NOTES

§9 The Virial Theorem.

Are there self-confined states of plasma?

Consider the moment of inertia

$$I = \frac{1}{2}\int_V d^3x\, \rho r^2 \quad \text{with fixed volume } V.$$

$$\frac{dI}{dt} = \frac{1}{2}\int d^3x\, r^2 \frac{\partial \rho}{\partial t} = -\frac{1}{2}\int d^3x\, r^2\, \nabla\cdot(\rho\vec{u}) =$$

$$= -\int d^3x \left[\nabla\cdot \frac{\rho\vec{u} r^2}{2} - \rho\vec{u}\cdot\underbrace{\nabla\frac{r^2}{2}}_{=\vec{r}}\right] = -\int_{\partial V} \underbrace{d\vec{S}\cdot\vec{u}\,\frac{\rho r^2}{2}}_{0 \text{ assuming } \vec{u}\perp d\vec{S}} + \int d^3x\, \rho\vec{u}\cdot\vec{r}$$

(no in/out flows)

$$\frac{d^2 I}{dt^2} = \int d^3x \Big(\underbrace{\frac{\partial}{\partial t}\rho\vec{u}}_{= -\nabla\cdot\hat{\mathbb{T}}}\Big)\cdot\vec{r} = -\int d^3x\, (\nabla\cdot\hat{\mathbb{T}})\cdot\vec{r} =$$

$$= -\int d^3x\, [\nabla\cdot(\hat{\mathbb{T}}\cdot\vec{r}) - T_{ij}\,\overbrace{\partial_i r_j}^{=\delta_{ij}}] =$$

$$= -\int_{\partial V} d\vec{S}\cdot\hat{\mathbb{T}}\cdot\vec{r} + \int d^3x\, \mathrm{tr}\,\hat{\mathbb{T}}$$

$$\hat{\mathbb{T}} = \rho\vec{u}\vec{u} + \left(p + \frac{B^2}{8\pi}\right)\mathbb{1} - \frac{\vec{B}\vec{B}}{4\pi}$$

$$\mathrm{tr}\,\hat{\mathbb{T}} = \rho u^2 + 3p + \frac{B^2}{8\pi} = 2\mathcal{E}_{kin} + \overbrace{3(\gamma-1)}^{=2 \text{ for } \gamma=5/3}\mathcal{E}_{th} + \mathcal{E}_M$$

$$\int_{\partial V} d\vec{S}\cdot\hat{\mathbb{T}}\cdot\vec{r} = \cancel{\int d\vec{S}\cdot\vec{u}\vec{u}\rho} + \int d\vec{S}\cdot\vec{r}\left(p + \frac{B^2}{8\pi}\right) - \cancel{\int d\vec{S}\cdot\frac{\vec{B}\vec{B}}{8\pi}}$$

forces on the boundary.

$$\frac{d^2 I}{dt^2} = 2\mathcal{E}_{kin} + 3(\gamma-1)\mathcal{E}_{th} + \mathcal{E}_M - \int_{\partial V} d\vec{S}\cdot\vec{r}\left(p + \frac{B^2}{8\pi}\right)$$

In steady state,

$$\int_{\partial V} d\vec{S}\cdot\hat{\mathbb{T}}\cdot\vec{r} = 2\mathcal{E}_{kin} + 3\overbrace{(\gamma-1)\mathcal{E}_{th}}^{=p} + \mathcal{E}_M > 0$$

So stresses on the boundary are never $\emptyset$ $\Rightarrow$ no self-confinement.

Things become more interesting if gravity is included:

$$\frac{\partial}{\partial t}\rho\vec{u} = \ldots + \rho\vec{g}$$

Now $\rho\vec{g} = -\rho\nabla\Phi \Rightarrow \nabla^2\Phi = -\nabla\cdot\vec{g} = 4\pi G\rho \Rightarrow \rho = +\frac{\nabla^2\Phi}{4\pi G}$

Then
$$\rho\vec{g} = -\frac{1}{4\pi G}(\nabla\Phi)\nabla^2\Phi =$$

$$= -\frac{1}{4\pi G}\nabla\cdot\left[\nabla\Phi\,\nabla\Phi\right] + \frac{1}{4\pi G}\underbrace{(\nabla\Phi)\cdot\nabla\nabla\Phi}_{=\frac{1}{2}\nabla(\nabla\Phi)^2} =$$

$$= -\nabla\cdot\left[\frac{\nabla\Phi\,\nabla\Phi}{4\pi G} - \frac{(\nabla\Phi)^2}{8\pi G}\mathbb{1}\right] = -\nabla\cdot\underbrace{\left[\frac{\vec{g}\vec{g}}{4\pi G} - \frac{g^2}{8\pi G}\mathbb{1}\right]}_{=\hat{\mathbb{T}}_G}$$

$\mathrm{tr}\,\hat{\mathbb{T}}_G = -\frac{g^2}{8\pi G}$, so we get

$$\frac{d^2I}{dt^2} = 2\mathcal{E}_{kin} + 3(\gamma-1)\mathcal{E}_{th} + \mathcal{E}_M + \mathcal{E}_G - \int d\vec{S}\cdot\vec{r}\left(p + \frac{B^2}{8\pi} - \frac{g^2}{8\pi G}\right)$$

$\mathcal{E}_G$ negative

So now in steady state can have

$$2\mathcal{E}_{kin} + 3(\gamma-1)\mathcal{E}_{th} + \mathcal{E}_M + \mathcal{E}_G = 0$$

Confinement can be achieved by gravity.

Note: <u>for a star</u>, $\mathcal{E}_M < |\mathcal{E}_G|$

$$\frac{B^2}{8\pi}\pi R^3 \lesssim \frac{GM^2}{R} \Rightarrow B^2R^4 \lesssim GM^2$$

$$B \lesssim \frac{G^{1/2}M}{R^2} \sim 10^8\,\frac{M/M_\odot}{(R/R_\odot)^2}\ \text{G}.$$

(substituted solar par.)

$$\mathcal{E}_G = +\frac{1}{2}\int d^3x\, \rho\, \Phi = +\frac{1}{2}\int d^3x\, \frac{\nabla^2 \Phi}{4\pi G}\, \Phi = \cancel{+}-\frac{1}{2}\int d^3x\, \frac{|\nabla\Phi|^2}{4\pi G} =$$

$$= -\frac{1}{2}\int d^3x\, \frac{g^2}{4\pi G}$$

$$\mathcal{E}_{total} = \mathcal{E}_{kin} + \mathcal{E}_{th} + \mathcal{E}_M + \mathcal{E}_G$$

$$2\mathcal{E}_{kin} + 3(\gamma-1)\mathcal{E}_{th} + \mathcal{E}_M + \mathcal{E}_G = 0$$

$$\mathcal{E}_M + \mathcal{E}_G = -2\mathcal{E}_{kin} - 3(\gamma-1)\,\mathcal{E}_{th}$$

$$\mathcal{E}_{total} = -\mathcal{E}_{kin} - (3\gamma-4)\,\mathcal{E}_{th} = -\mathcal{E}_{kin} - \mathcal{E}_{th}, \quad \gamma = \frac{5}{3}$$

NB: If star radiates, $\mathcal{E}_{total}$ decreases

$\mathcal{E}_{th}$ increases ($\mathcal{E}_{kin} = 0$)

Star heats up as it cools!

(Kulsrud p. 86)