§10. MHD Equilibrium (Static)

Suppose $\vec{u}=0$ and we are seeking stationary (equilibrium) solutions. Then

closed set

+ all other forces if any

$$\boxed{-\nabla p + \frac{1}{c}\vec{J}\times\vec{B} = 0}$$ force balance

$$\vec{J} = \frac{c}{4\pi}\nabla\times\vec{B}$$

$$\nabla\cdot\vec{B} = 0$$

These a eqns of static MHD equilibrium

NB: $\vec{B}\cdot\nabla p = 0$ mag. surfaces = surfaces of const pressure

$\vec{J}\cdot\nabla p = 0$ current flows along these surfaces

→ interesting consequence: if field lines are stochastic and fill the volume, then $p=$const.

The standard approach is to

1) work out the equilibrium appropriate to the problem at hand
(usually this involves solving the eqns above in some specific geometry:

e.g. – cylindrical (⟼ pinch solutions)
– axisymmetric (⟼ Grad-Shafranov eq)

2) study the stability of the equilibrium
– this is done using either eigenvalue analysis or, more generally, a technique known as the energy principle

It is often convenient to use Lagrangian MHD for that, taking $\rho_0, p_0, \vec{B}_0$ of the equilibrium and asking if the displacement $\vec{\xi}$ will grow.

These stability problems are very common and very important in AFD - see Dr Ogilvie's lectures. We shall not cover any of this.

3) A (stable) equilibrium in MHD may (and <u>will</u>) support waves. These are of fundamental importance for turbulence and we are about to study them in detail.

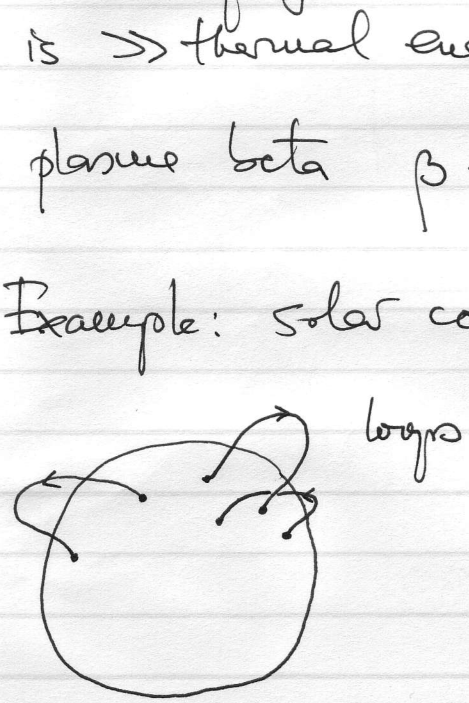
But before we do that, let us consider one very interesting particular case of MHD equilibrium:

<u>Force-free equilibrium</u>.

In some physical situations, magnetic energy is $\gg$ thermal energy (pressure):

plasma beta $\beta = \dfrac{p}{B^2/8\pi} \ll 1$

Example: solar corona: $\beta \sim 1 \ldots 10^{-6}$ $\left(\begin{array}{l} n \sim 10^9\ cm^{-3} \\ T \sim 10^2\ eV \\ B \sim 1 \ldots 10^3\ G \end{array} \right)$ photosphere, loops



In such cases, pressure is unimportant, so we get

$\vec{J} \times \vec{B} = 0$ force-free (Beltrami) fields arb. scalar fn with dim. of scale^{-1}.

This means $\vec{j} \parallel \vec{B} \Rightarrow \boxed{\nabla \times \vec{B} = \alpha(x)\vec{B}}$

(to solve, + $\nabla \cdot \vec{B} = 0$ and boundary conditions)

NB: $\nabla \cdot$ (above eqn):

$$\vec{B} \cdot \nabla \alpha = 0 \Rightarrow \alpha = \text{const on mag. surfaces.}$$

Again, if $\vec{B}$ is chaotic / fills space, $\alpha = \text{const}$ everywhere

NB: $\alpha = 0 \Rightarrow$ potential field (e.g. uniform, dipolar)

Popular particular case: $\alpha = \text{const}$ Linear force-free fields

Then $\nabla \times \Big| \; \nabla \times \vec{B} = \alpha \vec{B}$

$$-\nabla^2 \vec{B} = \alpha \nabla \times \vec{B} = \alpha^2 \vec{B}$$

So $(\nabla^2 + \alpha^2)\vec{B} = 0$ Helmholtz eqn

Lecture 12

7.11.05 In this context there exists a very nifty result:

- What is the mag.-field configuration when plasma relaxes to a state with minimum mag. energy subject to the topological constraint that helicity is conserved? (can't break linkages!)

