§15. Weak Turbulence of Alfvén Waves.

We have derived the following equation for the Elsasser potentials:

$$\frac{\partial \zeta^\pm}{\partial t} \mp v_A \frac{\partial \zeta^\pm}{\partial z} = \pm\frac{1}{2}\{\zeta^+,\zeta^-\} - \frac{1}{2}\nabla_\perp^{-2}\left(\{\zeta^+,\nabla_\perp^2\zeta^-\}+\{\zeta^-,\nabla_\perp^2\zeta^+\}\right) \quad (1)$$

In k space:

$$\frac{\partial \zeta_k^\pm}{\partial t} \mp i v_A k_\parallel \zeta_k^\pm = -\frac{1}{2}\sum_{k'}\hat{z}\cdot(\vec{k}\times\vec{k}')\left(\pm 1 - \frac{k_\perp'^2}{k_\perp^2} + \frac{|\vec{k}_\perp-\vec{k}_\perp'|^2}{k_\perp^2}\right)\zeta^-_{k'}\zeta^+_{k-k'}$$

$$\frac{\partial \zeta_k^+}{\partial t} - i v_A k_\parallel \zeta_k^+ = -\frac{1}{2}\sum_{k'}\hat{z}\cdot(\vec{k}\times\vec{k}')\left(1 - \cancel{\frac{k_\perp'^2}{k_\perp^2}} + 1 - 2\frac{\vec{k}_\perp\cdot\vec{k}_\perp'}{k_\perp^2} + \cancel{\frac{k_\perp'^2}{k_\perp^2}}\right)\zeta^-_{k'}\zeta^+_{k-k'}$$

$$= -\sum_{k'}\hat{z}\cdot(\vec{k}\times\vec{k}')\left(1 - \frac{\vec{k}_\perp\cdot\vec{k}_\perp'}{k_\perp^2}\right)\zeta^-_{k'}\zeta^+_{k-k'}$$

$$\frac{\partial \zeta_k^-}{\partial t} + i v_A k_\parallel \zeta_k^- = -\frac{1}{2}\sum_{k'}\hat{z}\cdot(\vec{k}\times\vec{k}')\left(-1 - \frac{k_\perp'^2}{k_\perp^2} + \frac{|\vec{k}_\perp-\vec{k}_\perp'|^2}{k_\perp^2}\right)\zeta^-_{k'}\zeta^+_{k-k'}$$

Change variables $\vec{k}''=\vec{k}-\vec{k}'$

$$= +\frac{1}{2}\sum_{k''}\hat{z}\cdot(\vec{k}\times\vec{k}'')\left(-1 - \frac{|\vec{k}_\perp-\vec{k}_\perp''|^2}{k_\perp^2} + \frac{k_\perp''^2}{k_\perp^2}\right)\zeta^+_{k''}\zeta^-_{k-k''}$$

rename $\vec{k}''\to\vec{k}'$

$$= -\sum_{k'}\hat{z}\cdot(\vec{k}\times\vec{k}')\left(1 - \frac{\vec{k}_\perp\cdot\vec{k}_\perp'}{k_\perp^2}\right)\zeta^+_{k'}\zeta^-_{k-k'}$$

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So, actually we have

$$\boxed{\frac{\partial \zeta_k^\pm}{\partial t} \mp i k_\parallel v_A \zeta_k^\pm = \sum_{k'} M_{kk'}\zeta^\mp_{k'}\zeta^\pm_{k-k'}} \quad (2)$$

where $M_{kk'} = -\hat{z}\cdot(\vec{k}\times\vec{k}')\left(1 - \frac{\vec{k}_\perp\cdot\vec{k}_\perp'}{k_\perp^2}\right)$ interaction (coupling) coefficient

We may write this in the following very generic form:

$$\frac{\partial \zeta_k^\pm}{\partial t} - i\omega_k^\pm \zeta_k^\pm = \epsilon \sum_{pq} M_{kpq}\, \zeta_p^\mp \zeta_q^\pm \underbrace{\delta_{k,p+q}}_{\text{"momentum conservation"}} \qquad (3)$$

where $\omega_k^\pm = \pm k_\parallel v_A$ mode frequency

$$M_{kpq} = -\hat{z}\cdot(\vec{k}\times\vec{p})\,\frac{\vec{k}_\perp\cdot\vec{q}_\perp}{k_\perp^2} \qquad (4)$$

and ϵ counts the # of powers of amplitude:

$$\zeta_k^\pm \to \epsilon\,\zeta_k^\pm \qquad (\epsilon = 1, \text{ but useful for accounting})$$

NB: NOT the same ϵ as in the prev. lecture!

