Lecture 2

24.01.05

Org. details
- office hours: after lectures or by appt.
- attendance list
- no lecture on 2.02.05
- first examples class 9.02.05 11:30, venue TBA

NOTES

§2 Kinetic Derivation of MHD Equations and the Limits of MHD Description

NB: summarise eqns from §1

Step. I) Start from particles of species $s\,(=i,e)$

position: $\vec{r}_i(t)$, $\qquad \dot{\vec{r}}_i = \vec{v}_i(t)$

velocity: $\vec{v}_i(t)$, $\qquad \dot{\vec{v}}_i = \dfrac{q_s}{m_s}\left[\vec{E}(t,\vec{r}_i) + \dfrac{1}{c}\vec{v}_i(t)\times\vec{B}(t,\vec{r}_i)\right]$

This information can be assembled into the Klimontovich distribution fn (exact)

$$\mathcal{F}_s(\vec{r},\vec{v},t) = \frac{1}{V}\sum_{i=1}^{N_s}\delta(\vec{r}-\vec{r}_i(t))\,\delta(\vec{v}-\vec{v}_i(t))$$

V: vol. of the system

$$\mathcal{F}_s(\vec{r},\vec{v},t)\,d^3\vec{r}\,d^3\vec{v} = \frac{\#\text{ particles } s \text{ with } \vec{r}_i,\vec{v}_i \text{ in } [\vec{r},\vec{r}+d\vec{r}]\cap[\vec{v},\vec{v}+d\vec{v}]}{\text{system volume}}$$

$$\int \mathcal{F}_s\,d^3\vec{r}\,d^3\vec{v} = \text{mean particle density}$$

Ref: Klimontovich

Klimontovich eqn (exact):

$$\frac{\partial \mathcal{F}_s}{\partial t} + \vec{v}\cdot\nabla\mathcal{F}_s + \frac{q_s}{m_s}\left(\vec{E}+\frac{1}{c}\vec{v}\times\vec{B}\right)\cdot\frac{\partial \mathcal{F}_s}{\partial \vec{v}} = 0 \qquad (1)$$

The fields are found from Maxwell's eqns:

$$\nabla\cdot\vec{E} = 4\pi\rho_e = 4\pi\sum_s q_s\int d^3\vec{v}\, F_s(\vec{r},\vec{v},t)$$

$$\nabla\cdot\vec{B} = 0$$

$$\frac{\partial\vec{B}}{\partial t} + c\nabla\times\vec{E} = 0$$

$$-\frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \nabla\times\vec{B} = \frac{4\pi}{c}\vec{j} = \frac{4\pi}{c}\sum_s q_s\int d^3\vec{v}\,\vec{v}\, F_s(\vec{r},\vec{v},t)$$

Everything here has variation on very small scales in the configuration space: interparticle distances.

Step II: Coarse-grain everything in the configuration space. Formally this means

$$f_s(\vec{r},\vec{v},t) \equiv \langle F_s\rangle = \int d^3\vec{r}'d^3\vec{v}'\,\underbrace{G(\vec{r}',\vec{v}',t)}_{\text{smoothing fn of some width}}\, F_s(\vec{r}-\vec{r}',\vec{v}-\vec{v}',t)$$

Similarly, coarse-grain the fields

$$\vec{E} = \langle\vec{E}\rangle + \delta\vec{E}\ , \quad \vec{B} = \langle\vec{B}\rangle + \delta\vec{B}$$

Since Maxwell's eqns are linear, coarse-grained fields satisfy Maxwell's eqns with coarse-grained distr. fn. used to calculate coarse-grained charge and current distributions.

