M05: Lecture 10 cont'd 2.11.05

M05

Ex. Class I: 7.11 @15:00 – 16:30 lecture @14:00?

§9. Conservation Laws

Let us come back to Eulerian MHD:

(1) $\dfrac{\partial \rho}{\partial t} = -\nabla\cdot(\rho\vec{u})$ conservation of mass

(2) $\rho\dfrac{d\vec{u}}{dt} = -\nabla p + \underbrace{\dfrac{(\nabla\times\vec{B})\times\vec{B}}{4\pi}}_{= -\nabla\frac{B^2}{8\pi} + \frac{\vec{B}\cdot\nabla\vec{B}}{4\pi}} + \nabla\cdot\hat{\Pi}$ viscous dissipation

↳ together with (1), this can be cast in a form that explicitly shows the conservation of momentum:

$$\frac{\partial}{\partial t}(\rho\vec{u}) = -\nabla\cdot\Big[\underbrace{\rho\vec{u}\vec{u}}_{\text{Reynolds stress}} + \underbrace{\Big(p}_{\text{pressure}} + \frac{B^2}{8\pi}\Big)\mathbb{1} - \underbrace{\frac{\vec{B}\vec{B}}{4\pi}}_{\text{maxwell stress}} - \underbrace{\hat{\Pi}}_{\text{viscous stress}}\Big]$$

How about energy?

Kinetic energy:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) = \frac{u^2}{2}\overset{(1)}{\frac{\partial \rho}{\partial t}} + \rho\vec{u}\cdot\overset{(2)}{\frac{\partial \vec{u}}{\partial t}} =$$

$$= \underbrace{-\frac{u^2}{2}\nabla\cdot(\rho\vec{u}) - \rho\vec{u}\cdot\nabla\frac{u^2}{2}}_{= -\nabla\cdot\left(\frac{1}{2}\rho u^2\vec{u}\right)} \underbrace{- \vec{u}\cdot\nabla p}_{= -\nabla\cdot(p\vec{u}) + p\nabla\cdot\vec{u}} + \underbrace{\vec{u}\cdot(\nabla\cdot\hat{\Pi})}_{= +\nabla\cdot(\hat{\Pi}\cdot\vec{u}) - \hat{\Pi}:\nabla\vec{u}} + \underbrace{\frac{\vec{u}\cdot[(\nabla\times\vec{B})\times\vec{B}]}{4\pi}}_{= -\frac{(\nabla\times\vec{B})\cdot(\vec{u}\times\vec{B})}{4\pi}}$$

use $\nabla\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\nabla\times\vec{a}) - \vec{a}\cdot(\nabla\times\vec{b})$

$$\nabla\cdot\left[\frac{(\vec{u}\times\vec{B})\times\vec{B}}{4\pi}\right] - \frac{1}{4\pi}\vec{B}\cdot[\nabla\times(\vec{u}\times\vec{B})]$$

$-c\vec{E} + \eta\nabla\times\vec{B}$ Ohm's law

So we have

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) = -\nabla\cdot\Big[\underbrace{\frac{1}{2}\rho u^2\vec{u}}_{\text{kinetic energy flux}} + \underbrace{p\vec{u}}_{\text{thermal flux}} - \underbrace{\hat{\Pi}\cdot\vec{u}}_{\text{visc. flux}} - \underbrace{\frac{(\vec{u}\times\vec{B})\times\vec{B}}{4\pi}}_{=\ \frac{c}{4\pi}\vec{E}\times\vec{B} + \frac{\eta}{4\pi}\vec{B}\times(\nabla\times\vec{B})}\Big] +$$

$\frac{c}{4\pi}\vec{E}\times\vec{B}$ Poynting flux; $\cancel{\frac{\eta}{4\pi}\vec{B}\times(\nabla\times\vec{B})}$ resistive flux

cancel

$$+\overbrace{p\nabla\cdot\vec{u}}^{=\ -\frac{p}{\rho}\frac{d\rho}{dt}} - \hat{\Pi}:\nabla\vec{u} - \underbrace{\cancel{\frac{1}{4\pi}\vec{B}\cdot\left[\nabla\times(\vec{u}\times\vec{B})\right]}}$$

compressional heating; viscous heating

energy exchange between $\vec{u}$ & $\vec{B}$

Now the ind. eqn:

$$\frac{\partial}{\partial t}\vec{B} = \nabla\times\left(\vec{u}\times\vec{B} - \eta\nabla\times\vec{B}\right)$$

