 Lecture 15 11.11.05

§14. Reduced MHD and the Decoupling of the Alfvén-Wave Cascade

If we think GS95 theory is reasonable at least on the qualitative level, the first main conclusion is that Alfvén-wave turbulence is expected to be anisotropic with

$$k_\parallel \ll k_\perp .$$

This gives us a small parameter:

$$\epsilon \sim \frac{k_\parallel}{k_\perp} \ll 1 .$$

The second main conclusion (in fact, conjecture) was the critical balance:

$$\underset{\substack{\text{typical}\\ \text{frequency}\\ \text{of fluctuations}}}{\omega} \sim \underset{\substack{\text{Alfvén}\\ \text{frequency}}}{k_\parallel v_A} \sim \underset{\substack{\text{"nonlinear}\\ \text{interaction frequency"}}}{k_\perp u} \quad \text{at each scale.}$$

Let us treat this conjecture not necessarily as a detailed scaling prediction but as an ordering assumption that allows us to order the size of the fluctuations wrt our small parameter:

$$\frac{u}{v_A} \sim \frac{k_\parallel}{k_\perp} \equiv \epsilon \ll 1$$

Thus, we shall consider small fluctuations, but not arbitrarily small and we'll expand the MHD equations systematically in ϵ.

Static unperturbed equilibrium state:

$$\rho_0 = \text{const} \qquad p_0 = \text{const} \qquad \vec{B}_0 = B_0\hat{z} = \text{const}$$

Fluctuations: let us order (we'll see that this gives consistent results):

$$\frac{\delta\rho}{\rho_0} \sim \frac{\delta p}{p_0} \sim \frac{\delta B_\parallel}{B_0} \sim \frac{\delta B_\perp}{B_0} \sim \frac{u_\perp}{v_A} \sim \frac{u_\parallel}{v_A} \sim \frac{k_\parallel}{k_\perp} \sim \epsilon$$

always true

Intensity of slow waves ~ that of Alfvén waves

Alfvén waves

critical balance

Also $\omega \sim k_\parallel v_A \sim \epsilon k_\perp v_A$ and $c_s^2 \sim v_A^2$, i.e. $\frac{p_0}{\rho_0} \sim v_A^2$ (ordering!)

- Momentum equation:

$$\rho\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)\vec{u} = -\nabla\left(p + \frac{B^2}{8\pi}\right) + \frac{\vec{B}\cdot\nabla\vec{B}}{4\pi}$$

Now $\rho = \rho_0 + \delta\rho$, $p = p_0 + \delta p$, $\vec{B} = B_0\hat{z} + \delta\vec{B}$ and take $\perp$ part:

Order wrt $k_\perp v_A^2$:

$$\underbrace{\frac{\partial \vec{u}_\perp}{\partial t}}_{\epsilon^2} + \underbrace{\vec{u}_\perp\cdot\nabla_\perp\vec{u}_\perp}_{\epsilon^2} + \underbrace{u_\parallel\nabla_\parallel\vec{u}_\perp}_{\epsilon^3} = -\frac{1}{\rho_0}\nabla_\perp\Big(\underbrace{\delta p}_{\epsilon} + \underbrace{\frac{B_0\delta B_\parallel}{4\pi}}_{\epsilon}\Big) + \underbrace{v_A^2\nabla_\parallel\frac{\delta\vec{B}_\perp}{B_0}}_{\epsilon^2} + \underbrace{v_A^2\frac{\delta\vec{B}_\perp}{B_0}\cdot\nabla_\perp\frac{\delta\vec{B}_\perp}{B_0}}_{\epsilon^2} \qquad (1)$$

Order ϵ: $\nabla_\perp\left(\delta p + \frac{B_0\delta B_\parallel}{4\pi}\right) = 0$

$$\frac{\delta p}{p_0} = -\frac{B_0^2}{4\pi p_0}\frac{\delta B_\parallel}{B_0} = -\gamma\frac{v_A^2}{c_s^2}\frac{\delta B_\parallel}{B_0} \qquad (2)$$

• Energy equation: $\frac{d}{dt}\frac{p}{\rho^\gamma} = 0$

$$\frac{d}{dt}\left(\frac{\delta p}{p_0} - \gamma\frac{\delta\rho}{\rho_0}\right) = 0$$

Using (2), $$\frac{d}{dt}\left(\frac{\delta\rho}{\rho_0} + \frac{V_A^2}{c_s^2}\frac{\delta B_\parallel}{B_0}\right) = 0 \qquad (3)$$

