§12. Finite-Amplitude Alfvén Waves.

Let us again consider the Lagrangian MHD:

$$\rho_0 \frac{\partial^2 \vec{\xi}}{\partial t^2} = -J(\nabla_0 \vec{x})^{-1} \cdot \nabla_0 \left(\frac{p_0}{J^\gamma} + \frac{|\vec{B}_0 \cdot \nabla_0 \vec{x}|^2}{8\pi J^2} \right) + \frac{1}{4\pi} \vec{B}_0 \cdot \nabla_0 \left(\frac{\vec{B}_0}{J} \cdot \nabla_0 \vec{x} \right)$$

where $\vec{x} = \vec{x}_0 + \vec{\xi}$. Let us assume incompressibility:

$$J = 1 = \text{const.}$$

Also take $\vec{B}_0 = B_0 \hat{z}$ as in the waves problem.

Then
$$\frac{\partial^2 \vec{\xi}}{\partial t^2} = -(\nabla_0 \vec{x})^{-1} \cdot \nabla_0 \underbrace{\tilde{p}}_{\text{total pressure}} + \underbrace{\frac{B_0^2}{4\pi \rho_0}}_{\equiv V_A^2} \nabla_{0\parallel}^2 \vec{x}$$

$$(\nabla_0 \vec{x}) \cdot \left(\frac{\partial^2 \vec{\xi}}{\partial t^2} - V_A^2 \nabla_{0\parallel}^2 \vec{\xi} \right) = -\nabla_0 \frac{\tilde{p}}{\rho_0}$$

NB: $\tilde{p}$ determined from the constraint $\boxed{J = 1}$

Now $\nabla_{0\parallel} \equiv \frac{\partial}{\partial z_0}$ and

$$\boxed{(\mathbb{1} + \nabla_0 \vec{\xi}) \cdot \left(\frac{\partial}{\partial t} - V_A \frac{\partial}{\partial z_0} \right) \left(\frac{\partial}{\partial t} + V_A \frac{\partial}{\partial z_0} \right) \vec{\xi} = -\nabla_0 \frac{\tilde{p}}{\rho_0}}$$

Exact solutions: $\vec{\xi} = \vec{\xi}_+ (x_0, y_0, z_0 - V_A t)$

$\vec{\xi} = \vec{\xi}_- (x_0, y_0, z_0 + V_A t)$

with the constraint that $\vec{\xi}_\pm (x_0, y_0, z_0)$ are such that $J = 1$. NB: they stay that way because time evolution simply amounts to transformation to a frame moving with velocity $\pm V_A$ along z axis.

$\Rightarrow$ may choose $\nabla_0 \tilde{p} = 0$.

[illegible] These are finite-amplitude Alfvén waves (wave packets). Unlike waves, they do <u>not</u> have superposition property:

$\vec{\xi}_-$ <u>or</u> $\vec{\xi}_+$ are solutions

$\vec{\xi}_- + \vec{\xi}_+$ is <u>not</u>!

– because, formally, J will contain the interaction between them.

Now notice that $\vec{\xi}_\pm$ is a solution when

not small!

$$0 = \left(\frac{\partial}{\partial t} \pm v_A \frac{\partial}{\partial z_0}\right)\vec{\xi} = \vec{u} \pm \frac{\delta\vec{B}}{\sqrt{4\pi\rho_0}}, \text{ where } \vec{B} = B_0\hat{z} + \delta\vec{B}$$

Indeed, $\delta\vec{B} = \vec{B}_0\cdot\nabla_0\vec{x} - \vec{B}_0 = B_0\nabla_{0\parallel}\vec{\xi}$!

Thus, we should find that $\vec{Z}^\pm = \vec{u} \pm \frac{\delta\vec{B}}{\sqrt{4\pi\rho_0}} = 0$

are exact sltns of MHD.

$\vec{Z}^\pm$ are called <u>Elsässer variables</u> and it turns out that MHD takes a very nice symmetric form when written in these variables:

Consider Eulerian incompressible MHD:

$$\begin{cases} \dfrac{d\vec{u}}{dt} = -\dfrac{\nabla\tilde{p}}{\rho_0} + \dfrac{\vec{B}\cdot\nabla\vec{B}}{4\pi\rho_0} + \nu\nabla^2\vec{u}, \quad \nabla\cdot\vec{u} = 0 \\[2ex] \dfrac{d\vec{B}}{dt} = \vec{B}\cdot\nabla\vec{u} + \eta\nabla^2\vec{B} \end{cases}$$