Similarly, fluctuating fields ... with δF

~~[illegible]~~

From Klimontovich eqn ①, get

$$\frac{\partial f_s}{\partial t} + \vec{v}\cdot\nabla f_s + \frac{q_s}{m_s}\left(\langle\vec{E}\rangle + \frac{1}{c}\vec{v}\times\langle\vec{B}\rangle\right)\cdot\frac{\partial f}{\partial\vec{v}} =$$

$$= \underbrace{-\left\langle \frac{q_s}{m_s}\left(\delta\vec{E} + \frac{1}{c}\vec{v}\times\delta\vec{B}\right)\cdot\frac{\partial\,\delta F_s}{\partial\vec{v}}\right\rangle}_{\text{collisions}} \equiv \sum_{s'}\hat{C}_{ss'}[f_s, f_{s'}]$$

In principle, there is a closure problem: we don't know how to determine δF_s in terms of f_s.

A closure scheme (BBGKY) leads to an approximation

get

$$② \quad \frac{\partial f_s}{\partial t} + \vec{v}\cdot\nabla f_s + \frac{q_s}{m_s}\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right)\cdot\frac{\partial f_s}{\partial\vec{v}} = \sum_{s'}\hat{C}_{ss'}[f_s, f_{s'}]$$

omit $\langle\ldots\rangle$

collision operator (Landau Lenard-Balescu)

– Vlasov-Landau equation, kinetic eqn. (Boltzmann equation for plasma)

++ Maxwell's equations with ρ_e and $\vec{J}$ based on f_s.

Step Ⅲ: Now take velocity-space moments of ②.
(so description in $\vec{r}$ space quantities)

Density $n_s(\vec{r},t) = \int d^3\vec{v}\, f_s(\vec{r},\vec{v},t)$ satisfies (exact)

$$③ \quad \frac{\partial n_s}{\partial t} + \nabla\cdot(n_s\vec{u}) = 0, \text{ where}$$

mean velocity $n_s\vec{u}_s(\vec{r},t) = \int d^3\vec{v}\,\vec{v}\, f_s(\vec{r},\vec{v},t)$ satisfies

$$m_s n_s\left(\frac{\partial\vec{u}_s}{\partial t} + \vec{u}_s\cdot\nabla\vec{u}_s\right) = -\nabla\cdot\hat{P}_s + q_s n_s\left(\vec{E} + \frac{\vec{u}_s\times\vec{B}}{c}\right) + \vec{R}_s$$

inter because collisions between same conserve ~~total~~ momentum $\longrightarrow$ interspecies collisions (friction)

Closure valid provided we assume that

$$\frac{1}{n\lambda_D^3} \ll 1 \qquad \text{where } \frac{1}{\lambda_D^2} = \sum_s \frac{4\pi q_s^2 n_s}{T_s} = \sum_s \frac{\omega_{ps}^2}{v_{ths}^2}$$

Debye length

... where $\hat{P}_s = m_s \int d^3\vec{v}\, (\vec{v}-\vec{u}_s)(\vec{v}-\vec{u}_s)\, f_s$

— pressure tensor for species s.

Define $p_s = \frac{1}{3} \mathrm{Tr}\, \hat{P}_s$ and $\hat{P}_s = p_s \mathbb{1} - \hat{\Pi}_s$

scalar pressure

viscous stress

So, we get

$$(4) \quad m_s n_s \left(\frac{\partial \vec{u}_s}{\partial t} + \vec{u}_s \cdot \nabla \vec{u}_s \right) = -\nabla p_s + \nabla \cdot \hat{\Pi}_s + q_s n_s \left(\vec{E} + \frac{\vec{u}_s \times \vec{B}}{c} \right) + \vec{R}_s$$

Again closure problem: must know p_s, $\hat{\Pi}_s$.

Taking the second moment of (2), get

Ohmic heating in here

$$(5) \quad \frac{\partial p_s}{\partial t} + \vec{u}_s \cdot \nabla p_s + \frac{5}{3} p_s \nabla \cdot \vec{u}_s - \frac{2}{3} \hat{\Pi}_s : \nabla \vec{u}_s + \frac{2}{3} \nabla \cdot \vec{q}_s = \frac{2}{3} Q_s$$

interspecies heat exchange

where heat flux $\vec{q}_s = \int d^3\vec{v}\, \frac{m_s |\vec{v}-\vec{u}_s|^2}{2} (\vec{v}-\vec{u}_s)\, f_s$

Note: temperature $T_s \doteq \frac{p_s}{n_s} = \frac{2}{3} \int d^3\vec{v}\, \frac{m_s |\vec{v}-\vec{u}_s|^2}{2} f_s$

This can go on forever:

$\vec{q}_s$ in terms of 4th moments etc.

