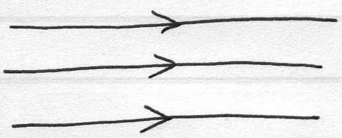
§11. MHD Waves

Consider the following homogeneous equilibrium:



$\vec{B}_0 = B_0\hat{z} = \text{const}$ $\qquad \vec{u}_0 = 0$ (static)

$\rho_0 = \text{const}$

$p_0 = \text{const}$

Now consider small perturbations of this equilibrium, specifically, small displacements $\vec{\xi}$.

These are Lagrangian displacements, so, to get an evolution equation for them, we may take the Lagrangian MHD and linearize it:

$$\rho_0 \frac{\partial^2 \vec{\xi}}{\partial t^2} = -J\left(\nabla_0 \vec{X}\right)^{-1} \cdot \nabla_0 \left(\frac{p_0}{J^\gamma} + \frac{|\vec{B}_0 \cdot \nabla_0 \vec{X}|^2}{8\pi J^2}\right) + \frac{1}{4\pi}\vec{B}_0 \cdot \nabla_0 \left(\frac{\vec{B}_0}{J}\cdot \nabla_0 \vec{X}\right)$$

$$= \frac{\gamma p_0}{J^\gamma}\left(\nabla_0 \vec{X}\right)^{-1}\cdot \nabla_0 J - \frac{B_0^2}{8\pi J}\left(\nabla_0 \vec{X}\right)^{-1}\cdot \nabla_0 |\nabla_{0\parallel} \vec{X}|^2 +$$

$\uparrow$ substitute the homogeneous equilibrium $\qquad \uparrow\ \nabla_{0\parallel} \equiv \frac{\partial}{\partial z_0}$

$$+ \frac{B_0^2}{4\pi J^2} |\nabla_{0\parallel}\vec{X}|^2 \left(\nabla_0 \vec{X}\right)^{-1}\cdot \nabla_0 J$$

$$+ \frac{B_0^2}{4\pi J}\nabla_{0\parallel}^2 \vec{X} - \frac{B_0^2}{4\pi J^2}\left(\nabla_{0\parallel}\vec{X}\right)\nabla_{0\parallel} J$$

Now $\nabla_0 \vec{X} = \nabla_0 \left(\vec{x}_0 + \vec{\xi}\right) = \mathbb{1} + \nabla_0 \vec{\xi}$

$$\left(\nabla_0 \vec{X}\right)^{-1} = \mathbb{1} - \nabla_0 \vec{\xi} + \dots$$

Since $(\nabla_0 \vec{X})^{-1}$ everywhere multiplies a ∇_0 of something, which must be 1-order (equilibrium quantities have no spatial variation!), we can, to the linear order, replace $(\nabla_0 \vec{x})^{-1} \approx \mathbb{1}$ everywhere.

[In other words, to linear order (and for the homogeneous equilibrium) the Lagrangian and the Eulerian coord. systems are the same.

Now let us linearize J: if you take $x_i = x_{0i} + \xi_i$ and use Newcomb's formula

$$J = \frac{1}{6} \epsilon_{ijk} \epsilon_{mnl} \frac{\partial x_i}{\partial x_{0m}} \frac{\partial x_j}{\partial x_{0n}} \frac{\partial x_k}{\partial x_{0l}},$$

you can show (exercise!) that

$$J = 1 + \nabla_0 \cdot \vec{\xi} + \ldots$$

[use the formula $\epsilon_{ijk}\epsilon_{inl} = \delta_{jn}\delta_{kl} - \delta_{jl}\delta_{kn}$]

Here is an alternative way to prove this:

in the Eulerian MHD, $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$

Then, for the Lagrangian density,

$$\frac{\partial \rho_L}{\partial t} = - \rho_L \nabla \cdot \vec{u} = - \rho_L \nabla \cdot \frac{\partial \vec{\xi}}{\partial t}$$

$\nabla \approx \nabla_0$ to linear order here

Let $\rho_L = \rho_0 + \delta\rho_L \Rightarrow \frac{\partial \delta\rho_L}{\partial t} \approx - \rho_0 \nabla_0 \cdot \frac{\partial \vec{\xi}}{\partial t}$

so $\delta\rho_L = - \rho_0 \nabla_0 \cdot \vec{\xi}$

But, on the other hand,

$$\rho_L = \rho_0 + \delta\rho_L = \frac{\rho_0}{J} \Rightarrow J = \frac{1}{1 + \frac{\delta\rho_L}{\rho_0}} \approx 1 - \frac{\delta\rho_L}{\rho_0} = 1 + \nabla_0 \cdot \vec{\xi}$$

