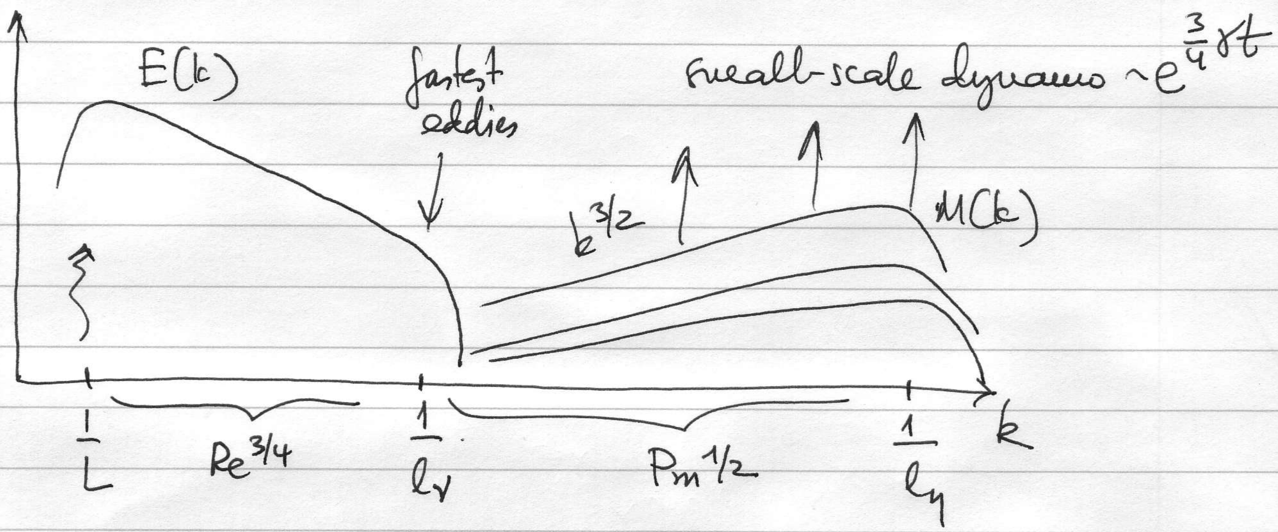
§24. Nonlinear Small-Scale Dynamo and Isotropic MHD Turbulence (overview)

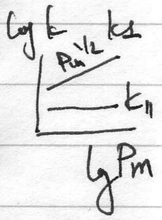


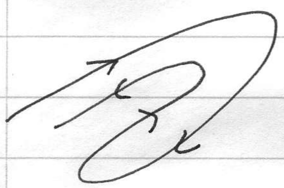
So we have learned that

1) Magnetic energy grows exp-lly at $\frac{3}{4}\gamma \sim \frac{\delta u_{l_\nu}}{l_\nu}$

2) Fields have a $k^{3/2}$ spectrum at $k_\eta \gg k \gg k_\nu$ and the bulk of the growing energy is sitting at k_η

3) What is the structure of the fields? Folded:





$k^{3/2}$ is the spectrum of direction reversals

so $k_\perp \sim k_\eta$, but $k_\parallel \sim k_\nu$ (scale of the flow)

I have not demonstrated this rigorously, but it is possible to do that – see papers posted on course log.

This is all true for the kinematic regime.
At some point, the energy will grow enough for the fields to be dynamically important.
When does that happen?

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\nabla p + \underline{\underline{\vec{B}\cdot\nabla\vec{B}}} + \nu\nabla^2\vec{u}$$

back reaction term.

$\vec{B}\cdot\nabla\vec{B} \sim k_\parallel B^2$ — does not know about direction reversals!

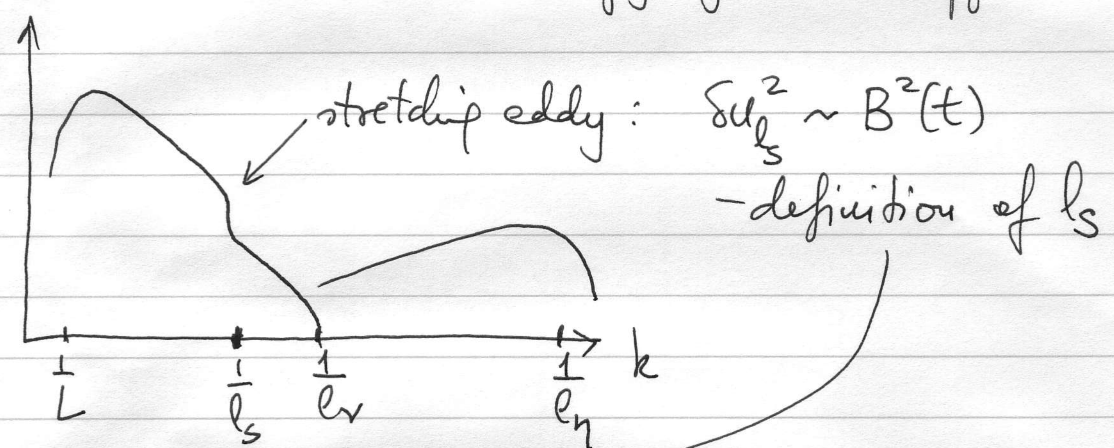
Becomes important when

$$\vec{B}\cdot\nabla\vec{B} \sim \underset{k_\nu}{k_\parallel} B^2 \sim \vec{u}\cdot\nabla\vec{u} \sim \delta u_{\ell_\nu}^2 k_\nu$$

$$B^2 \sim \delta u_{\ell_\nu}^2$$

What then? Well, we don't really know, but here is a physical argument

$B^2 \sim \delta u_{\ell_\nu}^2$ — eddies at l_ν are suppressed, but eddies at $l > l_\nu$ still stretching (more energetic, although slower) until energy grows to suppress them as well.



