Generally, $H_{\substack{\text{tube}\\ i}} = \Phi_{\substack{\text{tube}\\ i}}\ \Phi_{\substack{\text{through}\\ \text{hole in}\\ \text{tube}}} = \Phi_i \sum_j \Phi_j N_{ij}$

N_{ij}: # times tube j passes through tube i

Total helicity:

$$\boxed{H = \sum_{ij} \Phi_i \Phi_j N_{ij}}$$

counts the linkages.

Its conservation is related to flux conservation: to unlink flux tubes, need to break field lines!

NOTES!

§6. Conservation Laws in MHD.

Lecture 6 9.02.05

NB: drop dis. terms!

① $$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0$$

② $$\rho\frac{d\vec{u}}{dt} = -\nabla p + \frac{(\nabla\times\vec{B})\times\vec{B}}{4\pi} + \nabla\cdot\hat{\Pi}$$

③ $$\frac{1}{\gamma-1}\frac{dp}{dt} + \frac{\gamma p}{\gamma-1}\nabla\cdot\vec{u} + \nabla\cdot\vec{q} = \hat{\Pi}:\nabla\vec{u} + \eta\frac{|\nabla\times\vec{B}|^2}{4\pi}$$

($\nabla\cdot\vec{q}$: heat flux; $\hat{\Pi}:\nabla\vec{u}$: viscous heating; $\eta\frac{|\nabla\times\vec{B}|^2}{4\pi}$: Ohmic heating)

$\gamma = \frac{5}{3}$

$p = nT = \frac{\rho T}{m_i}$

$\left[\ \vec{q} = -\varkappa\nabla T \quad \varkappa\text{: thermal diffusivity}\ \right]$

[This comes from the interspecies heat exchange terms

$Q_i + Q_e = (\vec{u}_i - \vec{u}_e)\cdot\vec{R}_e$

$\vec{R}_e = m_e n_e \nu_{ei}(\vec{u}_i - \vec{u}_e)$

$\vec{u}_i - \vec{u}_e = \frac{1}{en_e}\vec{J}$]

④ $$\frac{d\vec{B}}{dt} = \vec{B}\cdot\nabla\vec{u} - \vec{B}\,\nabla\cdot\vec{u} + \eta\nabla^2\vec{B}$$

<u>Conservation of mass:</u> ① already in cons. form

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{\Gamma} = 0, \quad \vec{\Gamma} = \rho\vec{u}$$

$$\frac{\partial}{\partial t}\int_V d^3x\,\rho = -\int_{\partial V} d\vec{S}\cdot\vec{\Gamma}$$ — mass flux through boundaries.

Conservation of momentum: Using ① and ②,

$$\frac{\partial}{\partial t}(\rho\vec{u}) = \vec{u}\frac{\partial\rho}{\partial t} + \rho\frac{\partial\vec{u}}{\partial t} = \underbrace{-\vec{u}\,\nabla\cdot(\rho\vec{u}) - \rho\vec{u}\cdot\nabla\vec{u}}_{=-\nabla\cdot(\rho\vec{u}\vec{u})} - \nabla p + \underbrace{\frac{(\nabla\times\vec{B})\times\vec{B}}{4\pi}}_{=\nabla\cdot\left[-\frac{B^2}{8\pi}\mathbb{1} + \frac{\vec{B}\vec{B}}{4\pi}\right]} + \nabla\cdot\hat{\Pi} =$$

$$= -\nabla\cdot\Big[\underbrace{\rho\vec{u}\vec{u}}_{\text{Reynolds stress}} + (\underbrace{p}_{\text{pressure}} + \underbrace{\tfrac{B^2}{8\pi}}_{\text{mag. pressure}})\mathbb{1} - \frac{\vec{B}\vec{B}}{4\pi} - \underbrace{\hat{\Pi}}_{\text{viscous stress}}\Big] \equiv -\nabla\cdot\hat{\mathbb{T}}$$

maxwell stress

So $\displaystyle\frac{\partial}{\partial t}\int_V d^3x\,(\rho\vec{u}) = -\int_V d^3x\,\nabla\cdot\hat{\mathbb{T}} = -\int_{\partial V} d\vec{S}\cdot\hat{\mathbb{T}}$. Momentum flux through boundaries.