• Continuity equation: $\frac{d\rho}{dt} = -\rho\,\nabla\cdot\vec{u}$

$$\frac{d}{dt}\frac{\delta\rho}{\rho_0} = -\nabla\cdot\vec{u} \qquad (4)$$

• Induction equation: $\frac{d\vec{B}}{dt} = \vec{B}\cdot\nabla\vec{u} - \vec{B}\,\nabla\cdot\vec{u}$

∥ part: $$\frac{d\,\delta B_\parallel}{dt} = \vec{B}\cdot\nabla u_\parallel - B_0\,\nabla\cdot\vec{u}$$

$$\frac{d}{dt}\frac{\delta B_\parallel}{B_0} - \frac{\vec{B}}{B_0}\cdot\nabla u_\parallel = -\nabla\cdot\vec{u} \overset{\text{from (4)}}{=} \frac{d}{dt}\frac{\delta\rho}{\rho_0}$$

$$\frac{d}{dt}\left(\frac{\delta\rho}{\rho_0} - \frac{\delta B_\parallel}{B_0}\right) + \frac{\vec{B}}{B_0}\cdot\nabla u_\parallel = 0 \qquad (5)$$

Combine (3) and (5):

$$\frac{d}{dt}\frac{\delta\rho}{\rho_0} = -\frac{1}{1 + c_s^2/V_A^2}\frac{\vec{B}}{B_0}\cdot\nabla u_\parallel \qquad (6)$$

$$\frac{d}{dt}\frac{\delta B_\parallel}{B_0} = \frac{1}{1 + V_A^2/c_s^2}\frac{\vec{B}}{B_0}\cdot\nabla u_\parallel \qquad (7)$$

• Momentum equation, ∥ part:

$$\underbrace{\frac{du_\parallel}{dt}}_{\epsilon^2} = \underbrace{-\frac{1}{\rho_0}\nabla_\parallel \underbrace{\left(\delta p + \frac{B_0\,\delta B_\parallel}{4\pi}\right)}_{= 0 \text{ to } \epsilon \text{ order}}}_{\epsilon^3} + \underbrace{V_A^2\,\frac{\vec{B}}{B_0}\cdot\nabla\frac{\delta B_\parallel}{B_0}}_{\epsilon^2}$$

$$\frac{du_\parallel}{dt} = V_A^2 \frac{\vec{B}}{B_0}\cdot\nabla \frac{\delta B_\parallel}{B_0} \qquad (8)$$

Now $\frac{d}{dt} \approx \frac{\partial}{\partial t} + \vec{u}_\perp\cdot\nabla_\perp$

$$\frac{\vec{B}}{B_0}\cdot\nabla \approx \nabla_\parallel + \frac{\delta\vec{B}_\perp}{B_0}\cdot\nabla_\perp$$

So the nonlinear terms in (6-8) involve $\vec{u}_\perp$, $\delta\vec{B}_\perp$.

- Recall $\underbrace{\nabla\cdot\vec{u}}_{\parallel} = -\frac{d}{dt}\frac{\delta\rho}{\rho_0} \sim \epsilon^2 k_\perp V_A$

eq. (4)

$$\underset{\epsilon k_\perp V_A}{\nabla_\perp\cdot\vec{u}_\perp} + \underset{\epsilon^2 k_\perp V_A}{\nabla_\parallel u_\parallel}$$

To order ϵ,

$$\nabla_\perp\cdot\vec{u}_\perp = 0$$

We can, therefore, set $\boxed{\vec{u}_\perp = \hat{z}\times\nabla_\perp\phi}$

stream function

We also have

$$0 = \nabla\cdot\vec{B} = \underset{\epsilon}{\nabla_\perp\cdot\delta\vec{B}_\perp} + \underset{\epsilon^2}{\nabla_\parallel\delta B_\parallel} \Rightarrow \nabla_\perp\cdot\delta\vec{B}_\perp = 0$$

to order ϵ

Therefore $\boxed{\frac{\delta\vec{B}_\perp}{\sqrt{4\pi\rho_0}} = \hat{z}\times\nabla_\perp\psi}$

flux function.