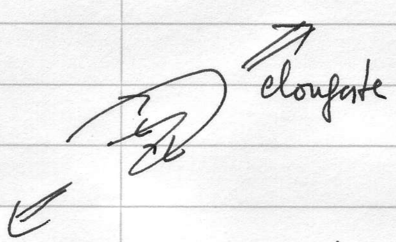
Then $\frac{d}{dt}B^2 \sim \frac{\delta u_{\ell_s}}{l_s} B^2 \sim \frac{\delta u_{\ell_s}}{l_s} \delta u_{\ell_s}^2 \sim \frac{\delta u_{\ell_s}^3}{l_s} \sim \epsilon$

Thus, $B^2 \sim \epsilon t$ — linear-in-time growth ("nonlinear dynamo")

$$\delta u_{l_s}^2 \sim (\epsilon l_s)^{2/3} \sim B^2 \sim \epsilon t$$

Field scales: $l_\parallel \sim l_s(t) \sim \sqrt{\epsilon}\, t^{3/2}$ grows

$$l_\perp \sim \left(\frac{\eta}{\delta u_{l_s}/l_s}\right)^{1/2} \sim \sqrt{\eta t} \text{ grows (slower!)}$$



(so the field is getting more anisotropic!)

Eventually, $l_s \sim L$ after $t \sim \dfrac{L^{2/3}}{\epsilon^{1/3}} \sim \dfrac{L}{\delta u_L}$ outer-eddy time

so $l_\parallel \sim L$

and $l_\perp \sim \sqrt{\dfrac{\eta L}{\delta u_L}} \sim L\, R_m^{-1/2}$

Note that $\dfrac{l_\perp^{(\mathrm{nlin})}}{l_\perp^{(\mathrm{kin})}} \sim \dfrac{L\, R_m^{-1/2}}{L\, Re^{-3/4} P_m^{-1/2}} \sim Re^{1/4}$

This is still within the subviscous range if

$l_\perp^{(\mathrm{nlin})} \ll l_\nu$, i.e. $Re^{1/4} \ll P_m^{1/2}$ or $\boxed{P_m \gg Re^{1/2}}$

— the "true meaning" of "Large P_m"

Very stringent, numerically!

Magnetic energy at this point is $B^2 \sim \delta u_L^2$.

Dynamo should saturate.

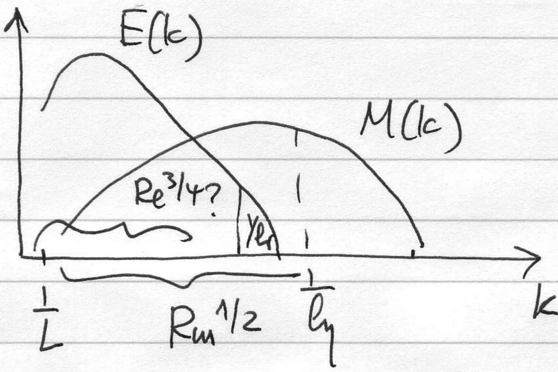
The precise mechanism of this saturation is not known, but note that:

1) The reason fields can back react is that folds have large-scale "parallel" coherence

2) One can construct a model that shows that saturation can occur by partial anisotropization of the velocity gradients wrt the fold direction — see papers posted on the web page.

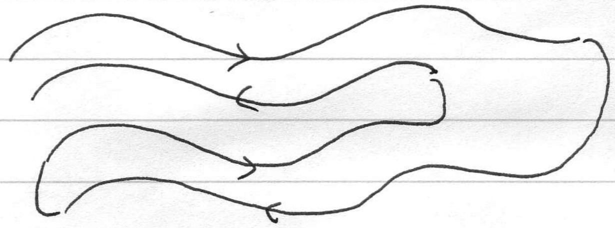
3) The field scale is still small: $l_\perp \sim L\, R_m^{-1/2}$.

- So, the spectra of isotropic MHD turbulence look like this:



Numerics seem to confirm this, but suffer from insufficient resolution.

- Motions in the inertial range: Alfvén waves, but not of the usual kind – they propagate along folds, not the external field.



Can derive them via linear theory for tensor $B^i B^j$.

$$\omega^2(\vec{k}) = \underbrace{\langle B^2 \rangle}_{\substack{\text{averaged}\\ \text{over small}\\ \text{scales.}}} \vec{k}\vec{k} : \underbrace{\hat{b}\hat{b}}_{\text{large-scale object}}$$

- How does this differ from the standard picture?

– a theory similar to that for aniso. turbulence, but assume $\delta B_\perp$ plays the role of the imposed field.

However

– assumption that m. energy is at L probably wrong

– locality of interactions in k space probably wrong.