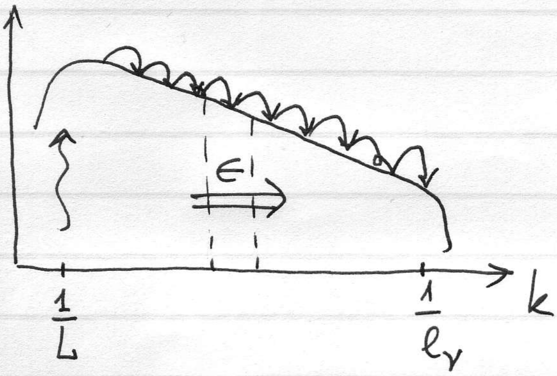
§13. Alfvén-Wave Turbulence: A Historical Overview.

Recall how we got Kolmogorov scaling for hydrodynamic turbulence:



- Scale invariance
- Locality of interactions (energy transfer) in k space

$$\Downarrow$$

$$\epsilon \sim \frac{\delta u_l^2}{\tau_l} \sim \text{const}$$

τ_l — cascade time

Dimensionally, only one time-like combination locally at scale l:

$$\tau_l \sim \frac{l}{\delta u_l} \quad \Rightarrow \quad \epsilon \sim \frac{\delta u_l^3}{l} \sim \text{const} \quad \Rightarrow \quad \boxed{\delta u_l \sim (\epsilon l)^{1/3}}$$

Physically, $\partial_t \vec{u} + \underbrace{\vec{u}\cdot\nabla\vec{u}}_{\sim\ \tau_l^{-1}} = -\nabla p + \nu\nabla^2\vec{u} + \vec{f}$

Now consider the hom. MHD equilibrium with a strong uniform external field $\vec{B}_0 = B_0\hat{z}$.

$$\partial_t \vec{Z}^{\pm} \mp \underbrace{V_A \nabla_{\parallel}}_{\tau_A^{-1}} \vec{Z}^{\pm} + \underbrace{\vec{Z}^{\mp}\cdot\nabla}_{\tau_s^{-1}} \vec{Z}^{\pm} = -\nabla p + \frac{\nu+\eta}{2}\nabla^2\vec{Z}^{\pm} + \frac{\nu-\eta}{2}\nabla^2\vec{Z}^{\mp}$$

where $\vec{Z}^{\pm} = \vec{u} \pm \frac{\delta\vec{B}}{\sqrt{4\pi\rho_0}}$ and $\nabla_{\parallel} = \frac{\partial}{\partial z}$

Exact slns: $\vec{Z}_- = 0 \quad \vec{Z}_+ = \vec{f}_+(x, y, z + V_A t)$

$\vec{Z}_+ = 0 \quad \vec{Z}_- = \vec{f}_-(x, y, z - V_A t)$

Let us think of an ensemble of spatially localised Alfvén-wave packets (parallel extent $l_\parallel$, perp. extent $l_\perp$) propagating parallel/antiparallel to $\vec{B}_0$.

Only counterpropagating ones can interact and thereby break (cascade) into smaller packets.

Again assume locality of interactions.

Kraichnan (1965)

Also assume $\underbrace{\delta z_l^+ \sim \delta z_l^-}_{\text{no imbalance}} \sim \underbrace{\delta u_l \sim \delta B_l/\sqrt{4\pi\rho_0}}_{\substack{\text{"Alfvénic" scaling} \\ \text{(same for } u \text{ and } B)}}$

Can we repeat the K41 argument?

$$\epsilon \sim \frac{\delta u_l^2}{\tau_l} \sim \text{const}$$

OK, but we do not know τ_l (time to cascade) because, physically speaking, 2 timescales are present:

strain ("eddy") time $\tau_s \sim \dfrac{l_\perp}{\delta u_l}$

Alfvén (wave) time $\tau_A \sim \dfrac{l_\parallel}{v_A}$

More formally speaking, there are now not one, but three dimensionless combinations at each scale l:

$$\frac{\epsilon l_\perp}{\delta u_l^3}, \quad \frac{\delta u_l}{v_A}, \quad \frac{l_\parallel}{l_\perp}$$

these 2 are new (due to presence of $\vec{B}_0$)

So can't solve MHD turbulence dimensionally. Further assumptions (besides locality) needed.

