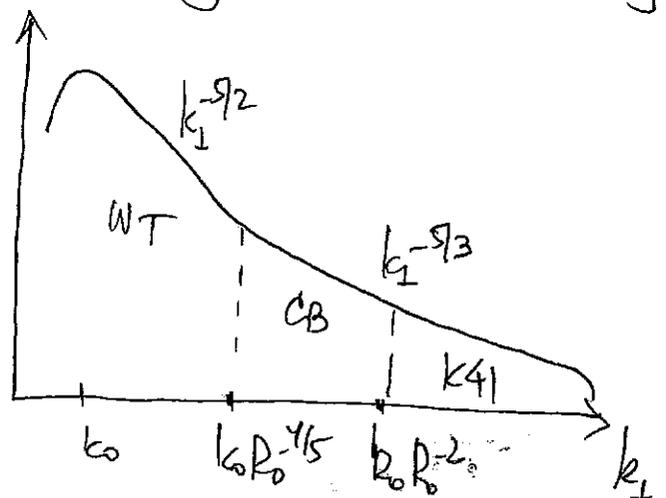
Turbulence get less anisotropic with scale and reverts to isotropy (as, in a hydro system, it must!) at

$$k_{||} \sim \epsilon^{1/3} \Omega^{-1} k_{\perp}^{5/3} \sim k_{\perp}$$

↓

$$k_{\perp} \sim \epsilon^{-1/2} \Omega^{3/2} \sim k_0 R_0^{-2}$$

[Possibly deformation scale]



Polarization Alignment

Mininni & al (09) Thiele & Müller (2009)

Latest numerical simulations suggest that the spectrum is, in fact $E \propto k_{\perp}^{-2}$ for rot. turbulence, not $k_{\perp}^{-5/3}$.

This is reminiscent of the MHD numerical simulation (forced) that consistently give $E \propto k_{\perp}^{-3/2}$, not $k_{\perp}^{-5/3}$.

Is this all wrong?

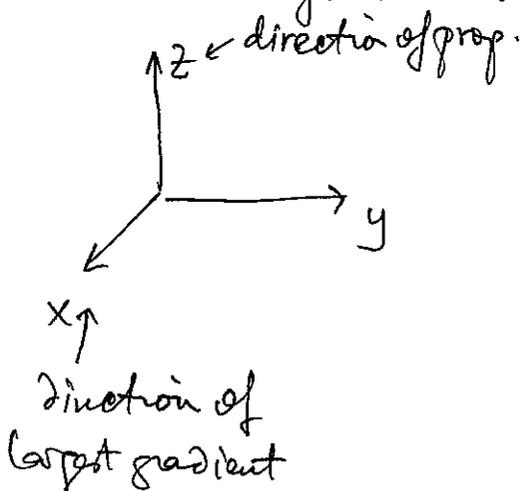
Note in particular that k^{-2} for ~~rot.~~ ^{rot.} and $k^{-3/2}$ for MHD are spectra we would get if we assumed isotropic weak turbulence [which, within the iso. assumption, remains weak at all scales, does not break down].

For MHD - Iroshnikov, Kraichnan 1965

For rotation -

Well, one thing we know definitely is that turbulence is not isotropic. So what is the solution?

Let us imagine that things are 3D-anisotropic



$$k_{\perp} \sim k_x \gg k_y \gg k_z \sim k_{\parallel}$$

$$\nabla_{\perp} \cdot \vec{u}_{\perp} = 0 \Rightarrow u_x \sim \frac{k_y}{k_x} u_y \ll u_y$$

$k_y = 0 \Rightarrow$ monochromatic inertial wave [or Alfvén wave] which is in fact an exact solution - zeroes out the nonlinearity.

If $k_y \ll k_x$, the nonlinearity is reduced:

$$\tau_{NL}^{-1} \sim k_y u_y \sim k_{\perp} u_{\perp} \theta(k_{\perp})$$

$$\begin{matrix} \nearrow & \uparrow & \nearrow \\ k_x & u_y & \theta \sim \frac{k_y}{k_x} \sim \frac{u_x}{u_y} \ll 1 \end{matrix}$$

Velocity angle - weakens the nonlinearity if small.

