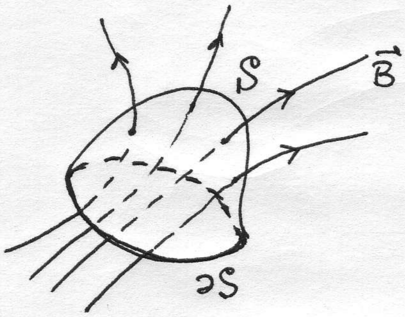
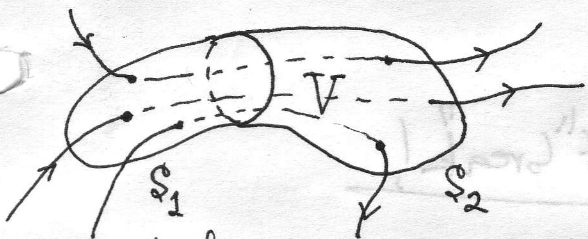
§4. Flux Freezing

Define magnetic flux $\Phi_S = \int_S \vec{B}\cdot d\vec{S}$



Lemma. Φ depends on the loop ∂S but on the choice of the surface spaning the loop (Φ is a well defined quantity)

Pf. Consider two surfaces spaning same loop:



$$\Phi_{S_1} = \int_{S_1} \vec{B}\cdot d\vec{S}$$

$$\Phi_{S_2} = \int_{S_2} \vec{B}\cdot d\vec{S}$$

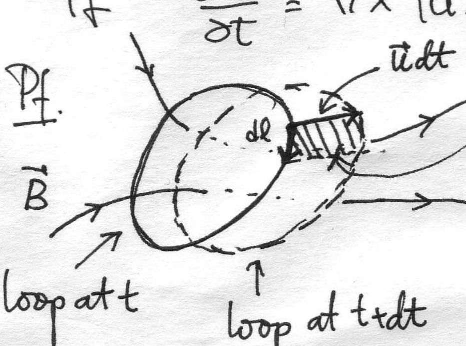
Flux ~~through~~ out of volume V enclosed by $S_1 \cup S_2$:

$$\underbrace{\int_{\partial V} \vec{B}\cdot d\vec{S} = \Phi_{S_2} - \Phi_{S_1} = \int_V \nabla\cdot\vec{B}\, dV}_{\text{Gauss theorem}} = 0 \quad (\text{field solenoidal})$$

QED

Theorem 1. (Alfvén; analog of Kelvin circulation theorem)
Flux through any loop moving with the fluid is conserved if $\frac{\partial \vec{B}}{\partial t} = \nabla\times(\vec{u}\times\vec{B})$ is satisfied

Pf.



$d\vec{S} = -\vec{dl}\times\vec{u}\,dt$ (into)

Let $S^{(t)}$ – surface spaning loop at time t

$S(t+dt) = S(t)$ + ribbon traced by the loop as it moved

$$\Phi(t) = \int_{S(t)} \vec{B}^{(t)}\cdot d\vec{S}$$

$$\Phi(t+dt) = \int_{S(t+dt)} \vec{B}^{(t+dt)}\cdot d\vec{S} = \int_{S(t)} \vec{B}(t+dt)\cdot d\vec{S} + \int_{\text{ribbon}} \vec{B}(t+dt)\cdot \underset{\vec{u}dt\times\vec{dl}}{\underset{\parallel}{d\vec{S}}} =$$

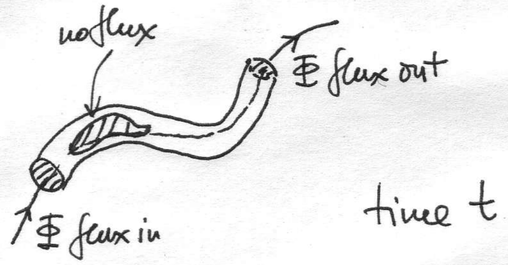
$$= \underbrace{\int_{S(t)} \vec{B}(t)\cdot d\vec{S}}_{\displaystyle \Phi(t)} + dt \int_{S(t)} \frac{\partial \vec{B}(t)}{\partial t}\cdot d\vec{S} + dt \underbrace{\oint_{\partial S(t)} \vec{B}(t)\cdot(\vec{u}\times d\vec{l})}_{\displaystyle -\oint_{\partial S(t)} (\vec{u}\times\vec{B})\cdot d\vec{l} \overset{\text{Stokes}}{=} -\int_{S(t)} [\nabla\times(\vec{u}\times\vec{B})]\cdot d\vec{S}} + O(dt^2)$$