Action principle: $\delta(\mathcal{E}_M - \lambda H) = 0$

mag. energy — Lagrange multiplier — helicity

$$\delta \int d^3x \left(\frac{B^2}{8\pi} - \lambda \vec{B} \cdot \vec{A} \right) = 0$$

Let's work it out.

$$\delta \mathcal{E}_M = \int d^3x \, \frac{2\vec{B}\cdot\delta\vec{B}}{8\pi} = \frac{1}{4\pi}\int d^3x \, \vec{B}\cdot(\nabla\times\delta\vec{A}) =$$

$$= \frac{1}{4\pi}\int d^3x \left[-\nabla\cdot(\vec{B}\times\delta\vec{A}) + (\nabla\times\vec{B})\cdot\delta\vec{A}\right] =$$

$$= -\frac{1}{4\pi}\cancel{\int_{\partial V} d\vec{S}\cdot(\vec{B}\times\delta\vec{A})} + \frac{1}{4\pi}\int d^3x \, \delta\vec{A}\cdot(\nabla\times\vec{B})$$

$$\delta H = \int d^3x \, (\vec{B}\cdot\delta\vec{A} + \underbrace{\vec{A}\cdot\delta\vec{B}}_{=\vec{A}\cdot(\nabla\times\delta\vec{A})})$$

$$\vec{A}\cdot(\nabla\times\delta\vec{A}) = -\nabla\cdot(\vec{A}\times\delta\vec{A}) + \overbrace{(\nabla\times\vec{A})}^{=\vec{B}}\cdot\delta\vec{A}$$

$$= -\cancel{\int_{\partial V} d\vec{S}\cdot(\vec{A}\times\delta\vec{A})} + 2\int d^3x \, \delta\vec{A}\cdot\vec{B}$$

Now $\frac{\partial}{\partial t}\delta\vec{B} = \nabla\times(\vec{u}\times\vec{B}) \implies \frac{\partial}{\partial t}\delta\vec{A} = \vec{u}\times\vec{B} = \frac{\partial\vec{\xi}}{\partial t}\times\vec{B}$

$$\implies \delta\vec{A} = \vec{\xi}\times\vec{B}$$

Then
$$\left.\begin{aligned}\vec{B}\times\delta\vec{A} &= B^2\vec{\xi} - \vec{B}\cdot\vec{\xi}\,\vec{B} \\ \vec{A}\times\delta\vec{A} &= \vec{A}\cdot\vec{B}\,\vec{\xi} - \vec{A}\cdot\vec{\xi}\,\vec{B}\end{aligned}\right| \implies$$
surface terms vanish if $\vec{B} \parallel \partial V$, $\vec{\xi} \parallel \partial V$ (fixed surface)

So $\delta(\mathcal{E}_M - \lambda H) = \int d^3x \, \delta\vec{A}\cdot\left[\frac{\nabla\times\vec{B}}{4\pi} - 2\lambda\vec{B}\right] = 0 \quad \forall \delta\vec{A}$

$\nabla\times\vec{B} = 8\pi\lambda\vec{B} = \alpha\vec{B}$ linear force-free field!

NB: $\alpha = \alpha(H)$ depends on the helicity (a given parameter).

This result is known as Woltjer Theorem

(a.k.a. J.B. Taylor relaxation)

So, recipe: 1) solve $\nabla^2\vec{B} + \alpha^2\vec{B} = 0$

2) calculate $H \mapsto$ get $\alpha = \alpha(H)$.