So now we are ready to linearize the Lagr. MHD:

$$\rho_0 \frac{\partial^2 \vec{\xi}}{\partial t^2} = \gamma p_0 \nabla_0 \nabla_0 \cdot \vec{\xi} - \frac{B_0^2}{8\pi} \nabla_0 \underbrace{|\hat{z} + \nabla_{0\parallel} \vec{\xi}|^2}_{\simeq\, 2\nabla_0 \nabla_{0\parallel} \xi_\parallel} + \frac{B_0^2}{4\pi} \nabla_0 \nabla_0 \cdot \vec{\xi}$$

$$+ \frac{B_0^2}{4\pi} \nabla_{0\parallel}^2 \vec{\xi} - \frac{B_0^2}{4\pi} \hat{z} \nabla_{0\parallel} \nabla_0 \cdot \vec{\xi}$$

NB: we used $\nabla_{0\parallel} \vec{x} = \hat{z} + \nabla_{0\parallel} \vec{\xi}$

$$\frac{\partial^2 \vec{\xi}}{\partial t^2} = \frac{\gamma p_0}{\rho_0} \nabla_0 \nabla_0 \cdot \vec{\xi} + \frac{B_0^2}{4\pi \rho_0} \Big[\nabla_0 \underbrace{\left(\nabla_0 \cdot \vec{\xi} - \nabla_{0\parallel} \vec{\xi}_\parallel \right)}_{=\, \nabla_{0\perp} \cdot \vec{\xi}_\perp} + \nabla_{0\parallel} \underbrace{\left(\nabla_{0\parallel} \vec{\xi} - \hat{z} \nabla_0 \cdot \vec{\xi} \right)}_{=\, \nabla_\parallel \vec{\xi}_\perp - \hat{z} \nabla_\perp \cdot \vec{\xi}_\perp} \Big]$$

$\frac{\gamma p_0}{\rho_0} \equiv C_s^2$ sound speed; $\frac{B_0^2}{4\pi\rho_0} \equiv V_A^2$ Alfvén speed

$$[\ldots] = \nabla_{0\perp} \nabla_{0\perp} \cdot \vec{\xi}_\perp + \nabla_\parallel^2 \vec{\xi}_\perp$$

$$= C_s^2 \nabla \nabla \cdot \vec{\xi} + V_A^2 \left(\nabla_\perp \nabla_\perp \cdot \vec{\xi}_\perp + \nabla_\parallel^2 \vec{\xi}_\perp \right)$$

← I dropped 0's on ∇ because $\nabla \approx \nabla_0$

Recall
$$\vec{B} = \vec{B}_0 + \delta \vec{B} = \frac{\vec{B}_0 \cdot (\mathbb{1} + \nabla_0 \vec{\xi})}{J} =$$

$$= B_0 \left(\hat{z} + \nabla_{0\parallel} \vec{\xi} \right) \left(1 - \nabla_0 \cdot \vec{\xi} + \ldots \right) \approx B_0 \hat{z} + B_0 \nabla_{0\parallel} \vec{\xi} - \hat{z} B_0 \nabla_0 \cdot \vec{\xi}$$

$$\delta \vec{B} \approx B_0 \left(\nabla_{0\parallel} \vec{\xi} - \hat{z} \nabla_0 \cdot \vec{\xi} \right) = B_0 \Big(\underbrace{\nabla_\parallel \vec{\xi}_\perp}_{\delta B_\perp} - \hat{z} \underbrace{\nabla_\perp \cdot \vec{\xi}_\perp}_{\delta B_\parallel} \Big)$$

This is the formal Lagrangian derivation of the linearized eqns.

However, all this can be obtained straight out of the Eulerian MHD using the fact that to linear order Eulerian and Lagrangian time/space derivatives are the same.