Magnetic energy:

$$\frac{\partial}{\partial t}\frac{B^2}{8\pi} = \frac{\vec{B}}{4\pi}\cdot\frac{\partial\vec{B}}{\partial t} = \cancel{\frac{1}{4\pi}\vec{B}\cdot\left[\nabla\times(\vec{u}\times\vec{B})\right]} - \overbrace{\frac{\eta}{4\pi}\vec{B}\cdot\left[\nabla\times(\nabla\times\vec{B})\right]}^{=\ -\frac{\eta}{4\pi}|\nabla\times\vec{B}|^2 + \cancel{\frac{\eta}{4\pi}\nabla\cdot\left[\vec{B}\times(\nabla\times\vec{B})\right]}}$$

Ohmic heating: $-\frac{\eta}{4\pi}|\nabla\times\vec{B}|^2$; resistive flux: $\cancel{\frac{\eta}{4\pi}\nabla\cdot\left[\vec{B}\times(\nabla\times\vec{B})\right]}$

again use $\nabla\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\nabla\times\vec{a}) - \vec{a}\cdot(\nabla\times\vec{b})$

So, we have

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \frac{B^2}{8\pi}\right) =$$

fluxes

$$= -\nabla\cdot\Big[\underbrace{\frac{1}{2}\rho u^2\vec{u}}_{\text{kinetic}} + \underbrace{p\vec{u}}_{\text{thermal}} - \underbrace{\hat{\Pi}\cdot\vec{u}}_{\text{viscous}} + \underbrace{\frac{c}{4\pi}\vec{E}\times\vec{B}}_{\text{Poynting}}\Big]$$

$$-\Big[\underbrace{-p\nabla\cdot\vec{u}}_{\text{compressional}} + \underbrace{\hat{\Pi}:\nabla\vec{u}}_{\text{viscous}} + \underbrace{\frac{\eta}{4\pi}|\nabla\times\vec{B}|^2}_{\text{Ohmic}}\Big]$$

heating terms

Lecture 11

4.11.05

[NB: Vanishes in the incompressible case, so we get

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \frac{B^2}{8\pi}\right) = -\nabla\cdot[\ldots] - \mu|\nabla\times\vec{u}|^2 - \eta\frac{|\nabla\times\vec{B}|^2}{4\pi}$$
]

Finally, recall the equation for pressure:

(3) $$\underbrace{\frac{d}{dt}\frac{p}{\rho^\gamma}}_{=} = \text{non-adiabatic terms} \begin{cases} \to \text{heat fluxes} \\ \to \text{heating terms} \end{cases}$$

$$\frac{1}{\rho^\gamma}\frac{dp}{dt} - \frac{\gamma}{\rho^{\gamma+1}}p\frac{d\rho}{dt} = \frac{1}{\rho^\gamma}\left(\frac{\partial p}{\partial t} + \vec{u}\cdot\nabla p + \gamma p \nabla\cdot\vec{u}\right) =$$

$$= \frac{1}{\rho^\gamma}\left(\frac{\partial p}{\partial t} + \nabla\cdot(p\vec{u}) + (\gamma-1)p\nabla\cdot\vec{u}\right)$$

So we have

$$\frac{\partial}{\partial t}\frac{p}{\gamma-1} = \underbrace{-\nabla\cdot\frac{p\vec{u}}{\gamma-1}}_{\text{thermal flux}} \underbrace{-\,p\nabla\cdot\vec{u}}_{\text{compressional heating}} + \text{non-adiabatic terms}$$

Comparing with the eqn for the kinetic + magnetic energy, we can, in fact, figure out what the non-adiabatic terms are:

$$\frac{\partial}{\partial t}\frac{p}{\gamma-1} = -\nabla\cdot\frac{p\vec{u}}{\gamma-1} \underbrace{-\,p\nabla\cdot\vec{u}}_{\text{compressional heating}} \underbrace{-\,\nabla\cdot\vec{q}}_{\text{heat flux}} + \underbrace{\hat{\Pi}:\nabla\vec{u}}_{\text{viscous heating}} + \underbrace{\frac{\eta}{4\pi}|\nabla\times\vec{B}|^2}_{\text{Ohmic heating}}$$

This can, of course, be derived independently from kinetic theory.

$$\vec{q} = -\varkappa\nabla T$$

thermal diffusivity

Final energy eqn:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \frac{B^2}{8\pi} + \frac{p}{\gamma-1}\right) = -\nabla\cdot\Big[\underbrace{\frac{1}{2}\rho u^2\vec{u}}_{\text{kinetic}} + \underbrace{\frac{\gamma}{\gamma-1}p\vec{u}}_{\text{thermal}} \underbrace{-\,\hat{\Pi}\cdot\vec{u}}_{\text{viscous}} + \underbrace{\frac{c}{4\pi}\vec{E}\times\vec{B}}_{\text{Poynting}} + \underbrace{\vec{q}}_{\text{heat}}\Big]$$

Besides mass, momentum and energy, MHD has topological invariants.