Measures must be taken to stop this madness!

STEP IV : Assume

$\nabla \sim \frac{1}{\ell}$, $\ell \gg \lambda_{mfp}$ - mean free path

Then $\quad \frac{\partial}{\partial t} \sim \frac{1}{\tau}$, $\underbrace{\frac{\tau}{\tau_c}}_{\text{coll. time}} \sim \frac{\ell}{u}\frac{v_{th}}{\lambda_{mfp}} \sim \frac{v_{th}}{u}\frac{\ell}{\lambda_{mfp}} \gg 1$, $v_{th} = \sqrt{\frac{T}{m}}$

In this (collisional) approximation, plasma is in local thermodynamic equilibrium and the distribution fn is, to lowest order in $\frac{\lambda_{mfp}}{\ell}$, a Maxwellian:

$$f_s(\vec{r},\vec{v},t) = \left[\frac{m_s}{2\pi T_s(\vec{r},t)}\right]^{3/2} n_s(\vec{r},t)\, e^{-\frac{m_s[\vec{v}-\vec{u}_s(t,\vec{r})]^2}{2T_s(\vec{r},t)}}$$

This can be shown to solve $\sum_{s'} C_{ss'}[f_s, f_{s'}] = 0$.

Then to lowest order in $\frac{\lambda_{mfp}}{\ell}$, we have

— isotropic pressure, $\hat{P}_s = \mathbb{1} p_s$, $\hat{\Pi}_s = 0$

— no heat flow $\vec{q}_s = 0$

— no momentum or heat exchange between species

$\vec{u}_e = \vec{u}_i$, $T_e = T_i$, $\vec{R}_s = 0$, $Q_s = 0$

To get collisional effects, go to next order and calculate everything.

Refs: Braginskii, Helander & Sigmar

The result is equs ③, ④, ⑤ with

$\hat{\Pi}_s, \vec{q}_s, \vec{R}_s, Q_s$ = some expressions in terms of n_s, $\vec{u}_s$, p_s and transport coefficients that depend on the collision frequencies/mean free path.

for lecture 3

Revision of last time:

Klimontovich distr. function,

$$① \quad F_s = \frac{1}{V}\sum_{i=1}^{N_s} \delta(\vec{r}-\vec{r}_i(t))\,\delta(\vec{v}-\vec{v}_i(t)) \quad \text{exact}$$

+ Maxwell's equations for field varying on interparticle distances

$$\Downarrow$$

Assume $\frac{1}{n\lambda_D^3} \ll 1$, coarse grain, get

$f_s(\vec{r},\vec{v},t)$ and $\vec{E}$, $\vec{B}$ varying on macroscopic distances.

$$② \quad \frac{\partial f_s}{\partial t} + \vec{v}\cdot\frac{\partial f_s}{\partial \vec{r}} + \frac{q_s}{m_s}\left(\vec{E}+\frac{1}{c}\vec{v}\times\vec{B}\right)\cdot\frac{\partial f_s}{\partial \vec{v}} = \underbrace{\sum_{s'} C_{ss'}[f_s, f_{s'}]}_{\text{coll. operator}}$$

Take moments:

$$③ \quad \frac{\partial n_s}{\partial t} + \nabla\cdot(n_s\vec{u}_s) = 0$$

$$④ \quad m_s n_s\left(\frac{\partial \vec{u}_s}{\partial t} + \vec{u}_s\cdot\nabla\vec{u}_s\right) = \underbrace{-\nabla p_s}_{\text{pressure}} + \underbrace{\nabla\cdot\hat{\Pi}_s}_{\text{visc stress}} + \underbrace{q_s n_s\left(\vec{E}+\frac{\vec{u}_s\times\vec{B}}{c}\right)}_{\text{Lorentz}} + \underbrace{\vec{R}_s}_{\text{interspecies friction}}$$