So, here is the straightforward linearisation of MHD about the homogeneous equilibrium:

$$\rho = \rho_0 + \delta\rho \qquad \vec{B} = \vec{B}_0 + \delta\vec{B}, \quad \vec{B}_0 = B_0\hat{z}$$

$$p = p_0 + \delta p \qquad \vec{u} = \frac{\partial\vec{\xi}}{\partial t} \quad (1\text{ order})$$

Then $\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0 \Rightarrow \frac{\partial\delta\rho}{\partial t} = -\rho_0\nabla\cdot\frac{\partial\vec{\xi}}{\partial t} \Rightarrow \boxed{\delta\rho = -\rho_0\nabla\cdot\vec{\xi}}$

$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)\frac{p}{\rho^\gamma} = 0 \Rightarrow \frac{\partial}{\partial t}\frac{\delta p}{p_0} = \gamma\frac{\partial}{\partial t}\frac{\delta\rho}{\rho_0} \Rightarrow \boxed{\delta p = -\gamma p_0\nabla\cdot\vec{\xi}}$$

$$\underbrace{\rho\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)\vec{u}}_{\Downarrow} = \underbrace{-\nabla\left(p + \frac{B^2}{8\pi}\right)}_{\parallel} + \underbrace{\frac{\vec{B}\cdot\nabla\vec{B}}{4\pi}}_{\parallel}$$

$$\rho_0\frac{\partial^2\vec{\xi}}{\partial t^2} \qquad -\nabla\delta p - \frac{B_0}{4\pi}\nabla\delta B_\parallel \qquad \frac{B_0}{4\pi}\left(\nabla_\parallel\delta\vec{B}_\perp + \hat{z}\nabla_\parallel\delta B_\parallel\right)$$

So
$$\frac{\partial^2\vec{\xi}}{\partial t^2} = \frac{\gamma p_0}{\rho_0}\nabla\nabla\cdot\vec{\xi} + \frac{B_0^2}{4\pi\rho_0}\left[-\nabla_\perp\frac{\delta B_\parallel}{B_0} + \nabla_\parallel\frac{\delta\vec{B}_\perp}{B_0}\right]$$

$$\left(\frac{\partial}{\partial t} + \vec{u}\cdot\nabla\right)\vec{B} = \vec{B}\cdot\nabla\vec{u} - \vec{B}\,\nabla\cdot\vec{u}$$

$$\Rightarrow \frac{\partial\delta\vec{B}}{\partial t} = B_0\nabla_\parallel\frac{\partial\vec{\xi}}{\partial t} - \hat{z}B_0\nabla\cdot\frac{\partial\vec{\xi}}{\partial t} \Rightarrow \boxed{\frac{\delta\vec{B}_\perp}{B_0} = \nabla_\parallel\vec{\xi}_\perp \qquad \frac{\delta B_\parallel}{B_0} = -\nabla_\perp\cdot\vec{\xi}_\perp}$$

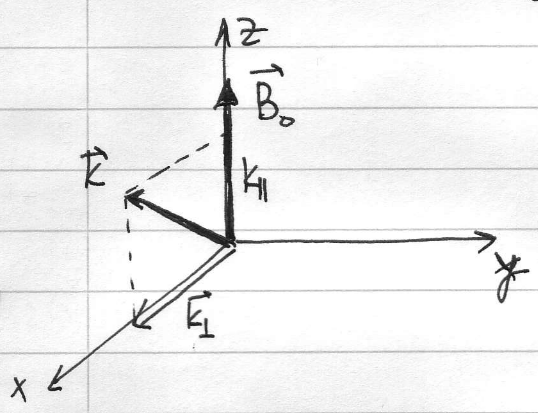
Finally,
$$\boxed{\frac{\partial^2\vec{\xi}}{\partial t^2} = \underbrace{\frac{\gamma p_0}{\rho_0}}_{\equiv C_s^2}\nabla\nabla\cdot\vec{\xi} + \underbrace{\frac{B_0^2}{4\pi\rho_0}}_{\equiv V_A^2}\left(\nabla_\perp\nabla_\perp\cdot\vec{\xi}_\perp + \nabla_\parallel^2\vec{\xi}_\perp\right)}$$

C_s^2 Sound speed; V_A^2 Alfvén speed

Consider wave-like slns: $\vec{\xi} \sim \vec{\xi}_{k\omega}\, e^{-i\omega t + i\vec{k}\cdot\vec{x}}$. Then

$$\omega^2 \vec{\xi} = c_s^2 \vec{k}\,\vec{k}\cdot\vec{\xi} + v_A^2 \left(\vec{k}_\perp \vec{k}_\perp \cdot \vec{\xi}_\perp + k_\parallel^2 \vec{\xi}_\perp\right)$$