With these definitions,

$$\forall f \qquad \frac{df}{dt} = \frac{\partial f}{\partial t} + (\hat{z}\times\nabla_\perp\phi)\cdot\nabla_\perp f = \frac{\partial f}{\partial t} + \{\phi, f\}$$

$$\frac{\vec{B}}{B_0}\cdot\nabla f = \nabla_\parallel f + \frac{1}{V_A}\{\psi, f\}$$

where $\{\phi, f\} = \frac{\partial\phi}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial\phi}{\partial y}\frac{\partial f}{\partial x}$

- Induction equation, $\perp$ part :

$$\frac{d}{dt}\frac{\delta\vec{B}_\perp}{B_0} = \frac{\vec{B}}{B_0}\cdot\nabla\vec{u}_\perp$$

$$\Downarrow$$

Exercise : check!

$$\frac{\partial\psi}{\partial t} + \{\phi,\psi\} = + v_A\frac{\partial\phi}{\partial z} \qquad (9)$$

- Momentum equation, $\perp$ part, order ϵ^2

$$\nabla_\perp \times \Big| \quad \frac{d}{dt}\vec{u}_\perp = -\frac{1}{\rho_0}\nabla_\perp\begin{pmatrix}\text{second-order}\\ \text{part of pressure}\end{pmatrix} + v_A^2\frac{\vec{B}}{B_0}\cdot\nabla\frac{\delta\vec{B}_\perp}{B_0}$$

$$\Downarrow$$

Exercise: check!

$$\frac{\partial}{\partial t}\nabla_\perp^2\phi + \{\phi,\nabla_\perp^2\phi\} = v_A\frac{\partial}{\partial z}\nabla_\perp^2\psi + \{\psi,\nabla_\perp^2\psi\} \qquad (10)$$

So, final set of equations:

$$\begin{cases}\dfrac{\partial}{\partial t}\nabla_\perp^2\phi + \{\phi,\nabla_\perp^2\phi\} = v_A\dfrac{\partial}{\partial z}\nabla_\perp^2\psi + \{\psi,\nabla_\perp^2\psi\}\\[2ex] \dfrac{\partial\psi}{\partial t} + \{\phi,\psi\} = v_A\dfrac{\partial\phi}{\partial z}\end{cases}$$

RMHD
closed set!

Eq. (8): $$\frac{\partial u_\parallel}{\partial t} + \{\phi,u_\parallel\} = v_A\left(v_A\frac{\partial}{\partial z}\frac{\delta B_\parallel}{B_0} + \left\{\psi,\frac{\delta B_\parallel}{B_0}\right\}\right)$$

Eq. (7): $$\frac{\partial}{\partial t}\frac{\delta B_\parallel}{B_0} + \left\{\phi,\frac{\delta B_\parallel}{B_0}\right\} = \frac{1}{1+v_A^2/c_s^2}\frac{1}{v_A}\left(v_A\frac{\partial u_\parallel}{\partial z} + \{\psi,u_\parallel\}\right)$$

Eq. (6): $$\frac{\partial}{\partial t}\frac{\delta\rho}{\rho_0} + \left\{\phi,\frac{\delta\rho}{\rho_0}\right\} = -\frac{1}{1+c_s^2/v_A^2}\frac{1}{v_A}\left(v_A\frac{\partial u_\parallel}{\partial z} + \{\psi,u_\parallel\}\right)$$

NB: $u_\parallel$ and $\delta B_\parallel$ are not coupled to $\delta\rho$
($\delta\rho$ passive)

We have found that ϕ, ψ variables interact only with each other, while $u_\parallel$, $\delta B_\parallel$, $\delta\rho$ variables do <u>not</u> self interact: they are slaved to the ϕ, ψ variables.

What does this mean physically?

(ϕ, ψ), or $(\vec{u}_\perp, \delta\vec{B}_\perp)$ are Alfvén-wave variables

$(u_\parallel, \delta B_\parallel, \delta\rho)$ are slow-wave variables

Thus, 1) Alfvén-wave cascade decouples

2) slow waves can only cascade by scattering off the Alfvén waves.

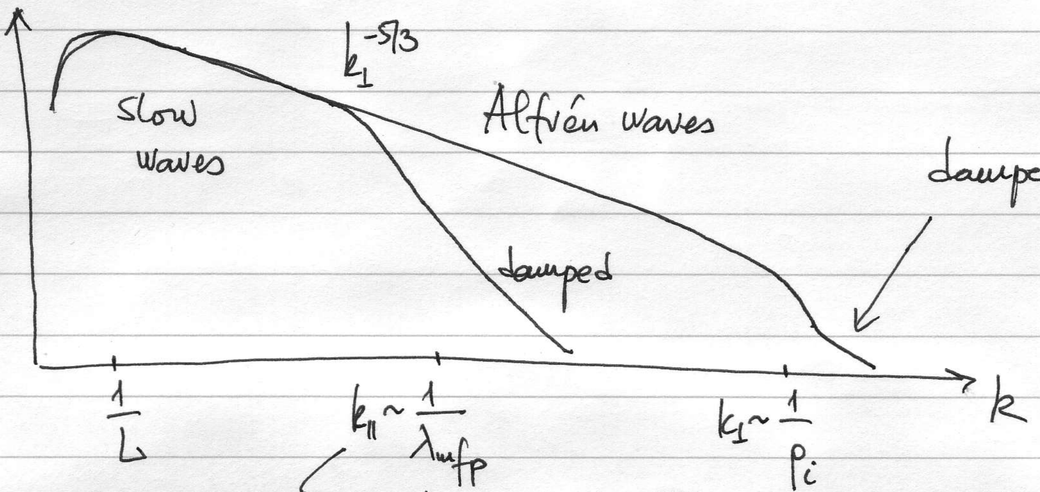
It turns out that in a magnetised weakly collisional plasma