Now suppose $\omega_k^\pm \gg \epsilon M_{kpq} \zeta_p^\mp$ ← Weak interaction

(i.e. linear frequency $\gg$ nonlinear frequency)

Now, to zeroth order,

$$\frac{\partial \zeta_k^\pm}{\partial t} = i\omega_k^\pm \zeta_k^\pm \qquad \text{free wave field}$$

Let $\zeta_k^\pm = \underbrace{\tilde{\zeta}_k^\pm(\epsilon t)}_{\text{slow time dep.}}\ \underbrace{e^{i\omega_k^\pm t}}_{\text{fast time dependence}}$

$$\frac{\partial \tilde{\zeta}_k^\pm}{\partial t} = \epsilon \sum_{pq} M_{kpq}\, \tilde{\zeta}_p^\mp \tilde{\zeta}_q^\pm\, \delta_{k,p+q}\, e^{i(\omega_p^\mp + \omega_q^\pm - \omega_k^\pm)t} \qquad (5)$$

Integrate this equation:

$$\tilde{\zeta}_k^\pm(t) = \underbrace{\tilde{\zeta}_k^\pm(0)}_{\text{0-th-order solution}} + \underbrace{\epsilon \sum_{pq}}_{\text{1st order}} M_{kpq}\, \delta_{k,p+q} \int_0^t dt'\, \tilde{\zeta}_p^\mp(t')\, \tilde{\zeta}_q^\pm(t')\, e^{i(\omega_p^\mp + \omega_q^\pm - \omega_k^\pm)t'}$$

If we want to work to first order only, we may replace $\tilde{\zeta}(t') \Rightarrow \tilde{\zeta}(0)$ in the 2nd term:

$$\tilde{\zeta}_k^\pm(t) = \tilde{\zeta}_k^\pm(0) + \epsilon \sum_{pq} M_{kpq}\, \delta_{k,p+q}\, \tilde{\zeta}_p^\mp(0)\, \tilde{\zeta}_q^\pm(0) \underbrace{\frac{e^{i(\omega_p^\mp+\omega_q^\pm-\omega_k^\pm)t} - 1}{i(\omega_p^\mp+\omega_q^\pm-\omega_k^\pm)}}_{\equiv\ \Delta_{kpq}(t)}$$

$$+\, O(\epsilon^2) \qquad (6)$$

Go back to the evolution equation for $\tilde{\zeta}_k^\pm$ and derive an equation for spectrum:

$$\langle \tilde{\zeta}_k^\pm \tilde{\zeta}_{k'}^\pm \rangle = C_k^\pm\, \delta_{k,-k'}\,, \qquad C_k^\pm = \langle |\tilde{\zeta}_k^\pm|^2 \rangle$$

NB: $C_k^\pm = C_{-k}^\pm$, real!

So,
$$\frac{\partial C_k^\pm}{\partial t} = \left\langle \tilde{\zeta}_{k}^{\pm *} \frac{\partial \tilde{\zeta}_k^\pm}{\partial t} \right\rangle + \text{c.c.} =$$

$$= \epsilon \sum_{pq} M_{kpq} \underbrace{\langle \tilde{\zeta}_p^\mp \tilde{\zeta}_q^\pm \tilde{\zeta}_{-k}^\pm \rangle}\, \delta_{k,p+q}\, e^{i(\omega_p^\mp+\omega_q^\pm-\omega_k^\pm)t} + \text{c.c.} \qquad (7)$$

into here substitute our sln (6):

$$\tilde{\zeta}_k^\pm(t) = \tilde{\zeta}_k^\pm(0) + \epsilon \sum_{pq} M_{kpq}\, \delta_{k,p+q}\, \Delta_{kpq}(t)\, \tilde{\zeta}_p^\mp(0)\, \tilde{\zeta}_q^\pm(0)$$

and only keep terms up to $O(\epsilon^2)$

$$\langle \tilde{\zeta}_p^\mp \tilde{\zeta}_q^\pm \tilde{\zeta}_{-k}^\pm \rangle = \cancel{\langle \tilde{\zeta}_p^\mp(0)\, \tilde{\zeta}_q^\pm(0)\, \tilde{\zeta}_{-k}^\pm(0) \rangle} +$$