So,

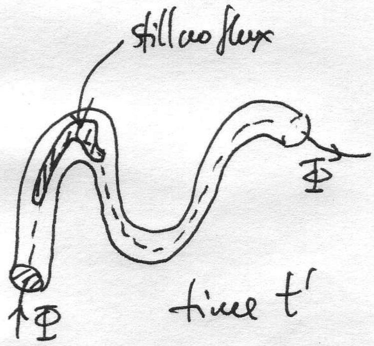
$$\frac{d\Phi}{dt} = \int_{S(t)} \underbrace{\left[\frac{\partial \vec{B}}{\partial t} - \nabla\times(\vec{u}\times\vec{B})\right]}_{\text{ind. eqn. in ideal MHD}}\cdot d\vec{S} = 0 \qquad \text{QED}$$

Corollary. Magnetic field lines are frozen into plasma flow.

Pf. 1 Define flux tube:

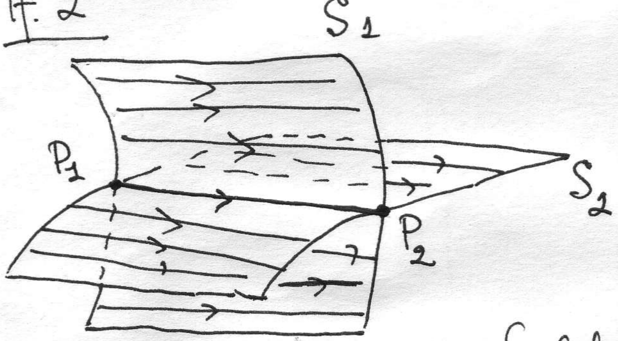


$\Longrightarrow$



Flux in same, flux out same, still no flux through the sides
⇓
field line still goes through.
QED

[Pf. 2



Consider two surfaces made of field lines.
Their intersection is also a field line connecting fluid particles P_1, P_2.

Flux through both surfaces is zero at all times (lines do not cross surfaces) ⟹ ~~lines~~ surfaces remain made up of field lines ⟹ P_1 and P_2 are still connected by a field line. QED.]

This corollary can be recast in a differential form:

<u>Theorem 2.</u> (Lundquist; analog of Helmholtz's 1st law) Fluid elements that lie on a field line initially, will remain on the same field line.

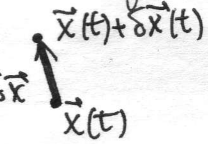
<u>Pf.</u> $$\frac{\partial\vec{B}}{\partial t} = \nabla\times(\vec{u}\times\vec{B}) = (\vec{B}\cdot\nabla)\vec{u} + \cancel{\vec{u}\,\nabla\cdot\vec{B}}^{\,0} - \vec{u}\cdot\nabla\vec{B} - \vec{B}\,\nabla\cdot\vec{u}$$

$$\frac{1}{\rho}\cdot \Big|\ \frac{d\vec{B}}{dt} = \vec{B}\cdot\nabla\vec{u} - \vec{B}\,\nabla\cdot\vec{u}$$

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\vec{u}) = -\vec{u}\cdot\nabla\rho - \rho\nabla\cdot\vec{u} \Rightarrow -\frac{1}{\rho}\frac{d\rho}{dt} = \nabla\cdot\vec{u}$$

$$\frac{1}{\rho}\frac{d\vec{B}}{dt} = \frac{\vec{B}}{\rho}\cdot\nabla\vec{u} + \frac{\vec{B}}{\rho^2}\frac{d\rho}{dt} \Rightarrow \boxed{\frac{d}{dt}\frac{\vec{B}}{\rho} = \frac{\vec{B}}{\rho}\cdot\nabla\vec{u}} \qquad (*)$$