1) Iroshnikov (1964) – Kraichnan (1965) Theory

- Assume weak interactions:

$$|\vec{Z}^{\mp} \cdot \nabla \vec{Z}^{\pm}| \ll |v_A \nabla_{\parallel} \vec{Z}^{\pm}|$$

$$\frac{\delta z^{\mp}_{\ell}}{\ell_{\perp}} \sim \frac{\delta u_{\ell}}{\ell_{\perp}} \ll \frac{v_A}{\ell_{\parallel}} \iff \tau_s \gg \tau_A$$

i.e. each interaction only decorrelates the waves just a bit before they pass each other and are gone in opposite directions:

– time to pass $\Delta t \sim \frac{\ell_{\parallel}}{v_A} \sim \tau_A$

$\ell_{\parallel}$: $\parallel$ extent of packet

v_A: phase speed.

– kick the amplitude gets in one interaction:

$$\Delta \delta z^{\pm}_{\ell} \sim \underbrace{\frac{\delta z^{+}_{\ell} \delta z^{-}_{\ell}}{\ell_{\perp}}}_{\sim \frac{1}{\tau_s}} \underbrace{\Delta t}_{\sim \tau_A} \sim \delta z^{+}_{\ell} \frac{\delta z^{-}_{\ell}}{\ell_{\perp}} \frac{\ell_{\parallel}}{v_A} \sim \delta z^{+}_{\ell} \frac{\tau_A}{\tau_s}$$

– the kicks are random ($\pm$) – assume they add up like a random walk:

$$\sum^{t} \Delta \delta z^{+}_{\ell} \sim \delta z^{+}_{\ell} \frac{\tau_A}{\tau_s} \underbrace{\sqrt{\frac{t}{\tau_A}}}_{= \sqrt{\#\text{ of kicks in time } t}}$$

– by definition, cascade time τ_{ℓ} is the time it takes for the sum of the kicks to become comparable to the amplitude itself.

I.e.: $\quad \tau_\ell \sim t$ o.t. $\displaystyle\sum^{t} \Delta \delta z_\ell^+ \sim \delta z_\ell^+$

$$\delta z_\ell^+ \frac{\tau_A}{\tau_s^-} \sqrt{\frac{\tau_\ell}{\tau_A}} \sim \delta z_\ell^+$$

$$\Downarrow$$

$$\tau_\ell \sim \frac{(\tau_s^-)^2}{\tau_A} \sim \frac{\ell_\perp^2 v_A}{\ell_\parallel (\delta z_\ell^-)^2} \sim \frac{\ell_\perp^2 v_A}{\ell_\parallel \delta u_\ell^2}$$

So, we have then

$$\epsilon^+ \sim \frac{(\delta z_\ell^+)^2}{\tau_\ell} \sim \frac{(\delta z_\ell^+)^2 (\delta z_\ell^-)^2 \ell_\parallel}{v_A \ell_\perp^2} \sim \text{const}$$

Since we are assuming $\delta z_\ell^+ \sim \delta z_\ell^- \sim \delta u_\ell \sim \delta B_\ell$, this gives $\boxed{\delta u_\ell \sim (\epsilon v_A)^{1/4} \ell_\perp^{1/2} \ell_\parallel^{-1/4}}$

- Finally, we need one more assumption to fix the scaling [1 assumption for each dim.-less quantity: locality + weak interactions + 1 more]

IK assumed that at $\ell \ll L$ ($\perp$ extent of the system), everything becomes isotropic: $\ell_\parallel \sim \ell_\perp \sim \ell$

Then $\boxed{\delta u_\ell \sim (\epsilon v_A)^{1/4} \ell^{1/4}} \iff \boxed{E(k) \sim (\epsilon v_A)^{1/2} k^{-3/2}}$

IK spectrum.

$\Rightarrow$ Now check that this result is consistent with the weak interaction assumption:

$$\frac{\tau_A}{\tau_s} \sim \frac{\ell_\parallel \delta u_\ell}{\ell_\perp v_A} \sim \left(\frac{\epsilon}{v_A^3}\right)^{1/4} \ell_\parallel^{3/4} \ell_\perp^{-1/2}$$

Use $l_\parallel \sim l_\perp \sim l$ and $\delta u_L \sim (\epsilon v_A)^{1/4} L^{1/4}$:

$$\frac{\tau_A}{\tau_S} \sim \frac{\delta u_L}{v_A}\left(\frac{l}{L}\right)^{1/4} \ll 1 \text{ at small scales OK.}$$

So IK appears consistent.