++ pressure equation ($p_s = n_s T$)

$$\Downarrow$$

Assume $\frac{\lambda_{mfp}}{\ell} \ll 1$ (collisional) $\Rightarrow$ local thermodynamical equilibrium to lowest order (f_s Maxwellian), isotropic pressure, $\hat{\Pi}_s = 0$, no heat flux, no exchange of heat/mom. between species ($\vec{u}_i = \vec{u}_e$, $T_i = T_e$)

To next order $O\left(\frac{\lambda_{mfp}}{\ell}\right)$, calculate viscous stress, interspecies friction, heat flux, heat exchange in terms of n_s, $\vec{u}_s$, p_s and transport coefficients. Ref: Braginskii

See revision of last time p.10 en regard.

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STEP V : Introduce MHD variables:

electrons do not carry mass { mass density $\rho = n_i m_i \; (+ \underbrace{n_e m_e}_{\text{small}})$

fluid velocity $\vec{u} = \vec{u}_i$ (NB: $\rho\vec{u} \approx n_i m_i \vec{u}_i + \underbrace{n_e m_e \vec{u}_e}_{\text{small}}$)

they do carry charge { current density $\vec{J} = en(\vec{u}_i - \vec{u}_e)$ $\quad q_i = e$

charge density $\rho_e = e(n_i - n_e)$ $\quad q_e = -e$, $n_e = n_i$

pressure $p = p_i + p_e$ etc.

$$③ \Rightarrow \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0 \quad \text{continuity eqn} \quad ⑥$$

NB: $\vec{R}_e + \vec{R}_i = 0$

$$④_i + ④_e \Rightarrow \rho\frac{d\vec{u}}{dt} = \underbrace{-\nabla p + \nabla\cdot\hat{\Pi}}_{\text{hold on.}} + \underbrace{\rho_e\vec{E}}_{\text{small when } u \ll c} + \frac{1}{c}\vec{J}\times\vec{B} \quad ⑦$$

$\frac{v}{c} \ll 1$ In the non-relat. approx, we showed in lecture 1 that the field eqns are

$$\vec{J} = \frac{c}{4\pi}\nabla\times\vec{B} \quad \text{Ampère's law}$$

$$\frac{\partial\vec{B}}{\partial t} = -c\nabla\times\vec{E} \quad \text{Faraday's law} \quad ⑧$$

We had to bring in Ohm's law in order to find $\vec{E}$ in terms of $\vec{B}$ and $\vec{J}$ and close Eq. ⑧.

In this kinetic derivation, we can derive Ohm's law from the electron momentum equation.

Target: $\vec{E} + \frac{1}{c}\vec{u}\times\vec{B} = \frac{1}{\sigma}\vec{J}$

$$\vec{u}_i - \vec{u}_e = \frac{\vec{j}}{en}$$

NB: for pressure, we use

$$Q_i + Q_e = (\vec{u}_i - \vec{u}_e)\cdot\vec{R}_e \quad \text{(can be shown)}$$

$$m_e n_e \nu_{ei} |\vec{u}_i - \vec{u}_e|^2 = \overbrace{\frac{m_e \nu_{ei}}{e^2 n}}^{=\frac{1}{\sigma}} j^2 = \frac{c^2}{4\pi\sigma}\frac{4\pi}{c^2} j^2 = \eta \frac{4\pi}{c^2} j^2$$

$$= \eta \frac{|\nabla\times\vec{B}|^2}{4\pi}$$

So

$$\frac{dp}{dt} + \underset{\gamma}{\underset{\shortparallel}{\frac{5}{3}}} p\nabla\cdot\vec{u} - \underset{(\gamma-1)}{\underset{\shortparallel}{\frac{2}{3}}} \hat{\Pi}:\nabla\vec{u} + \frac{2}{3}\nabla\cdot\vec{q}_s = \frac{2}{3}\eta\frac{j^2}{4\pi}$$