Compare with infinitesimal separation of fluid particles:



$$\frac{d}{dt}\vec{x}(t) = \vec{u}(\vec{x})$$

$$\frac{d}{dt}\vec{x}(t)+\delta\vec{x}(t) = \vec{u}(\vec{x}+\delta\vec{x})$$

So $\frac{d}{dt}\delta\vec{x}(t) = \vec{u}(\vec{x}+\delta\vec{x}) - \vec{u}(\vec{x}) = \delta\vec{x}\cdot\nabla\vec{u}$ same eq.

So, if initially particles are on the same line, $\delta\vec{x} = \alpha\vec{B}/\rho$, this relation will hold forever. QED.

A somewhat more $\overset{\text{(diff. geom.)}}{\text{mathematical}}$ formulation: consider fluid motion as coordinate transformation:

$$\vec{x}(t) = \vec{x}_0 + \int_0^t dt'\,\vec{u}(t',\vec{x}(t',\vec{x}_0)),\ \text{i.e. } \frac{d\vec{x}}{dt} = \vec{u}(t,\vec{x}),\ \vec{x}(0) = \vec{x}_0$$

Then solution of (*) is

$$\boxed{\frac{B_i}{\rho} = \frac{B_j(0)}{\rho(0)}\frac{\partial x_i}{\partial x_{0j}}}$$

<u>Cauchy solution</u> of the induction equation

$\equiv$ field transforms with the flow.

Pf. by substitution:

$$\frac{d}{dt}\frac{B_i}{\rho}(\vec{x},t) = \frac{\partial}{\partial t}\frac{B_i}{\rho}(\vec{x}(t,\vec{x}_0),t) = \frac{\partial}{\partial t}\frac{B_j}{\rho}(\vec{x}_0,0)\frac{\partial x_i(t,\vec{x}_0)}{\partial x_{0j}} =$$

to Lagrangian variables — substitute Cauchy sln

$$= \frac{B_j}{\rho}(\vec{x}_0,0)\frac{\partial}{\partial x_{0j}}\underbrace{\frac{\partial x_i(t,\vec{x}_0)}{\partial t}}_{=\,u_i(\vec{x}(t,\vec{x}_0),t)} =$$

$$= \frac{B_j}{\rho}(\vec{x}_0,0)\frac{\partial}{\partial x_{0j}}u_i(\vec{x}(t,\vec{x}_0),t) = \underbrace{\frac{B_j(\vec{x}_0,0)}{\rho}\frac{\partial x_k(t,\vec{x}_0)}{\partial x_{0j}}}_{=\,\frac{B_k}{\rho}(\vec{x},t)}\frac{\partial u_i(\vec{x},t)}{\partial x_k}$$

again Cauchy

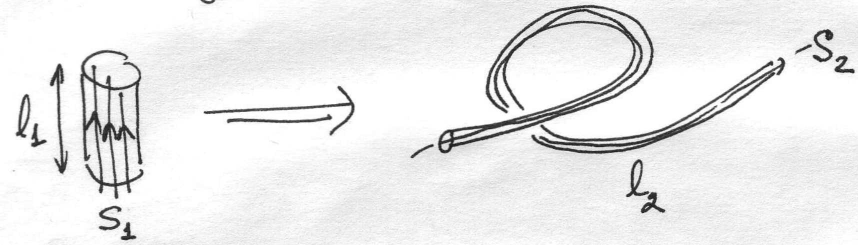
$$= \frac{B_k}{\rho}(\vec{x},t)\frac{\partial u_i(\vec{x},t)}{\partial x_k} \quad \text{QED.}$$

Ex:

EXAMPLES

Field amplification by the flow:

1)