1) slow waves are damped at $k_\parallel \lambda_{mfp} \sim 1$ by viscous damping $\left(\frac{k_\parallel \lambda_{mfp}}{\sqrt{\beta}} \sim 1 \right)$ and by collisionless kinetic damping $(k_\parallel \lambda_{mfp} \sim 1)$

2) Alfvén waves are not damped by this mechanism.

Even though MHD approximation is <u>not</u> valid at $k_\parallel \lambda_{mfp} > 1$, the RMHD set (9-10) remains valid for scales down to the ion gyroradius $\rho_i = \frac{v_{thi}}{\Omega_i} = \sqrt{\frac{kT}{m_i}} \frac{m_i c}{e B_0}$

So this is the picture:



$$k_\perp \sim \frac{1}{l_*(k_{\parallel 0}\lambda_{mfp})^{3/2}}$$

In astrophysical situations this interval is very wide (many astro plasmas have v. long λ_{mfp} but v. small ρ_i)

Lecture 16 14.11.05

Finally, let me back up what I have said about Alfvén-wave vs. slow-wave variables. Go back to waves and consider the limit $k_\parallel \ll k_\perp$. Mathematically, this is ~~quite~~ somewhat similar to the incompressible limit, even though we do not assume that $c_s \gg v_A$. Indeed,

$$\omega^2 = \frac{1}{2}k^2\left[c_s^2 + v_A^2 \pm \sqrt{(c_s^2+v_A^2)^2 - 4c_s^2 v_A^2 \frac{k_\parallel^2}{k^2}}\right] \approx$$

small!

$$= \frac{1}{2}k^2(c_s^2+v_A^2)\left[1 \pm \sqrt{1 - 4\frac{c_s^2 v_A^2}{(c_s^2+v_A^2)^2}\frac{k_\parallel^2}{k^2}}\right]$$

$$\approx \frac{1}{2}k^2(c_s^2+v_A^2)\left[1 \pm 1 \mp 4\frac{c_s^2 v_A^2}{(c_s^2+v_A^2)^2}\frac{1}{2}\frac{k_\parallel^2}{k^2}\right]$$

$$\omega^2 = k^2 (c_s^2 + V_A^2)$$

→ fast waves. We ordered them out by assuming $\omega \sim k_\parallel V_A \ll k V_A$ (in reality, they steepen into shocks and are strongly damped)

$$\text{or } \omega^2 = \frac{c_s^2 V_A^2}{(c_s^2 + V_A^2)} k_\parallel^2$$

→ slow waves, these remain

Exercise: derive these disp. relations from Eqs (6-10)

Alfvén waves are also there, unchanged, $\omega^2 = V_A^2 k_\parallel^2$

Now, for slow waves, we have $\omega^2 \xi_\parallel = c_s^2 k_\parallel \vec{k}\cdot\vec{\xi}$

$$k_\parallel^2 \frac{\cancel{c_s^2} V_A^2}{c_s^2+V_A^2} \xi_\parallel = \cancel{c_s^2} k_\parallel (k_\perp \xi_x + k_\parallel \xi_\parallel)$$

$$k_\parallel^2 \left(\frac{V_A^2}{c_s^2+V_A^2} - 1\right) \xi_\parallel = k_\parallel k_\perp \xi_x$$

$$\underset{(\epsilon)}{\xi_x} = \frac{k_\parallel}{k_\perp} \frac{c_s^2}{c_s^2+V_A^2} \xi_\parallel \ll \underset{(1)}{\xi_\parallel}$$

so, for slow waves in this regime, $\xi_\perp \ll \xi_\parallel$

$$\Downarrow$$

$$u_\perp \ll u_\parallel$$

Then

$$\frac{\delta B_\perp}{B_0} = i k_\parallel \underset{(\epsilon^2)}{\xi_x}$$

$$\frac{\delta B_\parallel}{B_0} = -i k_\perp \underset{(\epsilon)}{\xi_x}$$

$$\Rightarrow \delta B_\perp = -\frac{k_\parallel}{k_\perp} \delta B_\parallel$$

$$\delta B_\perp \ll \delta B_\parallel$$

and $$\frac{\delta\rho}{\rho_0} = -i k_\parallel \frac{V_A^2}{c_s^2 V_A^2} \xi_\parallel \sim \epsilon$$