Same way we needed CB to resolve the unknown relationship between ~~nonlinear~~ k_{\parallel} and k_{\perp} (or ω and τ_{NL}^{-1}), we now need another conjecture to resolve the unknown relationship between k_y and $k_x(k_{\perp})$.

If inertial waves propagate along perturbed vorticity lines ($2\Omega \hat{z} + \delta \omega_{\perp}$), then all directions have the angular uncertainty

$$\delta\theta \sim \frac{\delta \omega_{\perp}}{\Omega} \sim \theta \quad \left\{ \begin{array}{l} \text{Assume this, i.e. the alignment} \\ \text{is max. possible.} \end{array} \right.$$

In MHD, $\delta\theta \sim \frac{\delta B_{\perp}}{B_0} \sim \theta$ [Boldyrev 2006]

So this is the alignment hypothesis: nonlinear fluctuations want to be as close to exact wave-like solutions as they can manage, subject to CB.

Well,

$$\frac{k_y}{k_x} \lesssim \theta \sim \frac{\delta \omega_{\perp}}{\Omega} \sim \frac{k_x u_{\parallel}}{\Omega} \sim \frac{k_x u_{\perp}}{\Omega} \sim \frac{u_{\perp}}{v(k_{\perp})} \sim \frac{k_{\parallel}}{k_y} \quad \left\{ \begin{array}{l} u_{\parallel} \sim u_{\perp} \text{ (in-wave)} \\ \text{CB!} \end{array} \right.$$

also true for MHD.

Result of this:

$$k_y \sim (k_x k_{\parallel})^{1/2}, \quad \theta \sim \left(\frac{k_{\parallel}}{k_x}\right)^{1/2}$$

$$\text{So, } \tau_{NL}^{-1} \sim (k_{\perp} k_{\parallel})^{1/2} u_{\perp}$$

Go through the exact same derivation as before,

$$E(k_{\perp}) \sim [\varepsilon V(k_{\perp})]^{1/2} k_{\perp}^{-3/2} \begin{cases} \rightarrow (\varepsilon V_A)^{1/2} k_{\perp}^{-3/2} \text{ MHD} \\ \rightarrow (\varepsilon \Omega)^{1/2} k_{\perp}^{-2} \text{ rot.} \end{cases}$$

$$k_{\parallel} \sim \varepsilon^{1/2} [V(k_{\perp})]^{-3/2} k_{\perp}^{1/2} \begin{cases} \rightarrow \varepsilon^{1/2} V_A^{-3/2} k_{\perp}^{1/2} \text{ MHD} \\ \rightarrow \varepsilon^{1/2} \Omega^{-3/2} k_{\perp}^2 \text{ rot} \end{cases}$$

All the basic conclusions are the same as before.

Transition to B for rot. turbulence is at

$$k_{gc} \sim \varepsilon^{-1/4} k_{\parallel}^{1/2} \Omega^{3/2} \sim k_0 \text{Ro}^{-1}$$

Transition to isotropy is at

$$k_i \sim \varepsilon^{-1/2} \Omega^{3/2} \sim k_0 \text{Ro}^{-2} \text{ as before}$$

The parallel spectrum is [Rosby def-scale]

$$E(k_{\parallel}) \sim \varepsilon [V(k_{\perp})]^{-1} k_{\parallel}^{-2} \begin{cases} \rightarrow \varepsilon V_A^{-1} k_{\parallel}^{-2} \text{ MHD} \\ \text{(unchanged!)} \\ \rightarrow \varepsilon^{3/4} \Omega^{-1/4} k_{\parallel}^{-3/2} \text{ rot.} \end{cases}$$

NB: Pol. alignment confirmed in forced MHD. (for rot turbulence - why? knows?)
 Cha & Mallet '10 have discovered it disappears in the decaying case.
 Solar wind looks more $k_{\perp}^{-5/3}$.