Cons. of flux: $B_1 S_1 = B_2 S_2$

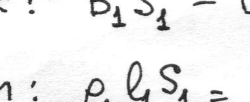
Cons. of mass: $\rho_1 \underbrace{l_1 S_1}_{V_1} = \rho_2 \underbrace{l_2 S_2}_{V_2}$ $\mapsto$ $\dfrac{B_2}{\rho_2 l_2} = \dfrac{B_1}{\rho_1 l_1}$

incompressible $\rho_1 = \rho_2$ $\mapsto$ $B_2 = \dfrac{l_2}{l_1} B_1$

Stretching

2) Uniform collapse - see ES1

3) Shear



$t=0$: $\vec{B} = B_0\hat{z}$, $\vec{u} = Az\hat{x}$ (uniform shear)

$$\frac{\partial \vec{B}}{\partial t} + \vec{u}\cdot\nabla\vec{B} = \vec{B}\cdot\nabla\vec{u} - \underbrace{\vec{B}\,\nabla\cdot\vec{u}}_{0}$$

$$\frac{\partial B_x}{\partial t} + \cancel{Az\frac{\partial B_x}{\partial x}} = AB_0$$

$$\frac{\partial B_z}{\partial t} + \cancel{Az\frac{\partial B_z}{\partial x}} = 0 \mapsto B_z = B_0 = \text{const}$$

$B_x = AB_0 t$ linear growth

also $B_y = 0 = \text{const}$

(also stretched)

Eigenfunctions of ind. eqn.

4) Dynamo Problem: A Quick Intro.

Can the induction equation lead to sustained (exponential) growth of magnetic field?

The problem is unsolved in general, but a lot is known

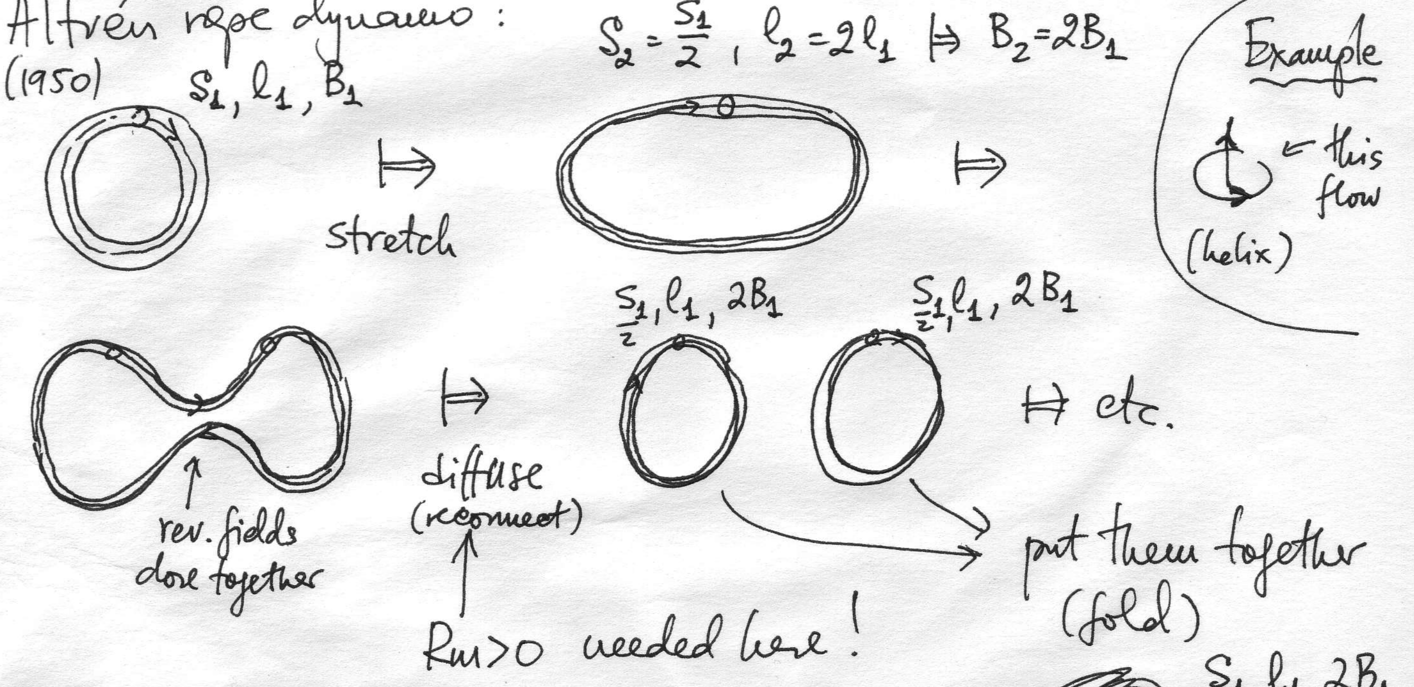
- velocity fields with high degree of symmetry are not dynamos (Antidynamo Theorems - see Proctor's course)
  In particular 2D v. field is not a dynamo - see ES1 (Zeldovich Theorem)
- numerical investigations show that flows with chaotic trajectories tend to be dynamos
  in particular, this is often true for non-steady flows.