[NB: $u_\parallel \sim \frac{\partial \xi_\parallel}{\partial t} \sim \omega \xi_\parallel \sim k_\parallel V_A \xi_\parallel \sim k_\perp u_\perp \xi_\parallel$ ← for Alf. waves

$u_\parallel \sim u_\perp \Rightarrow k_\perp \xi_\parallel \sim 1$]

↑ we ordered slow waves ~ Alfvén waves.

High-β limit: $c_s \gg v_A$ $\Rightarrow$ $\omega^2 \approx k_\parallel^2 v_A^2$ slow waves are pseudo-Alfvén waves

Eq. (7): $$\frac{d}{dt}\frac{\delta B_\parallel}{B_0} = \frac{\vec{B}}{B_0}\cdot\nabla u_\parallel$$

Eq. (6): $$\frac{d}{dt}\frac{\delta\rho}{\rho_0} = 0$$ — thus, density decouples from the rest is basically a passive scalar in the field of the Alfvén waves.

This is relevant to turbulence in the ISM, where what is actually measured is (electron) density fluctuations — these seem to have a $k^{-5/3}$ spectrum as a passive scalar should if the ambient turbulence has such a spectrum (recall example sheet 1!)

Low-β limit: $c_s \ll v_A$ $\Rightarrow$ $\omega^2 \approx k_\parallel^2 c_s^2$ slow waves are sound waves.

~~Eq. (6): $\frac{d}{dt}\frac{\delta\rho}{\rho_0} = -\frac{\vec{B}}{B_0}\cdot\nabla u_\parallel$~~

Eq. (7): $$\frac{d}{dt}\frac{\delta B_\parallel}{B_0} = 0$$

magnetic field is passive to lowest order (in β)
slow waves are much slower than the Alfvén waves with which they are interacting

Eq. (6): $$\frac{d}{dt}\frac{\delta\rho}{\rho_0} = -\frac{\vec{B}}{B_0}\cdot\nabla u_\parallel.$$

Elsässer form of the RMHD & slow-wave equations:

Introduce Elsässer potentials:

$$\zeta^\pm = \phi \pm \psi, \text{ so that } \vec{Z}_\perp^\pm = \hat{z}\times\nabla_\perp\zeta^\pm$$

Then (exercise: show this) Eqns (9-10) give:

$$\frac{\partial}{\partial t}\nabla_\perp^2\zeta^\pm \mp v_A\frac{\partial}{\partial z}\nabla_\perp^2\zeta^\pm =$$

$$= -\frac{1}{2}\left[\{\zeta^+, \nabla_\perp^2\zeta^-\} + \{\zeta^-, \nabla_\perp^2\zeta^+\} \mp \nabla_\perp^2\{\zeta^+,\zeta^-\}\right]$$

Again, no interaction except between counterpropagating Alfvén-wave packets.

For the slow waves, define

$$z_\parallel^\pm = u_\parallel \pm \frac{\delta B_\parallel}{\sqrt{4\pi\rho_0}}\sqrt{1+\frac{v_A^2}{c_s^2}}$$

Then Eqns (7-8) give

$$\frac{\partial z_\parallel^\pm}{\partial t} \mp \frac{v_A}{\sqrt{1+v_A^2/c_s^2}}\frac{\partial z_\parallel^\pm}{\partial z} = -\frac{1}{2}\left[\left(1\mp\frac{1}{\sqrt{1+v_A^2/c_s^2}}\right)\{\zeta^+, z_\parallel^\pm\} + \left(1\pm\frac{1}{\sqrt{1+v_A^2/c_s^2}}\right)\{\zeta^-, z_\parallel^\pm\}\right]$$

Thus, slow waves interact exclusively with counterpropagating Alfvén waves only in high-β limit ($v_A \ll c_s$).

Otherwise, a co-propagating Alfvén wave can catch up with a slow wave, ~~[illegible]~~ interact with it, and speed on (it's a jungle!)