Without loss of generality, let $\vec{k} = (k_\perp, 0, k_\parallel)$. Then



$$\begin{cases} \omega^2 \xi_x = c_s^2 k_\perp (k_\perp \xi_x + k_\parallel \xi_\parallel) + v_A^2 k^2 \xi_x \\ \omega^2 \xi_y = v_A^2 k_\parallel^2 \xi_y \\ \omega^2 \xi_\parallel = c_s^2 k_\parallel (k_\perp \xi_x + k_\parallel \xi_\parallel) \end{cases}$$

Note that

$$\frac{\delta\vec{B}}{B_0} = \frac{\delta(B\hat{b})}{B_0} = \frac{B_0 \delta\hat{b}}{B_0} + \frac{\hat{b}\,\delta B}{B_0} = \delta\hat{b} + \hat{z}\frac{\delta B}{B_0}$$

$$\equiv \frac{\delta\vec{B}_\perp}{B_0} + \hat{z}\frac{\delta B_\parallel}{B_0}$$

NB: $\hat{b}^2 = 1 = (\hat{z}+\delta\hat{b})\cdot(\hat{z}+\delta\hat{b}) = 1 + 2\hat{z}\cdot\delta\hat{b} + \ldots \Rightarrow \delta\hat{b} \perp \hat{z}$

Therefore $\delta\hat{b} = \dfrac{\delta\vec{B}_\perp}{B_0} = i k_\parallel \vec{\xi}_\perp \quad \left(\vec{\xi}_\perp = \begin{pmatrix}\xi_x \\ \xi_y \\ 0\end{pmatrix}\right)$

$$\frac{\delta B}{B_0} = \frac{\delta B_\parallel}{B_0} = -i k_\perp \xi_x$$

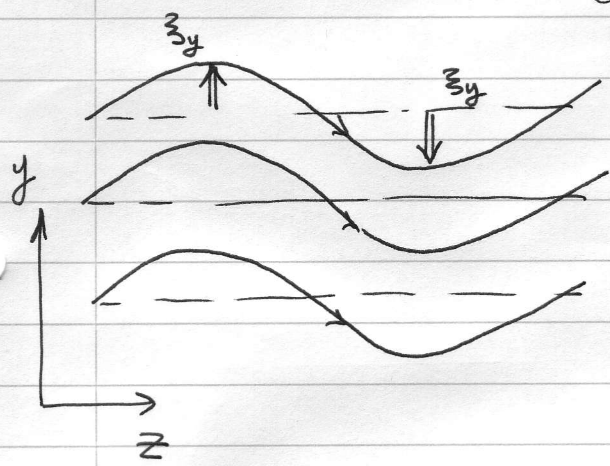
Also $\dfrac{\delta\rho}{\rho_0} = -i(k_x \xi_x + k_\parallel \xi_\parallel)$ and $\dfrac{\delta p}{p_0} = \gamma\dfrac{\delta\rho}{\rho_0}$.

1) The $\hat{y}$ eqn decouples: $\begin{pmatrix} 0 \\ \xi_y \\ 0 \end{pmatrix}$ is an eigenvector

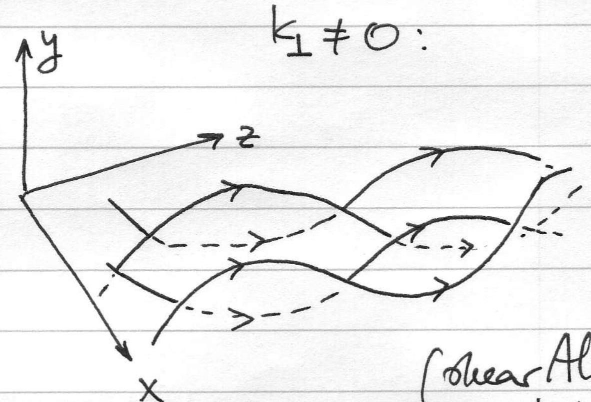
$$\boxed{\omega^2 = k_\parallel^2 V_A^2}$$ and $\delta\rho = 0$, $\delta p = 0$, $\delta B = 0$

$$\delta\hat{b} = i k_\parallel \xi_y \hat{y}$$

(field lines are curved ⟹ spring back)



These waves can have $k_\perp \neq 0$:



(shear Alfvén waves)