$$+\, \epsilon \sum_{rs} \Big[M_{prs}\, \delta_{p,r+s}\, \Delta_{prs}(t) \langle \cancel{\tilde{\zeta}_r^\mp(0)\, \tilde{\zeta}_s^\pm(0)}\, \tilde{\zeta}_q^\pm(0)\, \tilde{\zeta}_{-k}^\pm(0) \rangle$$

$$+\, M_{qrs}\, \delta_{q,r+s}\, \Delta_{qrs}(t) \langle \tilde{\zeta}_p^\mp(0)\, \tilde{\zeta}_r^\mp(0)\, \tilde{\zeta}_s^\pm(0)\, \tilde{\zeta}_{-k}^\pm(0) \rangle$$

$$+\, M_{-krs}\, \delta_{-k,r+s}\, \Delta_{-krs}(t) \langle \tilde{\zeta}_p^\mp(0)\, \tilde{\zeta}_q^\pm(0)\, \tilde{\zeta}_r^\mp(0)\, \tilde{\zeta}_s^\pm(0) \rangle \Big]$$

We shall further assume that to ϕ-order (no interaction) the + and − fields are uncorrelated (they arrived from opposite directions and have only just met).

Also, $\langle \tilde{\zeta}^{\pm}_{k}(0) \rangle = 0$.

Splitting the 3-d and 4-th order correlators, and using $\langle \tilde{\zeta}^{\pm}_{k}(0)\, \tilde{\zeta}^{\pm}_{k'}(0) \rangle = C^{\pm}_{k}\, \delta_{k,-k'}$, we get

$$\langle \tilde{\zeta}^{\mp}_{p}\, \tilde{\zeta}^{\pm}_{q}\, \tilde{\zeta}^{\pm}_{-k} \rangle =$$

$$= \epsilon \sum_{rs} \Big[M_{qrs}\, \delta_{q,r+s}\, \Delta_{qrs}(t)\, C^{\mp}_{p}\, C^{\pm}_{k}\, \delta_{r,-p}\, \delta_{s,k}$$
$$+ M_{-krs}\, \delta_{-k,r+s}\, \Delta_{-krs}(t)\, C^{\mp}_{p}\, C^{\pm}_{q}\, \delta_{r,-p}\, \delta_{s,-q} \Big]$$

$$= \epsilon \Big[M_{q,-p,k}\, \delta_{q,-p+k}\, \Delta_{q,-p,k}(t)\, C^{\mp}_{p}\, C^{\pm}_{k}$$
$$+ M_{-k,-p,-q}\, \delta_{-k,-p-q}\, \Delta_{-k,-p,-q}(t)\, C^{\mp}_{p}\, C^{\pm}_{q} \Big] \qquad (8)$$

Thus, we have been able to work out the 3-order correlator in terms of the 2-order ones.

<u>Note</u>: We split the 4-order correlators by assuming that + and − fields are uncorrelated. In other problems, this is usually done by assuming that the non-interacting fields (ϕ-order) have a Gaussian distribution

same as $\delta_{k,p+q}$

So we have

$$\frac{\partial C_k^\pm}{\partial t} = \epsilon^2 \sum_{pq} M_{kpq}\,\delta_{k,p+q}\Big[M_{q,-p,k}\,\delta_{q,-p+k}\,C_p^\mp C_k^\pm\,\Delta_{q,-p,k}(t)$$

$$+\underbrace{M_{-k,-p,-q}}_{\parallel\; M_{kpq}}\,\delta_{-k,-p-q}\,C_p^\mp C_q^\pm\,\Delta_{-k,-p,-q}(t)\Big]\,e^{i(\omega_p^\mp+\omega_q^\pm-\omega_k^\pm)t} + \text{c.c.}$$