Conservation of energy : Using ①, ②, ③, ④

$$\frac{\partial}{\partial t}\Big(\underbrace{\tfrac{1}{2}\rho u^2}_{\text{kinetic}} + \underbrace{\frac{p}{\gamma-1}}_{\text{thermal}} + \underbrace{\frac{B^2}{8\pi}}_{\text{magnetic}}\Big) = \frac{u^2}{2}\underbrace{\frac{\partial\rho}{\partial t}}_{①} + \rho\vec{u}\cdot\underbrace{\frac{\partial\vec{u}}{\partial t}}_{②} + \frac{1}{\gamma-1}\underbrace{\frac{\partial p}{\partial t}}_{③} + \frac{1}{4\pi}\vec{B}\cdot\underbrace{\frac{\partial\vec{B}}{\partial t}}_{④} =$$

$$= -\frac{u^2}{2}\nabla\cdot(\rho\vec{u}) - \rho\vec{u}\cdot(\vec{u}\cdot\nabla\vec{u}) - \vec{u}\cdot\nabla p + \vec{u}\cdot\frac{(\nabla\times\vec{B})\times\vec{B}}{4\pi} + (\nabla\cdot\hat{\Pi})\cdot\vec{u}$$

$$-\frac{1}{\gamma-1}\vec{u}\cdot\nabla p - \frac{\gamma}{\gamma-1}p\nabla\cdot\vec{u} - \nabla\cdot\vec{q} + \hat{\Pi}:\nabla\vec{u} + \eta\frac{|\nabla\times\vec{B}|^2}{4\pi}$$

$$+\frac{1}{4\pi}\vec{B}\cdot\left[\nabla\times\left(\vec{u}\times\vec{B} - \eta\nabla\times\vec{B}\right)\right] =$$

$$= -\underbrace{\nabla\cdot\tfrac{1}{2}\rho\vec{u}u^2}_{\text{kinetic en. flux}} - \underbrace{\frac{\gamma}{\gamma-1}\nabla\cdot(p\vec{u})}_{\text{th. en. flux}} - \underbrace{\nabla\cdot\vec{q}}_{\text{heat flux}} + \underbrace{\nabla\cdot(\hat{\Pi}\cdot\vec{u})}_{\text{visc. flux}} + \cancel{\eta\nabla\cdot[\vec{B}\times(\nabla\times\vec{B})]}$$

$$+\frac{1}{4\pi}\Big\{\underbrace{-(\nabla\times\vec{B})\cdot(\vec{u}\times\vec{B}) + \vec{B}\cdot[\nabla\times(\vec{u}\times\vec{B})]}_{=-\nabla\cdot[\vec{B}\times(\vec{u}\times\vec{B})]} + \eta|\nabla\times\vec{B}|^2 - \eta\vec{B}\cdot[\nabla\times(\nabla\times\vec{B})]\Big\} =$$

$$-\nabla\cdot[\vec{B}\times(\vec{u}\times\vec{B})] = -\nabla\cdot\left[\vec{B}\times(-c\vec{E} + \eta\nabla\times\vec{B})\right] = -c\nabla\cdot(\vec{E}\times\vec{B}) - \cancel{\eta\nabla\cdot[\vec{B}\times(\nabla\times\vec{B})]}$$

So,

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2+\frac{p}{\gamma-1}+\frac{B^2}{8\pi}\right)=-\nabla\cdot\left[\frac{1}{2}\rho u^2\vec u+\frac{\gamma}{\gamma-1}p\vec u+\vec q-\hat\Pi\cdot\vec u+\underbrace{\frac{c}{4\pi}\vec E\times\vec B}_{\text{Poynting vector}}\right]=-\nabla\cdot\vec{\mathcal F}.$$

$$\frac{\partial}{\partial t}\int_V d^3x\left(\frac{1}{2}\rho u^2+\frac{p}{\gamma-1}+\frac{B^2}{8\pi}\right)=-\int_{\partial V}d\vec S\cdot\vec{\mathcal F}=$$

$$=-\underbrace{\int_{\partial V}d\vec S\cdot\vec u\left(\frac{1}{2}\rho u^2+\frac{p}{\gamma-1}+\frac{B^2}{8\pi}\right)}_{\text{in/outflow of energy}}-\underbrace{\int_{\partial V}d\vec S\cdot\vec u\,p}_{\text{work done by pressure}}-\underbrace{\int_{\partial V}d\vec S\cdot\hat{\Pi}_M\cdot\vec u}_{\text{work done by mag. forces}}-$$

$$+\underbrace{\int_{\partial V}d\vec S\cdot\hat\Pi\cdot\vec u}_{\text{work done by visc. forces}}-\underbrace{\int_{\partial V}d\vec S\cdot\vec q}_{\text{in/out flow of heat}}$$

[Conservation of cross-helicity.

$\cancel{\nu=\eta=0:}$

$$\frac{\partial}{\partial t}\vec u\cdot\vec B=\vec u\cdot\frac{\partial\vec B}{\partial t}+\vec B\cdot\frac{\partial\vec u}{\partial t}=\vec u\cdot\left(-\vec u\cdot\nabla\vec B+\vec B\cdot\nabla\vec u-\vec B\,\nabla\cdot\vec u\right)+\vec B\cdot\frac{\partial\vec u}{\partial t}$$

$$=-\vec u\cdot(\nabla\vec B)\cdot\vec u+\vec B\cdot\nabla\frac{u^2}{2}-\vec u\cdot\vec B\,\nabla\cdot\vec u+\underbrace{\vec B\cdot\frac{\partial\vec u}{\partial t}}_{\frac{1}{\rho}\vec B\cdot\frac{\partial}{\partial t}\rho\vec u-\vec B\cdot\vec u\frac{1}{\rho}\frac{\partial\rho}{\partial t}}=$$

$$=-\vec u\cdot(\nabla\vec B)\cdot\vec u+\nabla\cdot\vec B\frac{u^2}{2}-\cancel{\vec u\cdot\vec B\,\nabla\cdot\vec u}+\frac{1}{\rho}\vec B\cdot\underbrace{\frac{\partial}{\partial t}\rho\vec u}_{=-\nabla\cdot(\rho\vec u\vec u)-\nabla\left(p+\frac{B^2}{8\pi}\right)+\nabla\cdot\frac{\vec B\vec B}{4\pi}}+\vec B\cdot\vec u\,\vec u\cdot\frac{\nabla\rho}{\rho}+\cancel{\vec B\cdot\vec u\,\nabla\cdot\vec u}$$

$$=\cancel{-\vec u\cdot(\nabla\vec B)\cdot\vec u}+\nabla\cdot\vec B\frac{u^2}{2}$$

$$-\nabla\cdot(\vec u\vec u\cdot\vec B)+\underbrace{\rho\vec u\cdot\left(\nabla\frac{\vec B}{\rho}\right)\cdot\vec u}_{=\cancel{\vec u\cdot(\nabla\vec B)\cdot\vec u}-\cancel{\vec B\cdot\vec u\,\vec u\cdot\frac{\nabla\rho}{\rho}}}-\nabla\cdot\left(p\frac{\vec B}{\rho}\right)+\underbrace{p\nabla\cdot\frac{\vec B}{\rho}}_{=-p\vec B\cdot\frac{\nabla\rho}{\rho^2}}-\cancel{\nabla\cdot\frac{B^2}{8\pi}\vec B}+\cancel{\nabla\cdot\frac{\vec B B^2}{8\pi}}+\cancel{\vec B\cdot\vec u\,\vec u\cdot\frac{\nabla\rho}{\rho}}$$

$$=-\nabla\cdot\left[\vec u\vec u\cdot\vec B-\vec B\frac{u^2}{2}+p\frac{\vec B}{\rho}\right]-p\vec B\cdot\frac{\nabla\rho}{\rho^2}\quad\ldots$$

only conserved if incompressible.]

I'd like to show you what happens in the <u>incompressible case</u>:

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\frac{\nabla p}{\rho} + \frac{(\nabla\times\vec{B})\times\vec{B}}{4\pi\rho} + \nu\nabla^2\vec{u}$$

Let $\frac{\vec{B}}{\sqrt{4\pi\rho}} \to \vec{B}$, $\frac{p}{\rho} + \frac{B^2}{8\pi} = \tilde{p}$. Then

$$\begin{cases} \frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\nabla\tilde{p} + \vec{B}\cdot\nabla\vec{B} + \nu\nabla^2\vec{u} + \vec{f}, \quad \nabla\cdot\vec{u} = 0 \\ \frac{\partial \vec{B}}{\partial t} + \vec{u}\cdot\nabla\vec{B} = \vec{B}\cdot\nabla\vec{u} + \eta\nabla^2\vec{B} \end{cases}$$

<u>Kinetic energy</u>:

$$\left[\frac{\partial}{\partial t}\frac{u^2}{2} = -\nabla\cdot\left[\vec{u}\frac{u^2}{2} + \tilde{p}\vec{u}\right] + (\vec{B}\cdot\nabla\vec{B})\cdot\vec{u} + \nu\vec{u}\cdot\nabla^2\vec{u} + \vec{f}\cdot\vec{u}\right].$$

$$\frac{\partial}{\partial t}\int_V d^3x\,\frac{u^2}{2} = \underbrace{\int d^3x(\vec{B}\cdot\nabla\vec{B})\cdot\vec{u}} - \nu\int d^3x\,|\nabla\vec{u}|^2 + \int d^3x\,\vec{f}\cdot\vec{u}$$

$$\int d^3x\left[\nabla\cdot(\vec{B}\vec{B}\cdot\vec{u})\right] - \int d^3x\,\vec{B}\vec{B}:\nabla\vec{u}$$

$$= -\underbrace{\int d^3x\,\vec{B}\cdot(\nabla\vec{u})\cdot\vec{B}}_{\text{exchange with m. field.}} - \underbrace{\nu\int d^3x\,|\nabla\vec{u}|^2}_{\text{Visc. diss.}} + \underbrace{\int d^3x\,\vec{f}\cdot\vec{u}}_{\text{power in}}$$

$$\left[\frac{\partial}{\partial t}\frac{B^2}{2} = -\nabla\cdot\left(\vec{u}\frac{B^2}{2}\right) + \vec{B}\cdot(\nabla\vec{u})\cdot\vec{B} + \eta\vec{B}\cdot(\nabla^2\vec{B})\right]$$

$$\frac{\partial}{\partial t}\int\frac{B^2}{2} = +\underbrace{\int d^3x\,\vec{B}\cdot(\nabla\vec{u})\cdot\vec{B}}_{\text{exchange with velocity}} - \eta\underbrace{\int d^3x\,|\nabla\vec{B}|^2}_{\text{"}j^2\text{" Ohmic dissipation}}$$

So $$\frac{\partial}{\partial t}\int d^3x\left(\frac{u^2}{2} + \frac{B^2}{2}\right) = -\nu\int d^3x\,|\nabla\vec{u}|^2 - \eta\int d^3x\,|\nabla\vec{B}|^2 + \int d^3x\,\vec{f}\cdot\vec{u}$$

Conservation of cross-helicity:

$$\frac{\partial}{\partial t}\vec{u}\cdot\vec{B} = \vec{u}\cdot\frac{\partial\vec{B}}{\partial t} + \vec{B}\cdot\frac{\partial\vec{u}}{\partial t} = \vec{u}\cdot\left(-\vec{u}\cdot\nabla\vec{B} + \vec{B}\cdot\nabla\vec{u} + \eta\nabla^2\vec{B}\right) + \vec{B}\cdot\left(-\vec{u}\cdot\nabla\vec{u} - \nabla\tilde{p} + \vec{B}\cdot\nabla\vec{B} + \nu\nabla^2\vec{u}\right)$$

$$= \cancel{-\vec{u}\cdot(\nabla\vec{B})\cdot\vec{u}} + \nabla\cdot\vec{B}\frac{u^2}{2} + \eta(\nabla^2\vec{B})\cdot\vec{u} - \underline{\nabla\cdot\vec{u}\,\vec{u}\cdot\vec{B}} + \cancel{\vec{u}\cdot(\nabla\vec{B})\cdot\vec{u}} - \underline{\nabla\cdot\vec{B}\tilde{p}} +$$

$$+ \nabla\cdot\vec{B}\frac{B^2}{2} + \nu(\nabla^2\vec{u})\cdot\vec{B} =$$

$$= \nabla\cdot\left[\vec{B}\left(\frac{u^2}{2} + \frac{B^2}{2}\right) - \vec{u}\,\vec{u}\cdot\vec{B} - \vec{B}\tilde{p}\right] + \eta(\nabla^2\vec{B})\cdot\vec{u} + \nu(\nabla^2\vec{u})\cdot\vec{B}$$

$$\frac{\partial}{\partial t}\int d^3x\,\vec{u}\cdot\vec{B} = -(\nu+\eta)\int d^3x(\nabla\vec{B}):(\nabla\vec{u})$$

a topological invariant like helicity – reflects conservation of linkages between vortex and flux tubes.