$$\hat{\Pi}:\nabla\vec{u} = \rho\nu\left[\nabla\vec{u} + (\nabla\vec{u})^T - \frac{2}{3}\nabla\cdot\vec{u}\,\mathbb{1}\right]:\nabla\vec{u} =$$

$$= \rho\nu\left[\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j} + \underbrace{\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}} - \frac{2}{3}|\nabla\cdot\vec{u}|^2\right] =$$

$$\frac{\partial}{\partial x_j} u_i \frac{\partial u_j}{\partial x_i} - u_i \frac{\partial^2 u_j}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j} u_i \frac{\partial u_j}{\partial x_i} - \frac{\partial}{\partial x_i} u_i \frac{\partial u_j}{\partial x_j} + |\nabla\cdot\vec{u}|^2$$

$$= \overset{=\mu}{\rho\nu}\left[|\nabla\vec{u}|^2 + \frac{1}{3}|\nabla\cdot\vec{u}|^2 + \nabla\cdot\left(\vec{u}\cdot\nabla\vec{u} - \nabla\cdot(\vec{u}\nabla\cdot\vec{u})\right)\right]$$

In ④$_e$, use $\vec{u}_e = \vec{u}_i - \frac{\vec{J}}{en} = \vec{u} - \frac{\vec{J}}{en}$ and get

$$\vec{E} + \frac{\vec{u}\times\vec{B}}{c} = \frac{\vec{J}\times\vec{B}}{cen} - \frac{\nabla p_e}{en} + \frac{\nabla\cdot\hat{\Pi}_e}{en} + \frac{\vec{R}_e}{en} - \frac{m_e}{e}\left(\frac{\partial\vec{u}_e}{\partial t} + \vec{u}_e\cdot\nabla\vec{u}_e\right)$$

Hall thermoelectric resistive electron inertia

$\beta = \frac{p}{B^2/8\pi}$ $\frac{1}{\beta}\frac{\rho_i}{l}$ $\frac{\rho_i}{l}$ $\frac{\lambda_{mfp}}{\rho_i}\left(\frac{\rho_i}{l}\right)^2$ $\frac{1}{\beta}\frac{\rho_i}{l}\sqrt{\frac{m_e}{m_i}}\frac{\rho_i}{\lambda_{mfp}}$ $\sqrt{\frac{m_e}{m_i}}\frac{\rho_i}{l}$

$\sqrt{\frac{m_e}{m_i}}\frac{\rho_i}{\lambda_{mfp}} = \frac{\rho_e}{\lambda_{mfp}} = \frac{\nu_{ei}}{\Omega_e}$

Electron-ion friction term: technically speaking, need a kinetic calculation, but roughly get

$$\vec{R}_e = m_e n_e \underset{\substack{\uparrow \\ \text{e-i coll. freq.}}}{\nu_{ei}} (\vec{u}_i - \vec{u}_e) = \frac{m_e}{e}\nu_{ei}\vec{J}$$

[From kinetics, we learn $\nu_{ei} \sim 10\,\frac{n_e}{T_e^{3/2}}\,\frac{1}{\text{sec}}$]

Assume a subsidiary ordering:

1) $\nabla \sim \frac{1}{l}$, $l \gg \rho_i = \frac{v_{thi}}{\Omega_i}$ ion gyro/cyclotron/Larmor radius

where $\Omega_i = \frac{q_i B}{m_i c}$ ion gyro/cyclotron freq.

