§8. Force-free Fields.

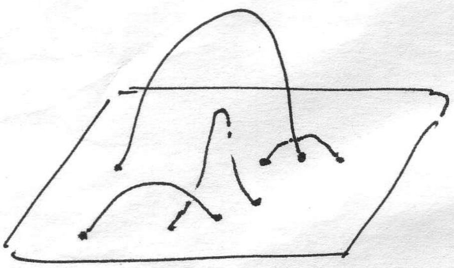
Reminder: in equilibrium, we have

$$\begin{cases} \nabla p = \frac{1}{c}\vec{J}\times\vec{B} & \text{force balance} \\ \vec{J} = \frac{c}{4\pi}\nabla\times\vec{B} & \text{Ampère} \\ \nabla\cdot\vec{B} = 0 & \text{solenoidality} \end{cases}$$

In some physical situations, ~~mag fields~~ magnetic energy is $\gg$ thermal energy:

$$\beta = \frac{p}{B^2/8\pi} \ll 1 \quad \text{plasma beta.}$$

Example: solar corona $\beta \sim 1 \ldots 10^{-6}$ ($n \sim 10^9\ \mathrm{cm}^{-3}$, $T \sim 10^2\ \mathrm{eV}$, $B \sim 1 \ldots 10^3\ \mathrm{G}$ — 1: photosph., 10^3: loops)



In such circumstances, pressure is unimportant, so we get

$$\vec{J}\times\vec{B} = 0 \quad \underline{\text{force-free fields}}.$$

$$\Downarrow$$

$$\vec{J} \parallel \vec{B} \xrightarrow{\text{Ampère}}$$

$$\text{I.}\ \boxed{\nabla\times\vec{B} = \alpha(\vec{x})\,\vec{B}}$$

To this, we must append $\nabla\cdot\vec{B}=0$ and <u>boundary conditions</u>

$\hookrightarrow \vec{B}\cdot\nabla\alpha = 0$ $\quad$ α = const on mag. surfaces.

NB: If $\vec{B}$ is chaotic (fills space), α = const everywhere.

NB: $\alpha = 0$: potential field (uniform, dipolar)

It is usually assumed $\alpha = \text{const}$ (linear force-free fields)

$$\nabla\times \Big|\ \nabla\times\vec B = \alpha \vec B$$

$$\underbrace{\nabla\times[\nabla\times\vec B]}_{-\nabla^2\vec B} = \alpha\,\nabla\times\vec B = \alpha^2\vec B$$

so $\boxed{(\nabla^2 + \alpha^2)\vec B = 0}$ Helmholtz eqn.

- What is the magnetic-field profile when plasma <u>relaxes</u> to a state with minimum mag. energy subject to the constraint that helicity is conserved?

Action principle for this is $\delta(\mathcal{E}_M - \lambda H) = 0$

$$\delta \int d^3x \left(\frac{B^2}{8\pi} - \lambda\,\vec B\cdot\vec A\right) = 0$$

$\uparrow$ Lagrange multiplier.

$$\delta\mathcal{E} = \int d^3x\, 2\frac{\vec B\cdot\delta\vec B}{8\pi} = \frac{1}{4\pi}\int d^3x\, \vec B\cdot(\nabla\times\delta\vec A) =$$

$$= \frac{1}{4\pi}\int d^3x\left[-\nabla\cdot(\vec B\times\delta\vec A) + (\nabla\times\vec B)\cdot\delta\vec A\right] =$$

$$= -\frac{1}{4\pi}\int_{\partial V} d\vec S\cdot(\vec B\times\delta\vec A) + \frac{1}{4\pi}\int_V d^3x\,\delta\vec A\cdot(\nabla\times\vec B)$$

$$\delta H = \int d^3x\,(\vec B\cdot\delta\vec A + \underbrace{\vec A\cdot\delta\vec B}) =$$

$$\vec A\cdot(\nabla\times\delta\vec A) = -\nabla\cdot(\vec A\times\delta\vec A) + \overbrace{(\nabla\times\vec A)}^{\vec B}\cdot\delta\vec A$$

$$= -\int_{\partial V} d\vec S\cdot(\vec A\times\delta\vec A) + 2\int_V d^3x\,\delta\vec A\cdot\vec B$$