Reality check. This was thought to be the right theory even though already in 1970s-80s, solar
till 1990's wind data was showing that

1) Fluctuations are not isotropic ($l_\parallel > l_\perp$)

2) Spectrum is closer to $k^{-5/3}$ than to $k^{-3/2}$

Simulations also show that $l_\parallel \gg l_\perp$!

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2) Weak turbulence theory

Let us consider weak interactions of counterpropagating Alfvén waves:

$$\omega(\vec{k}) = \pm k_\parallel v_A$$

Interactions occur if (1 meets 2 begets 3)

$$\left.\begin{array}{l} \vec{k}_1 + \vec{k}_2 = \vec{k}_3 \quad \mapsto \quad k_{\parallel 1} + k_{\parallel 2} = k_{\parallel 3} \\ \omega(\vec{k}_1) + \omega(\vec{k}_2) = \omega(\vec{k}_3) \quad \mapsto \quad k_{\parallel 1} - k_{\parallel 2} = \pm k_{\parallel 3} \end{array}\right| \Rightarrow \begin{array}{l} \text{say } + \\ k_{\parallel 2} = 0 \\ k_{\parallel 3} = k_{\parallel 1} \end{array}$$

Thus, $k_\parallel$ does not change in the interaction!

Let us then replace the isotropy assumption by its opposite:

$$l_{\parallel} \sim \frac{1}{k_{\parallel 0}} = \text{const}$$

$k_{\parallel 0}$ ← wave # at which waves are launched.

Then $\boxed{\delta u_l \sim (\epsilon v_A k_{\parallel 0})^{1/4} \, l_\perp^{1/2}} \iff \boxed{E(k) \sim (\epsilon k_{\parallel 0} v_A)^{1/2} k_\perp^{-2}}$

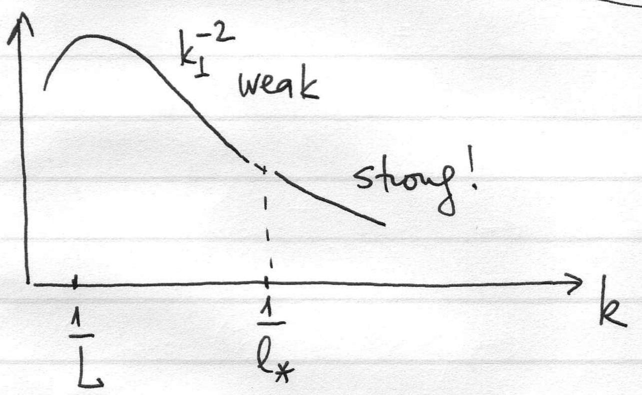
~~Unnamed~~ [This spectrum can, in fact, be obtained via a more formal procedure of expanding in small amplitude (later)]

⟹ Is this result still consistent with the weak interactions assumption?

$$\frac{\tau_A}{\tau_s} \sim \left(\frac{\epsilon}{v_A^3}\right)^{1/4} \frac{1}{k_{\parallel 0}^{3/4}} \, l_\perp^{-1/2} \ll 1 \ ?$$

$$l_\perp \gg \sqrt{\frac{\epsilon}{(k_{\parallel 0} v_A)^3}} \sim \left(\frac{\delta u_L}{v_A}\right)^2 \frac{1}{(k_{\parallel 0} L)^2} L \equiv l_*$$

$\delta u_L \sim (\epsilon v_A k_{\parallel 0})^{1/4} L^{1/2}$



Thus, while $l_* \ll L$

if $\frac{\delta u_L}{L} \ll k_{\parallel 0} v_A$

(i.e. if interactions are weak @ the injection scale),

there will always be a scale at which $\tau_A \sim \tau_s$ — interactions no longer weak! [Assume Re, Rm ≫ 1]

3) Goldreich – Sridhar (1995) Theory

GS replaced the assumptions of weak interactions ~~and (an)isotropy~~ by the assumption of <u>critical balance</u>:

$$\tau_s \sim \tau_A \quad \text{at each scale}$$

(strong interactions: $\vec{Z}^{\mp}\cdot\nabla\vec{Z}^{\pm} \sim v_A \nabla_\parallel \vec{Z}^{\pm}$)

Thus, $\dfrac{\delta u_\ell}{\ell_\perp} \sim \dfrac{v_A}{\ell_\parallel}$.