Δ's are the only complex part, everyth. else is real

$$M_{q,-p,k} = +\hat{z}\cdot(\underbrace{\vec{q}\times\vec{p}}_{\parallel\;\vec{k}-\vec{p}})\,\frac{\vec{q}_\perp\cdot\vec{k}_\perp}{q_\perp^2} = -\frac{k_\perp^2}{q_\perp^2}\,M_{kpq}$$

because of $\delta_{k,p+q}$

Now $$\Delta_{-k,-p,-q}(t) = -\frac{e^{-i(\omega_p^\mp+\omega_q^\pm-\omega_k^\pm)t}-1}{i(\omega_p^\mp+\omega_q^\pm-\omega_k^\pm)}$$ because $\omega_{-k}^\pm = -\omega_k^\pm$

$$\Delta_{q,-p,k}(t) = \frac{e^{i(-\omega_p^\mp+\omega_k^\pm-\omega_q^\pm)t}-1}{i(-\omega_p^\mp+\omega_k^\pm-\omega_q^\pm)} = \Delta_{-k,-p,-q}(t)$$

and

$$\Delta_{-k,-p,-q}(t)\,e^{i(\omega_p^\mp+\omega_q^\pm-\omega_k^\pm)t} + \text{c.c.} =$$

$$= -\frac{1-e^{i(\omega_p^\mp+\omega_q^\pm-\omega_k^\pm)t}}{i(\omega_p^\mp+\omega_q^\pm-\omega_k^\pm)} + \text{c.c} =$$

$$= \frac{e^{i(\omega_p^\mp+\omega_q^\pm-\omega_k^\pm)t}-e^{-i(\omega_p^\mp+\omega_q^\pm-\omega_k^\pm)t}}{i(\omega_p^\mp+\omega_q^\pm-\omega_k^\pm)} =$$

$$= \frac{2\sin\left[(\omega_p^\mp+\omega_q^\pm-\omega_k^\pm)t\right]}{\omega_p^\mp+\omega_q^\pm-\omega_k^\pm}$$

So we get

$$\frac{\partial C_k^{\pm}}{\partial t} = \epsilon^2 \sum_{pq} M_{kpq}^2 \, \delta_{k,p+q} \, C_p^{\mp}\left(C_q^{\pm} - \frac{k_\perp^2}{q_\perp^2} C_k^{\pm}\right) \underbrace{\frac{2\sin\left[(\omega_p^{\mp}+\omega_q^{\pm}-\omega_k^{\pm})t\right]}{\omega_p^{\mp}+\omega_q^{\pm}-\omega_k^{\pm}}}_{\approx\; 2\delta(\omega_p^{\mp}+\omega_q^{\pm}-\omega_k^{\pm})} \qquad (9)$$

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This is true for

$t \gg [\omega_k^{\pm}]^{-1}$ $\longrightarrow$ $2\delta(\omega_p^{\mp}+\omega_q^{\pm}-\omega_k^{\pm})$ as $t\to\infty$ (Riemann-Lebesgue lemma)

This δ function is the formal manifestation of the "energy conservation" I talked about in my lecture on dimensional theories of the Alfvén-wave turbulence: $\omega_k^{\pm} = \omega_p^{\mp}+\omega_q^{\pm}$ has to be satisfied.

Eq. (9) is called the **3**-wave kinetic equation.

Finally, $\omega_p^{\mp}+\omega_q^{\pm}-\omega_k^{\pm} = (\mp p_\parallel \pm \underbrace{q_\parallel}_{\cancel{k_\parallel}-p_\parallel} \mp k_\parallel) v_A = \mp 2 p_\parallel v_A$

$$2\delta(\mp 2 p_\parallel v_A) = \frac{1}{v_A}\delta(p_\parallel) = \frac{L_\parallel}{v_A}\delta_{p_\parallel,0}\,, \quad L_\parallel - \parallel \text{ box size}$$

Thus, interaction non-empty only if $p_\parallel = 0$, $q_\parallel = k_\parallel$!

OK, assemble everything: recall that

$$\zeta^{\pm} = \phi \pm \psi\,, \text{ so } \quad \hat{z}\times\nabla_\perp \zeta^{\pm} = \vec{u}_\perp \pm \delta\vec{B}_\perp/\sqrt{4\pi\rho_0} \equiv \vec{Z}_\perp^{\pm}$$

So we may define

$$\mathcal{E}_k^{\pm} = \langle|\vec{Z}_{\perp k}^{\pm}|^2\rangle = k_\perp^2 C_k^{\pm}.$$