NB: $\frac{\rho_e}{\rho_i} = \frac{v_{the}}{v_{thi}}\frac{\Omega_i}{\Omega_e} = \underbrace{\sqrt{\frac{T_e}{T_i}}}_{1 \text{ in collisional plasma}}\sqrt{\frac{m_e}{m_i}} \ll 1$

Then $\tau\Omega_i \sim \frac{l}{u}\Omega_i \sim \frac{l}{v_{thi}}\frac{v_{thi}}{u}\Omega_i \sim \frac{l}{\rho_i}\frac{v_{thi}}{u} \gg 1$

2) Also assume $u \sim v_{thi}$ (same order!), $\beta \lesssim 1$

3) Also assume $\rho_e \gg \lambda_{mfp}$ (or $\nu_{ei} \gg \Omega_e$) [$\Rightarrow \rho_i \gg \lambda_{mfp}$]

[NB: $l \gg \lambda_{mfp} \Rightarrow T_e \sim T_i$ - assumed before]

ES1 #2

Estimates:

$$\frac{\left|\frac{m_e}{e}\left(\frac{\partial \vec{u}_e}{\partial t}+\vec{u}_e\cdot\nabla\vec{u}_e\right)\right|}{\left|\frac{1}{c}\vec{u}\times\vec{B}\right|}\sim\frac{m_e u_e c}{e\tau u B}\sim\frac{u_e}{u}\frac{1}{\tau\Omega_e}\sim\frac{u_e}{v_{the}}\frac{v_{the}}{l\Omega_e}\sim$$

$$\sim\frac{u_e}{v_{the}}\frac{\rho_e}{l}\sim\frac{u_e}{v_{the}}\sqrt{\frac{m_e}{m_i}}\frac{\rho_i}{l}\sqrt{\frac{T_e}{T_i}}$$

NB $\frac{\rho_e}{\rho_i}=\sqrt{\frac{T_e}{T_i}}\sqrt{\frac{m_e}{m_i}}$

$$\frac{\left|\frac{\nabla\cdot\hat{\Pi}_e}{en}\right|}{\left|\frac{1}{c}\vec{u}\times\vec{B}\right|}\sim\frac{\frac{1}{l^2}\mu_e u_e c}{enuB}\sim\frac{u_e}{u}\frac{c m_e \cancel{n} v_{the}\lambda_{mfpe}}{l^2 eB\cancel{n}}\sim\frac{u_e}{u}\frac{v_{the}\lambda_{mfpe}}{l^2\Omega_e}\sim$$

$$\sim\frac{u_e}{v_{the}}\frac{v_{thi}}{u_e}\sqrt{\frac{T_e}{T_i}}\sqrt{\frac{m_i}{m_e}}\frac{\rho_e}{l}\frac{\lambda_{mfpe}}{l}\sim\frac{u_e}{v_{the}}\frac{v_{thi}}{u}\sqrt{\frac{T_e}{T_i}}\frac{\lambda_{mfpe}}{l}\frac{\rho_i}{l}$$

$$\frac{\left|\frac{\nabla p_e}{en}\right|}{\left|\frac{1}{c}\vec{u}\times\vec{B}\right|}\sim\frac{\cancel{n}T_e c}{le\cancel{n}uB}\sim\frac{c m_e v_{the}^2}{eBlu}\sim\frac{v_{the}}{u}\frac{\rho_e}{l}\sim\frac{v_{thi}}{u}\sqrt{\frac{T_e}{T_i}}\frac{\rho_i}{l}$$

$$\frac{\left|\frac{\vec{j}\times\vec{B}}{cen}\right|}{\left|\frac{1}{c}\vec{u}\times\vec{B}\right|}\sim\frac{\cancel{c}B^{\cancel{2}}\cdot c}{l\cancel{c}enu\cancel{B}}\sim\frac{cB}{lenu}\sim\frac{B^2 c m_i v_{thi}^2}{B m_i v_{thi}^2\, lenu}\sim\frac{B^2}{p_i}\frac{\rho_i}{l}\frac{v_{thi}}{u}\sim\frac{1}{\beta}\frac{v_{thi}}{u}\frac{\rho_i}{l}$$

$$\frac{\left|\frac{\vec{R}_e}{en}\right|}{\left|\frac{1}{c}\vec{u}\times\vec{B}\right|}\sim\frac{c m_e\nu_{ei}j}{e^2 nuB}\sim\frac{c m_e\nu_{ei}c\cancel{B}}{e^2 nul\cancel{B}}\sim\frac{c^2 m_e m_i B^2 v_{thi}^2\nu_{ei}}{e^2B^2\, m_i n v_{thi}^2\, ul}\sim\frac{B^2}{p_i}\frac{v_{thi}\nu_{ei}}{\Omega_e\Omega_i l}\frac{v_{thi}}{u}$$