Since now there is only one characteristic time, it is natural to also assume $\tau_\ell \sim \tau_s \sim \tau_A$.

This means that K41 is restored:

$$\epsilon \sim \frac{\delta u_\ell^2}{\tau_\ell} \sim \frac{\delta u_\ell^3}{\ell_\perp} \sim \text{const} \;\Rightarrow\; \boxed{\delta u_\ell \sim (\epsilon \ell_\perp)^{1/3}}$$

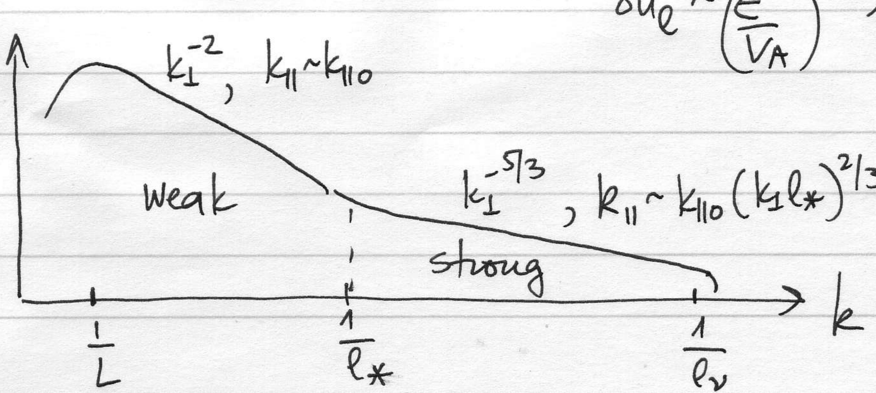
$$\boxed{E(k) \sim \epsilon^{2/3} k_\perp^{-5/3}}$$

Then critical balance gives

$$\ell_\parallel \sim \frac{v_A}{\delta u_\ell}\,\ell_\perp \sim \frac{v_A}{\epsilon^{1/3}}\,\ell_\perp^{2/3}$$, or, using $\ell_* = \dfrac{1}{k_{\parallel 0}^{3/2}}\left(\dfrac{\epsilon}{v_A^3}\right)^{1/2}$

"GS cone" $\boxed{\ell_\parallel \sim \dfrac{1}{k_{\parallel 0}}\left(\dfrac{\ell_\perp}{\ell_*}\right)^{2/3}}$ – cascade in $\ell_\parallel$ weaker, but <u>not</u> absent

$$\delta u_\ell \sim \left(\frac{\epsilon}{v_A}\right)^{1/2} \ell_\parallel^{1/2}$$



Finally, let us make sure that the interval (l_*, l_ν) is not empty, i.e. $l_* \gg l_\nu$.

As before, $l_\nu \sim \left(\frac{\nu^3}{\epsilon}\right)^{1/4}$

$$\frac{l_*}{l_\nu} \sim \frac{1}{k_{110}^{3/2}} \left(\frac{\epsilon}{V_A^3}\right)^{1/2} \left(\frac{\epsilon}{\nu^3}\right)^{1/4} = \left[\frac{\epsilon}{(k_{110} V_A)^2 \nu}\right]^{3/4}$$

Taking $\delta u_L \sim (\epsilon k_{110} V_A)^{1/4} L^{1/2}$, we get

$$\frac{l_*}{l_\nu} \sim \left[\frac{\delta u_L^4}{L^2 (k_{110} V_A)^3} \frac{1}{\nu}\right]^{3/4} \sim \left(\frac{\delta u_L}{V_A}\right)^{\frac{9}{4}} Re^{\frac{3}{4}} \frac{1}{(k_{110} L)^{\frac{9}{4}}} \gg 1$$

if $\boxed{Re \gg \left(\frac{V_A}{\delta u_L}\right)^3 (k_{110} L)^3}$

— this is an important technical point: it <u>does</u> matter in which order the strong-field $\left(\frac{V_A}{\delta u_L} \gg 1\right)$ and large-Re limits are taken!

(especially relevant for numerical simulations where Re is limited by resolution!)