With this definition the kinetic equation takes a nice compact form:

$$\boxed{\frac{\partial \tau_k^{\pm}}{\partial t} = \frac{L_\parallel}{v_A}\sum_{pq}\frac{|\hat{z}\cdot(\vec{k}_\perp\times\vec{p}_\perp)|^2(\vec{k}_\perp\cdot\vec{q}_\perp)^2}{k_\perp^2 p_\perp^2 q_\perp^2}\,\delta_{k,p+q}\,\delta_{p_\parallel,0}\,\tau_p^{\mp}\left(\tau_q^{\pm}-\tau_k^{\pm}\right)} \qquad (10)$$

Thus,

- interaction happens via "scattering" of waves with $k_\parallel \neq 0$ off $k_\parallel = 0$ modes
- There is no cascade in $k_\parallel$: $q_\parallel = k_\parallel$
  All values of $k_\parallel$ decouple from each other and for each $k_\parallel \neq 0$, there is a separate Eq. (10)

Clearly, we have hit a snag: Alfvén waves interact via the $k_\parallel = 0$ modes, which are __not__ waves themselves, rather they are 2D motions that do not propagate along z and, therefore, cannot be treated by the weak-interaction approximation: if we separated the $k_\parallel = 0$ modes in eq. (3) we get

$$\frac{\partial \zeta^{\pm}_{k_\perp, k_\parallel=0}}{\partial t} = \underbrace{\sum_{p_\perp, q_\perp} M_{kpq}\,\zeta^{\mp}_{p_\perp, p_\parallel=0}\,\zeta^{\pm}_{q_\perp, q_\parallel=0}\,\delta_{k_\perp, p_\perp+q_\perp}}_{\text{purely 2D part}} + \underbrace{\sum_{\substack{p_\perp, q_\perp\\ p_\parallel=-q_\parallel\neq 0}} M_{kpq}\,\tilde{\zeta}^{\mp}_p\,\tilde{\zeta}^{\pm}_q\,\delta_{k,p+q}\,e^{\mp 2 i p_\parallel v_A t}}_{\text{oscillatory part}}$$

purely 2D part
strong interaction
decoupled from
the waves.

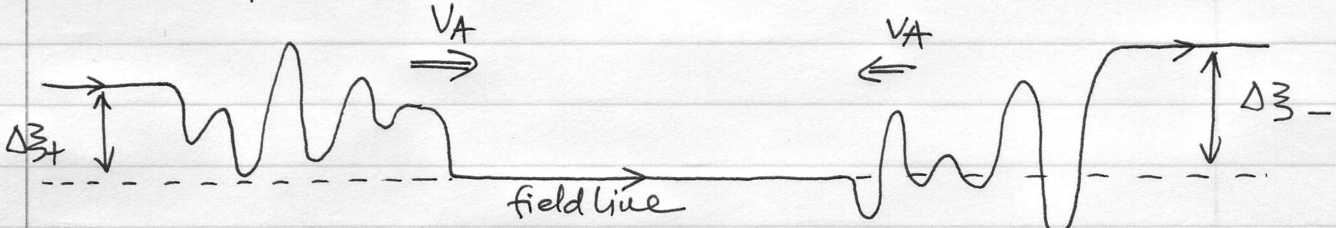
oscillatory part,
small as $t \to \infty$

↳ Decorrelation time like for strong turbulence:

$$\tau_0 \sim \left(k_\perp z^{\mp}_{k_\perp}\right)^{-1} \sim \left(k_\perp^2 \zeta^{\mp}_{k_\perp}\right)^{-1}$$

Interpretation of $k_\parallel = 0$ modes:

consider 2 counterpropagating wave packets that are, at $t=-\infty$, free Elsasser solutions of the MHD equations



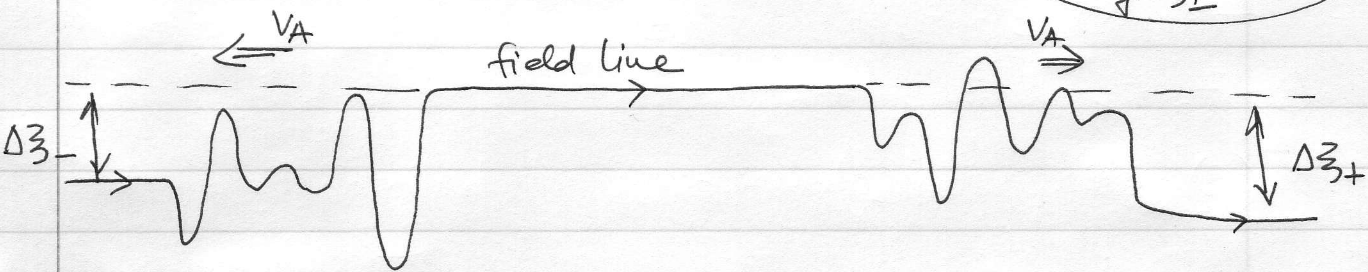
$\vec{\xi}_+(x_0, y_0, z_0 - V_A t)$ $\qquad$ $\vec{\xi}_-(x_0, y_0, z_0 + V_A t)$

displacements at $t=-\infty$ (before interaction)