Lecture 13 8.11.05

2) The rest of the waves:

$$\omega^2 \begin{pmatrix} \xi_x \\ \xi_\parallel \end{pmatrix} = \begin{pmatrix} C_s^2 k_\perp^2 + V_A^2 k^2 & C_s^2 k_\parallel k_\perp \\ C_s^2 k_\parallel k_\perp & C_s^2 k_\parallel^2 \end{pmatrix} \cdot \begin{pmatrix} \xi_x \\ \xi_\parallel \end{pmatrix}$$

$$\omega^4 - k^2 (C_s^2 + V_A^2)\omega^2 + C_s^2 V_A^2 k^2 k_\parallel^2 = 0$$

$$\omega^2 = \frac{1}{2} k^2 \left[C_s^2 + V_A^2 \pm \sqrt{(C_s^2 + V_A^2)^2 - 4 C_s^2 V_A^2 \frac{k_\parallel^2}{k^2}} \right]$$

($\frac{k_\parallel^2}{k^2} = \cos^2\theta$)

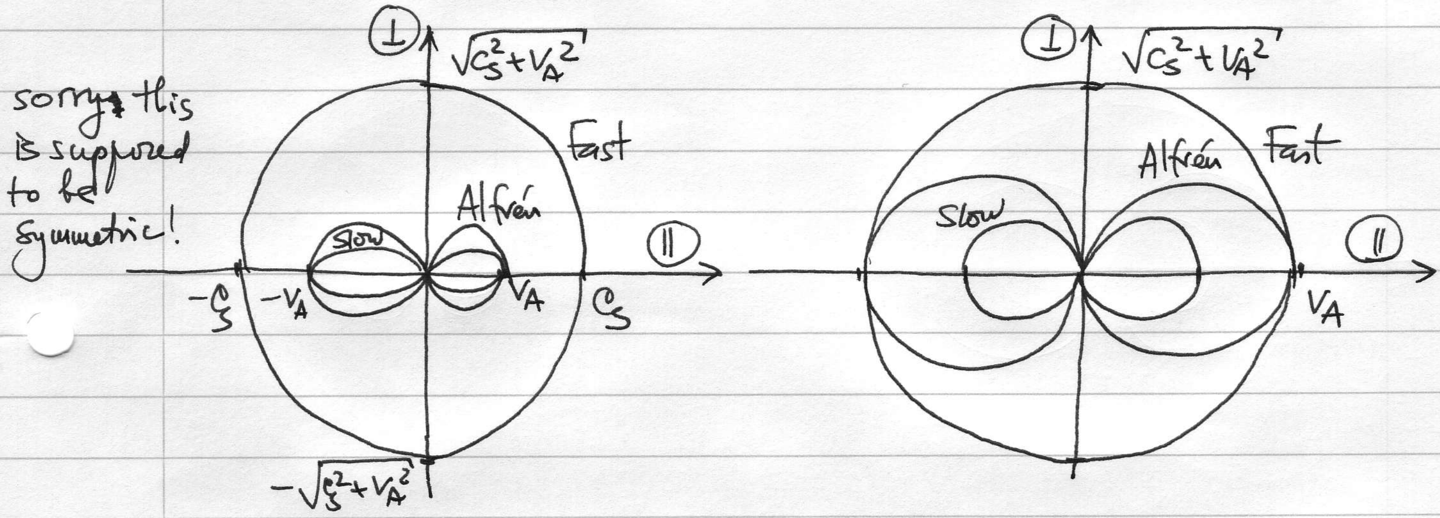
⊕ fast wave
⊖ slow wave
(magnetosonic)

NB: $\frac{C_s^2}{V_A^2} = \frac{\gamma p_0 4\pi \rho_0}{\rho_0 B_0^2} = \frac{\gamma}{2} \frac{p_0}{B_0^2/8\pi} = \frac{\gamma}{2}\beta$

Friedrichs diagram : polar plot with radius $= \frac{\omega}{k}$ (phase velocity)

angle $= \theta$

$c_s^2 > v_A^2$ ("high β") $\qquad$ $c_s^2 < v_A^2$ ("low β")



To understand physics, it is useful to consider a few special cases.