Two types of dynamo:

- Slow dynamo. Magnetic field grows exp-lly with

$$\gamma \sim \frac{u}{l} Rm^{-\alpha} \to 0 \quad \text{as } Rm \to \infty$$

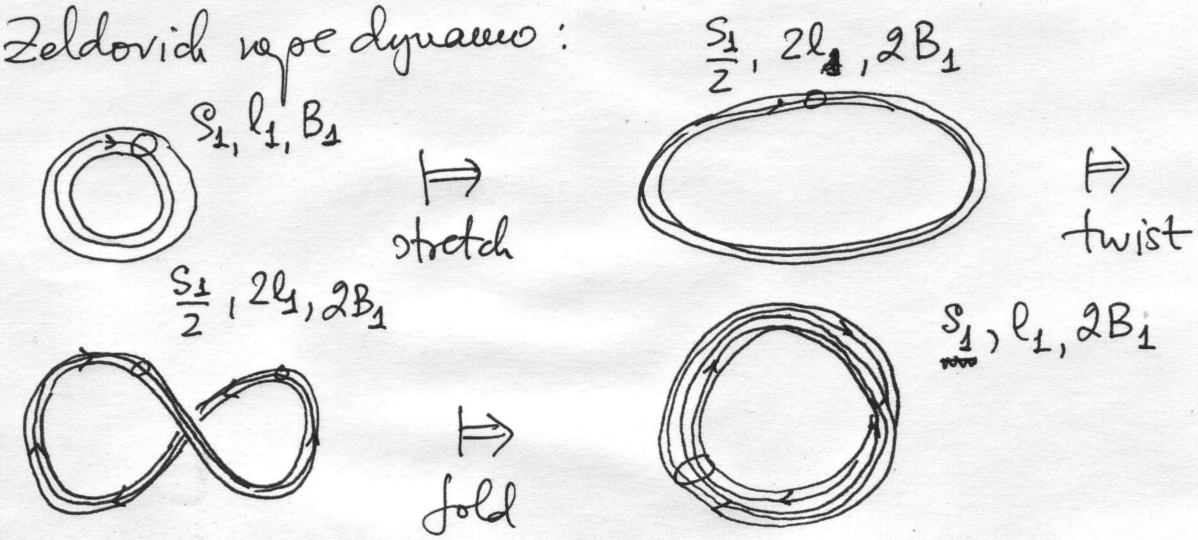
Alfvén rope dynamo:
(1950)



H. Alfvén, Tellus 2, 74 (1950)

• Fast dynamo. $\gamma \sim \frac{u}{l}$ ($\to$ const as $Rm \to \infty$)

Zeldovich rope dynamo:



Zeldovich 1972, Lecture in Cracow

Note: Since flux is conserved, how can we double the field going through ~~an~~ a loop?

Answer: it's conserved through a moving loop, but we can double flux through a fixed loop.

– In turbulent flows, ~~magnetic field growth~~ dynamo is often understood as exp-l growth of the avg. mag. energy (not as an exp-lly growing eigenfunction of the induction equation)