$$\sim\frac{1}{\beta}\frac{v_{thi}}{u}\frac{\rho_i}{l}\frac{\nu_{ei}}{\Omega_e}\sim\frac{1}{\beta}\frac{v_{thi}}{u}\frac{\rho_i}{l}\frac{\rho_e}{\lambda_{mfpe}}\sim\frac{1}{\beta}\frac{v_{thi}}{u}\frac{\rho_i}{l}\sqrt{\frac{m_e}{m_i}}\frac{\rho_i}{\lambda_{mfpe}}\frac{T_i^2}{T_e^2}$$

$$\nu_{ee}:\nu_{ei}:\nu_{ii}:\nu_{ie}\sim 1:1:\left(\frac{T_e}{T_i}\right)^{3/2}\sqrt{\frac{m_e}{m_i}}:\frac{m_e}{m_i}\Rightarrow\frac{\lambda_{mfpe}}{\lambda_{mfpi}}\sim\frac{v_{the}}{v_{thi}}\frac{\nu_{ii}}{\nu_{ei}}\sim\frac{T_e^2}{T_i^2}$$

With these assumptions, all terms except resistive drop out and we get

$$\vec{E} + \frac{\vec{u}\times\vec{B}}{c} = \frac{m_e \nu_{ei}}{e^2 n}\vec{J} \equiv \frac{1}{\sigma}\vec{J}$$

– our old Ohm's law.

Note that we also need $\rho_i \gg \lambda_{mfp}$ to get the old isotropic expression for the viscous stress:

$$\hat{\Pi} = \rho\nu\left[\nabla\vec{u} + (\nabla\vec{u})^T - \frac{2}{3}\nabla\cdot\vec{u}\,\mathbb{1}\right]$$

where kin. viscosity $\nu \sim v_{th_i}\lambda_{mfp}$

Summary of assumptions:

$l \gg \rho_i \gg \rho_e \gg \lambda_{mfp}$	collisional, particles unmagnetised.
$u \sim v_{th_i} \ll c$	non-relativistic

Extensions of MHD:

- if particles are magnetised, $\rho_i \lesssim \lambda_{mfp}$,
  - viscous stress becomes anisotropic
  - Ohm's law much more complicated

  } Braginskii theory...

- if $\frac{1}{\beta}\rho_i \gtrsim l \gg \lambda_{mfp} \gg \rho_e$ (β small or ρ_i big)
  - Hall term important $\Rightarrow$ Electron MHD
- if neutrals are present, 3-fluid theory

(for low ionisations, e-i fluid drifts wrt neutrals $\Rightarrow$ ambipolar diffusion)

Note that we also need $l \gg \lambda_D = \frac{v_{th}}{\omega_p} = \sqrt{\frac{T}{4\pi e^2 n}}$ same for e, i

But $$\frac{\rho_i^2}{\lambda_D^2} \sim \frac{\omega_{pi}^2}{\Omega_i^2} \sim \frac{4\pi \cancel{e^2} n\, m_i^{\cancel{2}} c^2}{\cancel{m_i} \cancel{e^2} B^2} \sim \frac{4\pi \rho_i c^2}{B^2} \sim$$

$$\sim \frac{\rho_i v_{th}^2}{B^2}\frac{c^2}{v_{th}^2} \sim \beta\frac{c^2}{v_{th}^2} \gg 1$$ if β not too small

$$\frac{\rho_e^2}{\lambda_D^2} \sim \frac{4\pi e^2 n m_e^2 c^2}{m_i e^2 B^2} \sim \frac{m_e n c^2}{B^2} \sim \frac{m_e}{m_i}\beta\frac{c^2}{v_{th}^2} \gg 1$$

if $\frac{v_{th}^2}{c^2} \ll \frac{m_i}{m_e}$