Parallel propagation: $k_\perp = 0$

Then $\begin{pmatrix} \xi_x \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ \xi_\parallel \end{pmatrix}$ are eigenvectors:

$$\omega^2 \xi_x = k_\parallel^2 v_A^2 \xi_x$$

$$\boxed{\omega^2 = k_\parallel^2 v_A^2}$$

(high β: slow
low β: fast)

like Alfvén wave

$$\omega^2 \xi_\parallel = c_s^2 k_\parallel^2 \xi_\parallel$$

$$\boxed{\omega^2 = k_\parallel^2 c_s^2}$$

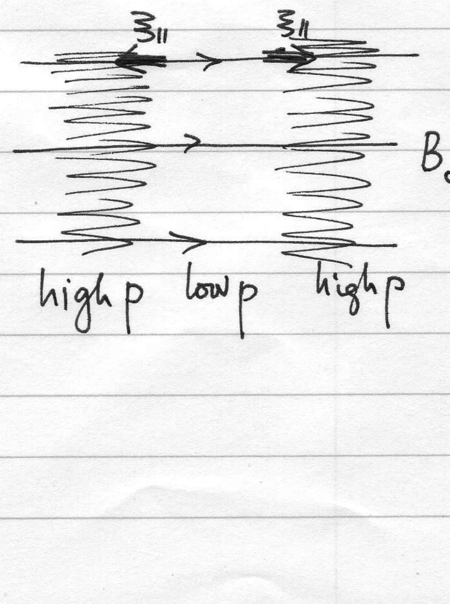
(high β: fast
low β: slow)

Sound wave

$\delta b = 0 \quad \delta B = 0$

$$\frac{\delta \rho}{\rho_0} = -ik_\parallel \xi_\parallel$$

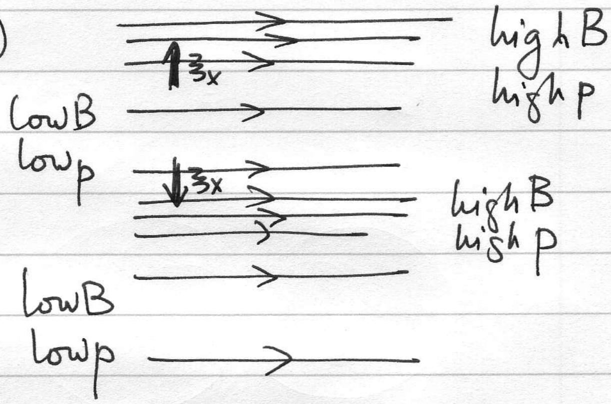
$$\frac{\delta p}{p_0} = \gamma \frac{\delta \rho}{\rho_0}$$



Perpendicular propagation : $k_\parallel = 0$

$\begin{pmatrix} \xi_x \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector: $\omega^2 \xi_x = (c_s^2 + v_A^2) k_\perp^2 \xi_x$

$$\boxed{\omega^2 = (c_s^2 + v_A^2) k_\perp^2} \quad \text{(fast)}$$



magnetosonic waves

$$\delta \hat{b} = 0$$

$$\frac{\delta B}{B_0} = -i k_\perp \xi_x$$

$$\frac{\delta \rho}{\rho_0} = -i k_\perp \xi_x , \quad \frac{\delta p}{p_0} = \gamma \frac{\delta \rho}{\rho_0}$$

Incompressible limit: $c_s^2 \gg v_A^2$ (very high β)

$$\omega^2 \approx \frac{1}{2} k^2 c_s^2 \left[1 \pm \sqrt{1 - 4 \frac{v_A^2}{c_s^2} \frac{k_\parallel^2}{k^2}} \right] \approx$$

$$\approx \frac{1}{2} k^2 c_s^2 \left[1 \pm 1 \mp 2 \frac{v_A^2}{c_s^2} \frac{k_\parallel^2}{k^2} \right]$$

(+) Fast : $\omega^2 = k^2 c_s^2$ sound wave (isotropic)

(−) Slow : $\omega^2 = k_\parallel^2 v_A^2$ pseudo-Alfvén wave.

$$\begin{cases} \omega^2 \xi_x = \underline{c_s^2 k_\perp \mathbf{k} \cdot \vec{\xi}} + \underline{k^2 v_A^2 \xi_x} \\ \omega^2 \xi_\parallel = \underline{c_s^2 k_\parallel \mathbf{k} \cdot \vec{\xi}} \end{cases}$$

↑ dropping this gives sound wave (corresponds to $\omega^2 = k^2 c_s^2$)

Let $\omega^2 = k_\parallel^2 V_A^2$. Then

$$k_\parallel^2 V_A^2 \xi_\parallel = c_s^2 k_\parallel \vec{k}\cdot\vec{\xi} \Rightarrow \vec{k}\cdot\vec{\xi} = \frac{V_A^2}{c_s^2} k_\parallel \xi_\parallel \approx -\frac{V_A^2}{c_s^2} k_\perp \xi_x$$

small (nearly incompressible!)