Let $\vec{B} = B_0 \hat{z} + \delta\vec{B}$ and $\vec{z}^{\pm} = \vec{u} \pm \dfrac{\delta\vec{B}}{\sqrt{4\pi\rho_0}}$. Then

$$\frac{\partial \vec{z}^{\pm}}{\partial t} = \frac{\partial \vec{u}}{\partial t} \pm \frac{1}{\sqrt{4\pi\rho_0}} \frac{\partial \delta\vec{B}}{\partial t} =$$

$$= -\vec{u}\cdot\nabla\vec{u} - \frac{\nabla\tilde{p}}{\rho_0} + \frac{B_0 \nabla_{\parallel} \delta\vec{B}}{4\pi\rho_0} + \frac{\delta\vec{B}\cdot\nabla\delta\vec{B}}{4\pi\rho_0} + \nu\nabla^2\vec{u}$$

$$\mp \frac{\vec{u}\cdot\nabla\delta\vec{B}}{\sqrt{4\pi\rho_0}} \pm \frac{B_0\nabla_{\parallel}\vec{u}}{\sqrt{4\pi\rho_0}} \pm \frac{\delta\vec{B}\cdot\nabla\vec{u}}{\sqrt{4\pi\rho_0}} \pm \eta\nabla^2 \frac{\delta\vec{B}}{\sqrt{4\pi\rho_0}} =$$

$$= \pm v_A \nabla_{\parallel} \vec{z}^{\pm} - \vec{z}^{\mp}\cdot\nabla\vec{u} \mp \vec{z}^{\mp}\cdot\nabla \frac{\delta\vec{B}}{\sqrt{4\pi\rho_0}} - \frac{\nabla\tilde{p}}{\rho_0} +$$

$$+ \nu\nabla^2 \frac{\vec{z}^{+}+\vec{z}^{-}}{2} \pm \eta\nabla^2 \frac{\vec{z}^{+}-\vec{z}^{-}}{2} =$$

$$= \pm v_A \nabla_{\parallel}\vec{z}^{\pm} - \vec{z}^{\mp}\cdot\nabla\vec{z}^{\pm} - \frac{\nabla\tilde{p}}{\rho_0} + \frac{\nu\pm\eta}{2}\nabla^2\vec{z}^{+} + \frac{\nu\mp\eta}{2}\nabla^2\vec{z}^{-}$$

Finally

$$\nabla\cdot\vec{z}^{\pm} = 0 \qquad \boxed{\frac{\partial \vec{z}^{\pm}}{\partial t} \mp v_A\nabla_{\parallel}\vec{z}^{\pm} + \vec{z}^{\mp}\cdot\nabla\vec{z}^{\pm} = -\frac{\nabla\tilde{p}}{\rho_0} + \frac{\nu+\eta}{2}\nabla^2\vec{z}^{\pm} + \frac{\nu-\eta}{2}\nabla^2\vec{z}^{\mp}}$$

If, say, $\vec{z}_- = \vec{u} - \dfrac{\delta\vec{B}}{\sqrt{4\pi\rho_0}} = 0$, then $\vec{z}_+$ satisfies

$$\frac{\partial \vec{z}^{+}}{\partial t} - v_A\nabla_{\parallel}\vec{z}^{+} = -\cancel{\frac{\nabla\tilde{p}}{\rho_0}} + \frac{\nu+\eta}{2}\nabla^2\vec{z}^{+}$$

0 because $\nabla\cdot\vec{z}^{\pm}$ satisfied always if satisfied initially.

So $\vec{z}^{+}$ is a wave + diffusion.

$\vec{u} = \pm\delta\vec{B}$ are called Alfvénic (or Elsässer) states.

Physics: these solutions exist because the nonlinearity never couples $\vec{z}^+$ with $\vec{z}^+$ or $\vec{z}^-$ with $\vec{z}^-$, in other words, only counterpropagating wave packets interact.

In a pure E. state, there are simply isolated wave packets that propagate along z with velocity v_A — they never catch up with each other (everyone has the same velocity) and never interact.

Energetics of Elsässer variables:

$$|\vec{z}^\pm|^2 = u^2 \pm 2\vec{u}\cdot\frac{\delta\vec{B}}{\sqrt{4\pi\rho_0}} + \frac{\delta B^2}{4\pi\rho_0}$$

So,

$$\int d^3x \frac{\rho_0}{2}\left[|\vec{z}^+|^2 + |\vec{z}^-|^2\right] = \underbrace{\int d^3x \left(\frac{1}{2}\rho_0 u^2 + \frac{\delta B^2}{8\pi}\right)}_{\text{total energy}}$$

$$\int d^3x \frac{\rho_0}{2}\left[|\vec{z}^+|^2 - |\vec{z}^-|^2\right] = \underbrace{\int d^3x \frac{2\rho_0}{\sqrt{4\pi\rho_0}}\,\vec{u}\cdot\delta\vec{B}}_{\text{cross-helicity}}$$

Both are conserved quantities, ~~[illegible]~~ so the "energy" of each Elsasser variable is conserved $\int d^3x \frac{1}{2}\rho_0 |\vec{z}^\pm|^2 = \text{const.}$

Cross-helicity is a measure of the imbalance between the $+$ and $-$ fields.