Steps in the field line: $\Delta\xi_\pm = \frac{1}{L_\parallel}\int_{-\infty}^{+\infty} dz\, \xi_\pm = \xi_\pm(k_\parallel = 0)$

$k_\parallel = 0$ part of $\xi_\pm$

After the interaction:



$\vec{\xi}_-(x_0, y_0, z + V_A t)$ $\qquad$ $\vec{\xi}_+(x_0, y_0, z_0 - V_A t)$

The field line has been displaced by $\Delta\xi_+ - \Delta\xi_-$.

These overall 2D motions of the field lines appear to mediate interactions.

But are they realistic? What if they conflict with boundary conditions at $z = \pm L_\parallel$?

It has been argued that the condition for the 3-wave interactions of this sort to be correctly described by eq. (10), the characteristic time over which $\zeta_k^\pm$ evolves (to a stationary state) via eq. (10) must be shorter than the time for the waves to traverse the box, $\tau_\parallel = \dfrac{L_\parallel}{v_A}$.

must be for weak interaction assumption

The characteristic time in eq. (10) is

$$\tau_\parallel \gg t \sim \left[\frac{L_\parallel}{v_A} k_\perp^2 \zeta^\mp_{p_\perp, p_\parallel = 0}\right]^{-1} \sim \frac{\tau_0^2}{\tau_\parallel} \gg \tau_A \sim \frac{1}{k_\parallel v_A}$$

assume $p_\perp \sim k_\perp$ (local interaction)

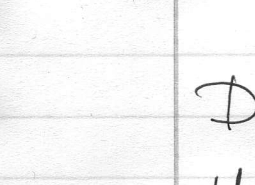
Thus,

$$\tau_\parallel \gg \tau_0 \gg (\tau_\parallel \tau_A)^{1/2}$$

τ_0: decorrelation time of the $k_\parallel = 0$ modes.

Does this convince you? Think about this problem! You see that we have gone through the Alfvén-wave theory and arrived at the point where current theoretical uncertainties start. Time to move on to another subject! But before that, let me explain (in the spirit of presenting some useful techniques) how to solve eq. (10) <u>if</u> we assume that $k_\parallel = 0$ are recoverable from it as a continuous limit $k_\parallel \to 0$.

In other words, let



$$\zeta_k^\pm = f^\pm(k_\parallel)\,\frac{E^\pm(k_\perp)}{2\pi k_\perp L_\perp^2}$$

$E^\pm(k_\perp)$: spectrum in the $\perp$ plane

$L_\perp$ = $\perp$ size of the box

where $f^\pm(0) = 1$ and $f^\pm$ is continuous across $k_\parallel$.

Then the $k_\parallel$ dependence in eq. (10) falls out — it is determined purely by how turbulence is driven.

Now convert the sum over mode into a continuous integral:

$$\sum_{pq} \delta_{k,p+q}\,\delta_{p_\|,0} \mapsto L_\perp^4 \int d^2p_\perp \int d^2q_\perp \frac{1}{L_\perp^2}\,\delta(\vec{k}_\perp - \vec{p}_\perp - \vec{q}_\perp) =$$

$$= L_\perp^2 \int d^2p_\perp = L_\perp^2 \int_0^\infty dp_\perp p_\perp \int_0^{2\pi} d\varphi =$$

(and replace $\vec{q}_\perp = \vec{k}_\perp - \vec{q}_\perp$)

can define φ as the angle between $\vec{k}_\perp$ and $\vec{p}_\perp$

$$= 2L_\perp^2 \int_0^\infty dp_\perp p_\perp \int_0^\pi d\varphi$$

because everything only depends on $\cos\varphi$, so contributions from $(0,\pi)$ and $(\pi,2\pi)$ are the same

Note that $q_\perp^2 = |\vec{k}_\perp - \vec{p}_\perp|^2 = k_\perp^2 + p_\perp^2 - 2k_\perp p_\perp \cos\varphi$

so $\cos\varphi = \dfrac{k_\perp^2 + p_\perp^2 - q_\perp^2}{2k_\perp p_\perp}$, $\sin\varphi = (1-\cos^2\varphi)^{1/2}$

$$-\sin\varphi\, d\varphi = -\frac{q_\perp dq_\perp}{k_\perp p_\perp} \quad \text{at const } p_\perp.$$

Thus, we can replace $\int_0^\pi d\varphi$ with $\int dq_\perp$.

Integration limits: $\varphi = 0 \quad 1 = \dfrac{k_\perp^2 + p_\perp^2 - q_\perp^2}{2k_\perp p_\perp} \mapsto q_\perp = |k_\perp - p_\perp|$

$\varphi = \pi \quad -1 = \dfrac{k_\perp^2 + p_\perp^2 - q_\perp^2}{2k_\perp p_\perp} \mapsto q_\perp = k_\perp + p_\perp$

Using all this, eq. (10) becomes

$$\frac{\partial}{\partial t} \cancel{f^\pm(k_\|)} \frac{E^\pm(k_\perp)}{\cancel{2\pi k_\perp \Delta_\perp^2}} = \frac{L_\|}{v_A} 2\cancel{L_\perp^2} \int_0^\infty dp_\perp \cancel{p_\perp} \int_{|k_\perp - p_\perp|}^{k_\perp + p_\perp} \frac{dq_\perp q_\perp}{\cancel{k_\perp p_\perp}\sin\varphi} \frac{\cancel{k_\perp^2 p_\perp^2}\sin^2\cancel{\varphi}\,(k_\perp^2 - k_\perp p_\perp \cos\varphi)^2}{\cancel{k_\perp^2 p_\perp^2} q_\perp^2}$$

$$\times \frac{E^\mp(p_\perp)}{\cancel{2\pi p_\perp \Delta_\perp^2}} \left(\frac{E^\pm(q_\perp)}{2\pi q_\perp \cancel{L_\perp^2}} - \frac{E^\pm(k_\perp)}{2\pi k_\perp \cancel{L_\perp^2}} \right) \cancel{f^\pm(k_\|)}$$

$$\frac{\partial E^{\pm}(k_\perp)}{\partial t} = \frac{L_{\parallel}}{\pi v_A} \int_0^\infty dp_\perp \int_{|k_\perp - p_\perp|}^{k_\perp + p_\perp} dq_\perp \sin\varphi \left(1 - \frac{p_\perp}{k_\perp}\cos\varphi\right)^2 \frac{k_\perp^3}{p_\perp q_\perp^2} \times$$

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$$\times E^{\mp}(p_\perp)\left[k_\perp E^{\pm}(q_\perp) - q_\perp E^{\pm}(k_\perp)\right] \equiv I \quad (11)$$

Let us look for power-law solutions:

$$E^{\pm}(k_\perp) = C^{\pm} k_\perp^{n_\pm}$$

Then $$I = \frac{L_{\parallel}}{v_A} C^{\pm} C^{\mp} \frac{1}{\pi} \iint dp_\perp dq_\perp \left(1-\cos^2\varphi\right)^{1/2} \left(1 - \frac{p_\perp}{k_\perp}\cos\varphi\right)^2 \frac{k_\perp^3}{p_\perp q_\perp^2} \times$$

$$\times p_\perp^{n_\mp}\left(k_\perp q_\perp^{n_\pm} - q_\perp k_\perp^{n_\pm}\right) =$$

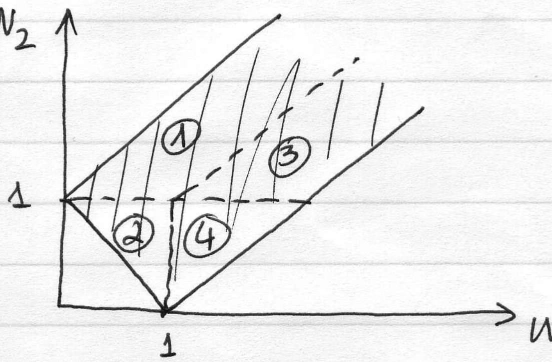
$$= \frac{L_{\parallel}}{v_A}\frac{C^{\pm}C^{\mp}}{\pi} k_\perp^{n_\pm + n_\mp + 3} \iint dw_1 dw_2 \left[1 - \left(\frac{1+w_1^2-w_2^2}{2w_1}\right)^2\right]^{1/2} \left(\frac{1-w_1^2+w_2^2}{2w_2}\right)^2 \times$$