Then

$$k_\parallel^2 V_A^2 \xi_x = c_s^2 k_\perp \left(-\frac{V_A^2}{c_s^2} k_\perp \xi_x\right) + k^2 V_A^2 \xi_x = k_\parallel^2 V_A^2 \xi_x$$

no longer negligible

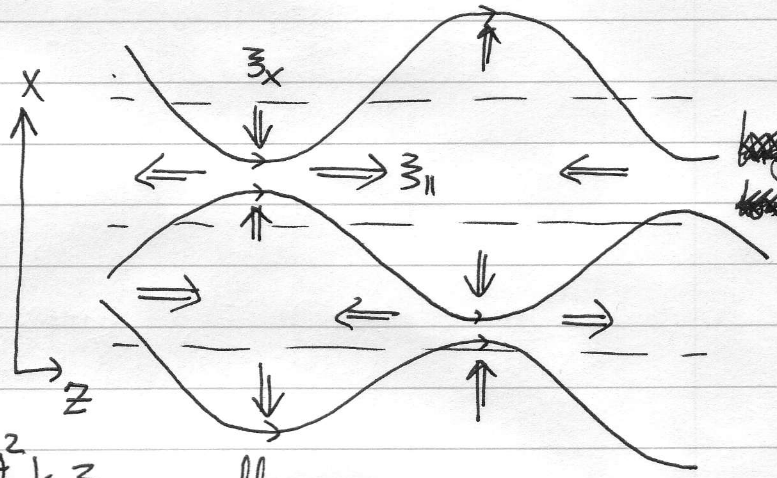
So: $\vec{\xi}_\perp = \xi_x \hat{x} \parallel \vec{k}_\perp$

$$\xi_\parallel \approx -\frac{k_\perp}{k_\parallel}\xi_x$$

$$\delta\hat{b} = i k_\parallel \xi_x \hat{x}$$

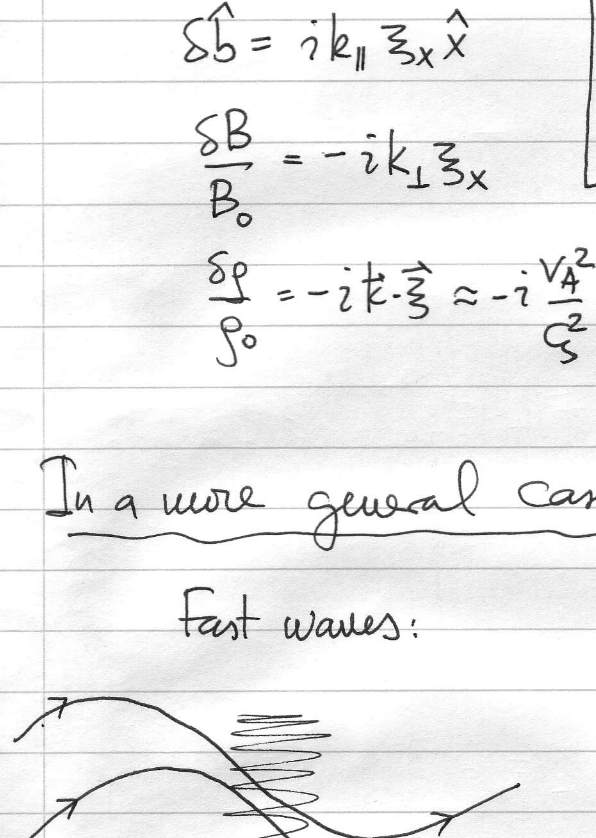
$$\frac{\delta B}{B_0} = -i k_\perp \xi_x$$

$$\frac{\delta\rho}{\rho_0} = -i\vec{k}\cdot\vec{\xi} \approx -i\frac{V_A^2}{c_s^2}k_\perp \xi_x \text{ small}$$



In a more general case, basically

Fast waves:



Slow waves: