§5. Helicity.

Usual approach when we've got ourselves some new equations is to see what they conserve. We have seen that the induction equation has a local invariant – flux. Now we turn to global invariants (quantities whose volume integrals remain constant – subject to dissipative terms and sources).

The induction equation turns out to possess one invariant that involves m-field only (indifferent to the velocity field) : helicity.

$$\boxed{H = \int_V d^3x\, \vec{B}\cdot\vec{A}}\ , \text{ where } \vec{B} = \nabla\times\vec{A}$$

- Helicity is well defined.

$\vec{A}$ is defined up to gauge transformation $\vec{A} \mapsto \vec{A} + \nabla\chi$

Then
$$H \mapsto H + \int_V \vec{B}\cdot\nabla\chi\, d^3x = H + \int_V \nabla\cdot(\vec{B}\chi)\, d^3x =$$
$$= H + \int_{\partial V} d\vec{S}\cdot(\vec{B}\chi) = H \quad \text{if } \vec{B}\cdot\hat{n} = 0 \text{ on the boundary}$$

- Helicity is conserved.

$$\frac{\partial \vec{B}}{\partial t} = \nabla\times\left(\vec{u}\times\vec{B} - \eta\,\nabla\times\vec{B}\right), \quad \vec{B} = \nabla\times\vec{A}$$

Uncurl:
$$\frac{\partial \vec{A}}{\partial t} = \vec{u}\times\vec{B} - \eta\,\nabla\times\vec{B} + \underbrace{\nabla\chi}_{\text{arb. scalar fn.}}$$

$$\frac{\partial}{\partial t}\vec{A}\cdot\vec{B} = \vec{B}\cdot\left(\cancel{\vec{u}\times\vec{B}} - \eta\,\nabla\times\vec{B} + \nabla\chi\right) + \vec{A}\cdot\left[\nabla\times(\vec{u}\times\vec{B}) - \eta\,\nabla\times\vec{B}\right]$$
$$= \underbrace{\vec{B}\cdot\nabla\chi}_{=\ \nabla\cdot(\vec{B}\chi)} - \eta\,\vec{B}\cdot(\nabla\times\vec{B}) + \vec{A}\cdot\left[\nabla\times(\vec{u}\times\vec{B})\right] - \eta\,\vec{A}\cdot\left[\nabla\times(\nabla\times\vec{B})\right]$$

Now use $\vec{A}\cdot(\nabla\times(\ldots)) = (\ldots)\cdot(\nabla\times\vec{A}) - \nabla\cdot\left[\vec{A}\times(\ldots)\right]$

$$\frac{\partial}{\partial t}\vec{A}\cdot\vec{B} = \nabla\cdot(\vec{B}\chi) - \eta\,\vec{B}\cdot(\nabla\times\vec{B}) + \cancel{(\vec{u}\times\vec{B})\cdot\underbrace{(\nabla\times\vec{A})}_{=\vec{B}}} - \nabla\cdot\Big[\underbrace{\vec{A}\times(\vec{u}\times\vec{B})}_{=\left(\vec{A}\cdot\vec{B}\,\vec{u} - \vec{A}\cdot\vec{u}\,\vec{B}\right)}\Big]$$

$$-\eta\,(\nabla\times\vec{B})\cdot\underbrace{(\nabla\times\vec{A})}_{=\vec{B}} + \eta\,\nabla\cdot\left[\vec{A}\times(\nabla\times\vec{B})\right]$$

$$= \nabla\cdot\left[\vec{B}\chi - \vec{u}\,\vec{A}\cdot\vec{B} + \vec{B}\,\vec{A}\cdot\vec{u} + \eta\,\vec{A}\times(\nabla\times\vec{B})\right] - 2\eta\,\vec{B}\cdot(\nabla\times\vec{B})$$

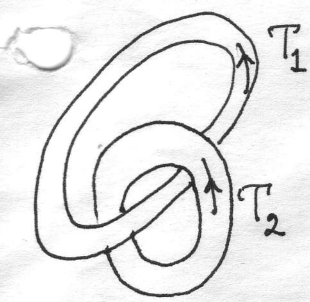
Integrate:

$$\frac{\partial}{\partial t}\int_V d^3x\,\vec{A}\cdot\vec{B} = \underbrace{\int_{\partial V} d\vec{S}\cdot[\ldots]}_{=0} - 2\eta\underbrace{\int_V d^3x\,\vec{B}\cdot(\nabla\times\vec{B})}_{\text{current helicity } (\propto \vec{B}\cdot\vec{J})}$$

So $\boxed{\dfrac{dH}{dt} = -2\eta\int_V d^3x\,\vec{B}\cdot\vec{J}}$, letting $\vec{J} = \nabla\times\vec{B} = \frac{4\pi}{c}\vec{\mathcal{J}}$.

- Helicity is a topological invariant.

Consider two linked flux tubes



$$H_1 = \int_{T_1} d^3x\,\vec{A}\cdot\vec{B} = \int_{T_1} \underbrace{d\vec{l}\cdot d\vec{S}}_{=dl\,\hat{b}\cdot\hat{b}\,dS}\,\vec{A}\cdot\underbrace{\vec{B}}_{=\hat{b}B} =$$

$$= \int_{T_1} \vec{A}\cdot\underbrace{\hat{b}\,dl}_{=d\vec{l}}\;\underbrace{B\,\hat{b}}_{=\vec{B}}\cdot\underbrace{\hat{b}\,dS}_{=d\vec{S}} = \int_{T_1} \vec{A}\cdot d\vec{l}\;\vec{B}\cdot d\vec{S} =$$

$$= \underbrace{\Phi_{\text{through tube}}}_{=\Phi_1}\;\underbrace{\oint\vec{A}\cdot d\vec{l}}_{=\int d\vec{S}'\cdot(\nabla\times\vec{A})} = \Phi_1\Phi_2$$

$$\oint\vec{A}\cdot d\vec{l} = \int d\vec{S}'\cdot(\nabla\times\vec{A}) = \int d\vec{S}'\cdot\vec{B} = \Phi_{\text{through hole}} = \Phi_2$$

$d\vec{S}'$: surface spanning the loop

Generally, $H_{\text{tube } i} = \Phi_{\text{tube } i}\, \Phi_{\substack{\text{through} \\ \text{hole in} \\ \text{tube}}} = \Phi_i \sum_j \Phi_j N_{ij}$

\# times tube j passes through tube i

Total helicity:

$$\boxed{H = \sum_{ij} \Phi_i \Phi_j N_{ij}}$$

counts the ~~bigger~~ linkages.

Its conservation is related to flux conservation: to unlink flux tubes, need to break field lines!

NOTES!

Lecture 6 9.02.05

§6. Conservation Laws in MHD.

NB: drop dis. terms!

① $$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0$$

② $$\rho\frac{d\vec{u}}{dt} = -\nabla p + \frac{(\nabla\times\vec{B})\times\vec{B}}{4\pi} + \nabla\cdot\hat{\Pi}$$

③ $$\frac{1}{\gamma-1}\frac{dp}{dt} + \frac{\gamma}{\gamma-1}p\nabla\cdot\vec{u} + \underbrace{\nabla\cdot\vec{q}}_{\text{heat flux}} = \underbrace{\hat{\Pi}:\nabla\vec{u}}_{\text{viscous heating}} + \underbrace{\eta\frac{|\nabla\times\vec{B}|^2}{4\pi}}_{\text{ohmic heating}}$$

$\gamma = \frac{5}{3}$

$p = nT = \frac{\rho T}{m_i}$

[$\vec{q} = -\varkappa\nabla T$; $\varkappa$: thermal diffusivity]

[This comes from the interspecies heat exchange terms

$Q_i + Q_e = (\vec{u}_i - \vec{u}_e)\cdot\vec{R}_e$

$\vec{R}_e = m_e n_e \nu_{ei}(\vec{u}_i - \vec{u}_e)$

$\vec{u}_i - \vec{u}_e = \frac{1}{en_e}\vec{J}$]

④ $$\frac{d\vec{B}}{dt} = \vec{B}\cdot\nabla\vec{u} - \vec{B}\nabla\cdot\vec{u} + \eta\nabla^2\vec{B}$$

<u>Conservation of mass</u>: ① already in cons. form

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{\Gamma} = 0, \quad \vec{\Gamma} = \rho\vec{u}$$

$$\frac{\partial}{\partial t}\int_V d^3x\,\rho = -\int_{\partial V} d\vec{S}\cdot\vec{\Gamma}$$

– mass flux through boundaries.