Now $\frac{\partial}{\partial t}\delta\vec B = \nabla\times[\vec u\times\vec B] \Rightarrow \frac{\partial}{\partial t}\delta\vec A = \vec u\times\vec B = \frac{\partial\vec\xi}{\partial t}\times\vec B$

$$\delta\vec A = \vec\xi\times\vec B \Rightarrow \begin{array}{l} \vec B\times\delta\vec A = B^2\vec\xi - \vec B\cdot\vec\xi\,\vec B \\ \vec A\times\delta\vec A = \vec A\cdot\vec B\,\vec\xi - \vec A\cdot\vec\xi\,\vec B \end{array} \Bigg| \Rightarrow$$

surface terms vanish if $\vec B \nparallel \partial V$ and $\vec\xi \nparallel \partial V$ (fixed surface)

So, $\delta(\mathcal{E} - \lambda H) = \int_V d^3x\, \delta\vec{A} \cdot \left[\frac{\nabla\times\vec{B}}{4\pi} - 2\lambda\vec{B}\right] = 0 \quad \forall \delta\vec{A}$

$$\nabla\times\vec{B} = 8\pi\lambda\vec{B} = \alpha\vec{B}$$ linear force-free field.

NB: $\alpha = \alpha(H)$ depends on the helicity of the configuration.

This is Woltjer theorem (a.k.a. JB Taylor relaxation)

- Consider cylindrical symmetry: $\frac{\partial}{\partial\theta} = 0, \frac{\partial}{\partial z} = 0$

$\Downarrow$

$\vec{B} = (0, B_\theta, B_z)$ from $\nabla\cdot\vec{B} = 0$

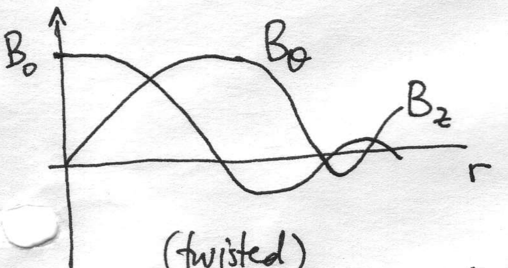
$\hat{z}\cdot \Big|\ \nabla^2\vec{B} + \alpha^2\vec{B} = 0$

$$\frac{d^2}{dr^2}B_z + \frac{1}{r}\frac{d}{dr}B_z + \alpha^2 B_z = 0$$

Solution? Bessel eqn! $\boxed{B_z = B_0 J_0(\alpha r)}$

Now $\alpha B_\theta = (\nabla\times\vec{B})_\theta = -\frac{\partial B_z}{\partial r} = -B_0\alpha J_0'(\alpha r) = \alpha B_0 J_1(\alpha r)$

$\boxed{B_\theta = B_0 J_1(\alpha r)}$



$\nabla\times\vec{B} = \alpha\vec{B} = \alpha\nabla\times\vec{A} \Rightarrow \vec{B} = \alpha\vec{A}$

(twisted)
Helical fields: $H = \int \vec{B}\cdot\vec{A}\, d^3x = \frac{1}{\alpha}\int B^2 d^3x =$

$$= \frac{1}{\alpha}2\pi\cdot L_z B_0^2 \int_0^a dr\, r\left[J_0^2(\alpha r) + J_1^2(\alpha r)\right] =$$

$$= \frac{\pi B_0^2 a^2 L_z}{\alpha^2}\left[J_0(\alpha a) + 2J_1^2(\alpha a) + J_2^2(\alpha a) - \frac{2}{\alpha a}J_1(\alpha a)J_2(\alpha a)\right]$$

$\frac{B_0^2 V}{\alpha^2}$

$\Downarrow$

$\alpha = \alpha(H)$

Note that $\alpha = \frac{8\pi\mathcal{E}}{H} \sim \frac{1}{l}$