$\boxed{\text{NOTES}}$

§7 MHD Equilibrium.

Suppose everything is stationary and there are no flows ($\vec{u}=0$).

Then $\boxed{-\nabla p + \frac{1}{c}\vec{J}\times\vec{B} = 0}$ + all other forces and $\vec{J} = \frac{c}{4\pi}\nabla\times\vec{B}$

pressure Lorentz $\nabla\cdot\vec{B}=0$

- NB: $\vec{B}\cdot\nabla p = 0$ mag. surfaces = surfaces of const pressure

  $\vec{J}\cdot\nabla p = 0$ current flows along these surfaces

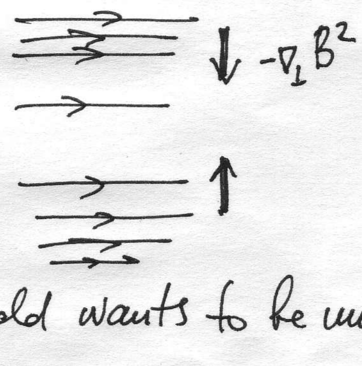
  if field lines stochastic (fill the volume), then $p=$ const.

- Magnetic forces:

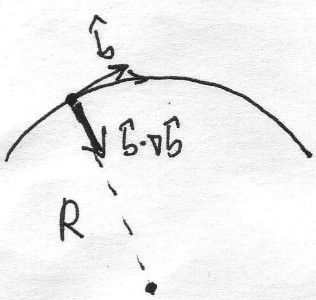
$$\frac{1}{c}\vec{J}\times\vec{B} = \frac{(\nabla\times\vec{B})\times\vec{B}}{4\pi} = \underbrace{\frac{\vec{B}\cdot\nabla\vec{B}}{4\pi}}_{\text{mag. tension}} - \underbrace{\nabla\frac{B^2}{8\pi}}_{\text{mag. pressure}} =$$

$$B\hat{b}\cdot\nabla B\hat{b} = B^2\hat{b}\cdot\nabla\hat{b} + \hat{b}\nabla_\parallel\frac{B^2}{2}$$

$$= \underbrace{\frac{B^2}{4\pi}\hat{b}\cdot\nabla\hat{b}}_{\text{curvature force}} - \underbrace{\nabla_\perp\frac{B^2}{8\pi}}_{\text{mag. pressure}}$$



field wants to be uniform



$\hat{b}\cdot\nabla\hat{b} \perp \hat{b}$

field lines want to straighten

Magnetic field is hard to compress and to bend.

• Equilibrium in cylindrical geometry (r, θ, z)

Assume cyl. symmetry: $\frac{\partial}{\partial\theta} = 0 \quad \frac{\partial}{\partial z} = 0$ (1D)

1) $\nabla\cdot\vec{B} = 0 \Rightarrow \frac{1}{r}\frac{\partial}{\partial r} r B_r = 0 \Rightarrow r B_r = \text{const} \Rightarrow B_r = 0$

2) $\nabla\times\vec{B} = \frac{4\pi}{c}\vec{j} \Rightarrow j_r = 0$

$$j_\theta = -\frac{c}{4\pi}\frac{\partial B_z}{\partial r}$$

$$j_z = \frac{c}{4\pi}\frac{1}{r}\frac{\partial}{\partial r} r B_\theta$$

3) $\nabla p = \frac{1}{c}\vec{j}\times\vec{B} \Rightarrow$

$$\frac{\partial p}{\partial r} = \frac{1}{c}\left(j_\theta B_z - j_z B_\theta\right) = \frac{1}{4\pi}\left(-B_z\frac{\partial B_z}{\partial r} - \frac{B_\theta}{r}\frac{\partial}{\partial r} r B_\theta\right)$$

$$= -\frac{\partial}{\partial r}\frac{B_z^2}{8\pi} - \frac{B_\theta^2}{4\pi r} - \frac{\partial}{\partial r}\frac{B_\theta^2}{8\pi}$$

So $\boxed{\frac{\partial}{\partial r}\left(p + \frac{B^2}{8\pi}\right) = -\frac{B_\theta^2}{4\pi r}}$ – [general screw pinch]

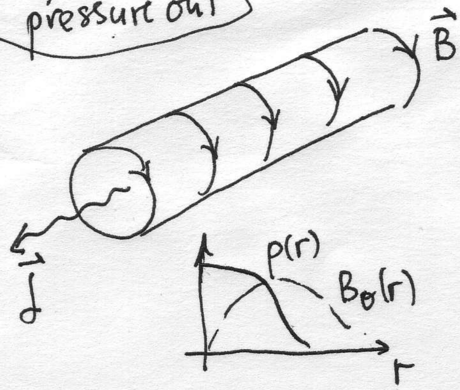
mag. pressure — curvature force.

z pinch: $B_z = 0$, $j_\theta = 0$ current $j_z = \frac{c}{4\pi}\frac{1}{r}\frac{\partial}{\partial r} r B_\theta$

$$B_\theta = \frac{4\pi}{c}\frac{1}{r}\int_0^r dr' r' j_z(r')$$

$$\frac{\partial p}{\partial r} = -\frac{1}{c} j_z B_\theta = -\frac{4\pi}{c^2}\frac{j_z(r)}{r}\int_0^r dr' r' j_z(r')$$

pressure out — pinch in



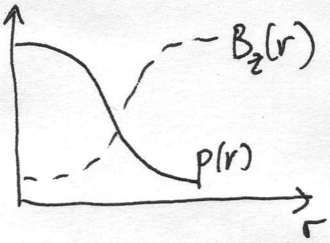
you can think of the pinch effect as coming from attraction of like currents or from the curvature force of mag. loops.

NB: experiments at IC, Sandia

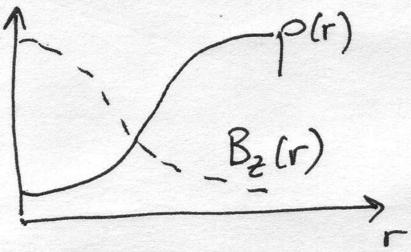
θ pinch. $B_\theta = 0$, $j_z = 0$ current $j_\theta = -\frac{c}{4\pi}\frac{\partial B_z}{\partial r}$

$$\frac{\partial}{\partial r}\left(p + \frac{B_z^2}{8\pi}\right) = \cancel{\text{const}}\ 0$$

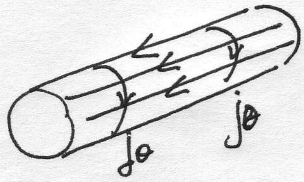
pressure confined:

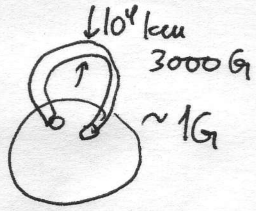


flux confined:



This is what happens with flux tubes in the sun: flux emergence, sunspots
(buoyant)





(2D): Assume axial symmetry only $\frac{\partial}{\partial \theta} = 0$.
Derive Grad-Shafranov equation (ES1) #9