One of them (first-order) was flux.

In differential form, its conservation is expressed by the induction equation.

We will now derive two second-order ones

1) Helicity. This involves magnetic field only:

$$H = \int_V d^3x\, \vec{B}\cdot\vec{A} \quad \text{where} \quad \vec{B} = \nabla\times\vec{A}$$

• Helicity is well-defined:

$\vec{A}$ is defined up to the gauge transformation

$$\vec{A} \mapsto \vec{A} + \nabla\chi$$

Then $H \mapsto H + \int_V \vec{B}\cdot\nabla\chi\, d^3x = H + \int_V \nabla\cdot(\vec{B}\chi)\, d^3x =$

$$= H + \int_{\partial V} d\vec{S}\cdot(\vec{B}\chi) = H \quad \text{if } \vec{B}\cdot\hat{n} = 0$$

on the boundary

• Helicity is conserved:

$$\frac{\partial\vec{B}}{\partial t} = \nabla\times\left(\vec{u}\times\vec{B} - \eta\nabla\times\vec{B}\right)$$

Uncurl: $\dfrac{\partial\vec{A}}{\partial t} = \vec{u}\times\vec{B} - \eta\nabla\times\vec{B} + \nabla\chi$ arbitrary scalar function

$$\frac{\partial}{\partial t}(\vec{A}\cdot\vec{B}) = \underbrace{\vec{B}\cdot\left(\cancel{\vec{u}\times\vec{B}} - \eta\nabla\times\vec{B} + \nabla\chi\right)}_{= -\eta\underline{\vec{B}\cdot(\nabla\times\vec{B})} + \underbrace{\vec{B}\cdot\nabla\chi}_{=\nabla\cdot\vec{B}\chi}} +$$

$$+\vec{A}\cdot\left[\nabla\times\left(\vec{u}\times\vec{B} - \eta\nabla\times\vec{B}\right)\right] =$$

again use $\nabla\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\nabla\times\vec{a}) - \vec{a}\cdot(\nabla\times\vec{b})$

$$= \nabla\cdot\left[\vec{A}\times\left(\vec{u}\times\vec{B} - \eta\nabla\times\vec{B}\right)\right] + \underbrace{(\nabla\times\vec{A})}_{=\vec{B}}\cdot\left(\cancel{\vec{u}\times\vec{B}} - \underline{\eta\nabla\times\vec{B}}\right)$$

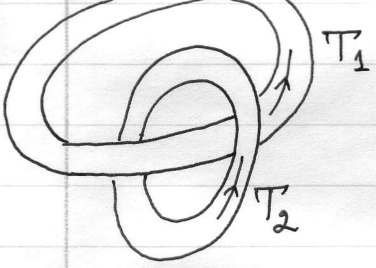
$$= \nabla\cdot\left[\vec{B}\chi + \vec{A}\times\left(\vec{u}\times\vec{B} - \eta\nabla\times\vec{B}\right)\right] - 2\eta\,\vec{B}\cdot(\nabla\times\vec{B})$$

So

$$\frac{d}{dt}\int d^3x\,\vec{A}\cdot\vec{B} = -2\eta\underbrace{\int d^3\,\vec{B}\cdot(\nabla\times\vec{B})}_{\text{current helicity}}$$

- Helicity is a ~~topo~~ topological quantity.

Consider linked flux tubes:



$$H_1 = \int_{T_1} d^3_x\,\vec{A}\cdot\vec{B} = \int_{T_1}\underbrace{d\vec{l}}_{=dl\hat{b}}\cdot\underbrace{d\vec{S}}_{=dS\hat{b}}\,\vec{A}\cdot\underbrace{\vec{B}}_{=\hat{b}B} =$$

$$= \int_{T_1}\vec{A}\cdot\hat{b}\,dl\;B\,\hat{b}\cdot\hat{b}\,dS =$$

$$= \int_{T_1}\vec{A}\cdot d\vec{l}\;\vec{B}\cdot d\vec{S} = \underbrace{\Phi_{\text{through tube}}}_{=\Phi_1}\underbrace{\int\vec{A}\cdot dl}_{=} = \Phi_1\Phi_2$$

$$\int d\vec{S}'\cdot(\nabla\times\vec{A}) = \int d\vec{S}'\cdot\vec{B} = \underbrace{\Phi_{\text{through hole}}}_{=\Phi_2}$$

surface spanning the loop

In general, $H_{\text{tube } i} = \Phi_{\text{tube } i}\, \Phi_{\text{through hole in tube } i} = \Phi_i \sum_j \Phi_j N_{ij}$

times tube j passes through tube i

Total helicity

$$H = \sum_{ij} \Phi_i \Phi_j N_{ij}$$

counts the linkages.