$$\times w_1^{n_\mp - 1} w_2 \left(w_2^{n_\pm - 1} - 1\right) \quad (12)$$

$$w_1 = \frac{p_\perp}{k_\perp}, \quad w_2 = \frac{q_\perp}{k_\perp}$$

$$\cos\varphi = \frac{1+w_1^2-w_2^2}{2w_1}$$

Integration here is over this domain:



Now introduce Zakharov transformation:

$$w_1' = \frac{w_1}{w_2}, \quad w_2' = \frac{1}{w_2} \quad \Leftrightarrow \quad w_1 = \frac{w_1'}{w_2'}, \quad w_2 = \frac{1}{w_2'}$$

This transformation maps the domain of integration onto itself (regions ① ⇔ ②, ③ ⇔ ④ are swapped)

Let us write I in the Zakharov-transformed form:

$$I = +\frac{L_\parallel}{V_A}\frac{C^\pm C^\mp}{\pi} k_\perp^{n_\pm+n_\mp+3}\iint dw_1' dw_2' \left(+\frac{1}{w_2'^3}\right)\underbrace{\left[1-\left(\frac{1+\frac{w_1'^2}{w_2'^2}-\frac{1}{w_2'^2}}{2\frac{w_1'}{w_2'}}\right)^2\right]^{1/2}} \times$$

$$\times \underbrace{\left(\frac{1-\frac{w_1'^2}{w_2'^2}+\frac{1}{w_2'^2}}{2\frac{1}{w_2'}}\right)^2}_{=\left(\frac{1-w_1'^2+w_2'^2}{2w_2'}\right)^2} \underbrace{\left(\frac{w_1'}{w_2'}\right)^{n_\mp-1}\frac{1}{w_2'}\left(\frac{1}{w_2'^{n_\pm-1}}-1\right)}_{=-w_1'^{n_\mp-1}(w_2')^{-n_\mp-n_\pm+1}(w_2'^{n_\pm-1}-1)} \quad = \frac{1}{w_2'}\left[1-\left(\frac{1+w_1'^2-w_2'^2}{2w_1'}\right)^2\right]^{1/2}$$

check!

$$= -\frac{L_\parallel}{V_A}\frac{C^\pm C^\mp}{\pi} k_\perp^{n_\pm+n_\mp+3}\iint dw_1' dw_2' \left[1-\left(\frac{1+w_1'^2-w_2'^2}{2w_1'}\right)^2\right]^{1/2}\left(\frac{1-w_1'^2+w_2'^2}{2w_2'}\right)^2$$

$$\times (w_1')^{n_\mp-1} w_2' \left(w_2'^{n_\pm-1}-1\right)\underbrace{(w_2')^{-n_\mp-n_\pm-4}} \qquad (13)$$

Exactly the same as before the transformation except for $\ominus$ sign and this factor

For st. solution, need $I = -I = 0$

use (12) use (13)

~~Finally, we can add these 2 forms of I and divide by 2~~:

get

$$\iint dw_1 dw_2 \left[1-\left(\frac{1+w_1^2-w_2^2}{2w_1}\right)^2\right]^{1/2}\left(\frac{1-w_1^2+w_2^2}{2w_2'}\right)^2 \times$$

$$\times w_1^{n_\mp-1} w_2\left(w_2^{n_\pm-1}-1\right)\left(1-w_2^{-n_\pm-n_\mp-4}\right) = 0$$

(st. solution) if $n_\pm + n_\mp = -4$

But, if + and − waves are symmetric (ϕ x-helicity),

$$n_\pm = n_\mp = -2 \Rightarrow \boxed{E^\pm(k_\perp) \propto k_\perp^{-2}}$$

Note that $n_\pm = 1$ will not work because then the rhs of (11) diverges.

What we have seen above is a fairly general method of obtaining power-law stationary spectra in weak turbulence theory.

For further methods/insights, read the monograph by Zakharov, Lvov & Falkovich.

(see also papers linked on the course blog).