To unlink flux tubes, need to break field lines
$\Rightarrow$ break flux conservation $\Rightarrow$ need resistivity.

2) Cross-helicity.

We may now define another quantity:

$$C = \int_V d^3x\, \vec{u}\cdot\vec{B}$$

Since $\nabla\times\vec{u} = \vec{\omega}$ and $\vec{\omega}$ is similar to $\vec{B}$ (in ideal MHD, it satisfies the same equation — prove this!), we may interpret C as counting linkages between flux and vortex tubes.
Is this conserved?

$$\frac{\partial}{\partial t}(\vec{u}\cdot\vec{B}) = \underbrace{\vec{u}\cdot\left[\nabla\times(\vec{u}\times\vec{B}) - \eta\nabla\times\vec{B}\right]} + \vec{B}\cdot\left[-\vec{u}\cdot\nabla\vec{u} - \frac{\nabla p}{\rho} + \cancel{\frac{(\nabla\times\vec{B})\times\vec{B}}{4\pi\rho}} + \frac{\nabla\cdot\hat{\Pi}}{\rho}\right] =$$

$$\boxed{\nabla\cdot\left[\vec{u}\times(\vec{u}\times\vec{B} - \eta\nabla\times\vec{B})\right] + (\nabla\times\vec{u})\cdot(\vec{u}\times\vec{B} - \eta\nabla\times\vec{B})}$$

$$= \nabla\cdot\left[\vec{u}\times(\vec{u}\times\vec{B}-\eta\nabla\times\vec{B})\right] - \eta(\nabla\times\vec{u})\cdot(\nabla\times\vec{B})$$

$$+ \underbrace{(\nabla\times\vec{u})\cdot(\vec{u}\times\vec{B})}_{=} - \cancel{(\vec{u}\cdot\nabla\vec{u})}\cdot\vec{B} - \nabla\cdot\frac{\vec{B}p}{\rho} - \frac{p}{\rho^2}\vec{B}\cdot\nabla\rho + \nabla\cdot\left(\hat{\Pi}\cdot\frac{\vec{B}}{\rho}\right) - \hat{\Pi}:\nabla\frac{\vec{B}}{\rho} =$$

$$\epsilon_{ijk}(\partial_j u_k)\epsilon_{imn}u_m B_n =$$
$$= (\delta_{jm}\delta_{kn}-\delta_{jn}\delta_{km})(\partial_j u_k)u_m B_n$$
$$= (\vec{u}\cdot\nabla\vec{u})\cdot\vec{B} - \vec{B}\cdot(\nabla\vec{u})\cdot\vec{u}$$
$$= \cancel{(\vec{u}\cdot\nabla\vec{u})}\cdot\vec{B} - \nabla\cdot\left(\vec{B}\frac{u^2}{2}\right)$$

$$= \nabla\cdot\overbrace{\left[\vec{u}\times(\vec{u}\times\vec{B}-\eta\nabla\times\vec{B}) - \frac{u^2}{2}\vec{B} - p\frac{\vec{B}}{\rho} + \hat{\Pi}\cdot\frac{\vec{B}}{\rho}\right]}^{\text{fluxes}}$$

$$- \underbrace{\hat{\Pi}:\nabla\frac{\vec{B}}{\rho}}_{\text{viscous losses}} - \underbrace{\eta(\nabla\times\vec{u})\cdot(\nabla\times\vec{B})}_{\text{resistive losses}} + \underbrace{p\nabla\cdot\frac{\vec{B}}{\rho}}_{\text{This term needs to be } \emptyset \text{ in order for } C \text{ to be conserved.}}$$

This is true in some situations, most interestingly when $\rho = \text{const}$ (incompressible fluid)

Then
$$\frac{\partial}{\partial t}(\vec{u}\cdot\vec{B}) = \nabla\cdot[\ldots] - \nu\underbrace{\nabla\vec{u}:\nabla\vec{B}}_{} - \eta\underbrace{(\nabla\times\vec{u})\cdot(\nabla\times\vec{B})}_{}$$

same!

$$\frac{\partial C}{\partial t} = -(\eta+\nu)\int d^3x\, \vec{\omega}\cdot(\nabla\times\vec{B})$$

vorticity current

$$(\nabla\times\vec{u})\cdot(\nabla\times\vec{B}) = \epsilon_{ijk}\epsilon_{imn}\partial_j u_k \partial_m B_n$$
$$= (\partial_j u_k)(\partial_j B_k) - (\partial_n u_m)(\partial_m B_n)$$
$$= \nabla\vec{u}:\nabla\vec{B} - \underbrace{\partial_n\partial_m u_m B_n}_{\text{vanishes under integration}}$$