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§23. Small-Scale Dynamo: Spectrum of M. Energy

So, we have understood (hopefully) why the small-scale dynamo might work. But, at least on the level of formalism, we did this for a linear velocity field constant in time [qualitatively, our results have wider applicability].

Now let us consider the case when the velocity field of the viscous eddies is random (in time): this corresponds to

- turbulence (visc. eddies decorrelate in time τ_ν)
- chaotic deterministic flows (Lagrangian trajectories are chaotic so the Lagr. corr. time is finite)

In principle, the same formalism applies, but now FTLE's $\frac{\zeta_i(t)}{2t}$ have some distribution. If find this distribution, we can average wrt it and find the evolution of $\overline{B^2}(t)$. However, it is usually v. hard or even impossible to calculate the distribution of FTLE's for a general velocity field.

There exists, however a very useful solvable model due to Kazantsev (1967) and Kraichnan (1968): velocity that is a Gaussian white noise:

$$\boxed{\langle u^i(t,\vec{x})\, u^j(t',\vec{x}')\rangle = \delta(t-t')\, \varkappa^{ij}(\vec{x}-\vec{x}')}$$

Kazantsev Model

where $\varkappa^{ij}(\vec{y})$ is the correlation tensor that can be specified subject to symmetry constraints (isotropy e.g.) and incompressibility.

Since we are working with linear velocity fields, all we need is, in fact, the distribution of velocity gradients:

$$\left\langle \frac{\partial u^i}{\partial x^m}(t,\vec{x})\, \frac{\partial u^j}{\partial x'^n}(t',\vec{x}')\right\rangle = \langle \sigma^i_m(t)\, \sigma^j_n(t')\rangle = \delta(t-t')\, \varkappa^{ij}_{mn},$$

where
$$\varkappa^{ij}_{mn} = -\left.\frac{\partial^2 \varkappa^{ij}(\vec{y})}{\partial y^m \partial y^n}\right|_{\vec{y}=0} \equiv \varkappa_2\, T^{ij}_{mn}$$
dim-less tensor

parameter with dimensions of inverse time, $\varkappa_2 \sim \frac{1}{\tau_\nu}$ ← turnover rate of the visc. eddies

For an isotropic, incompressible field, the general form of T^{ij}_{mn} is

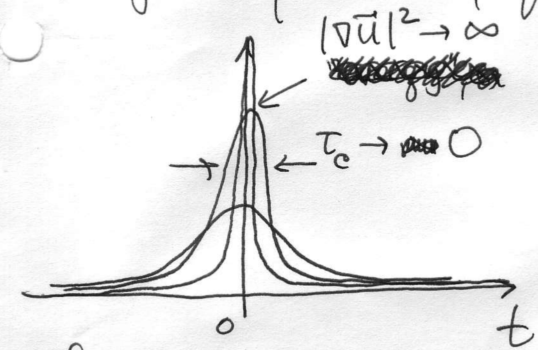
$$T^{ij}_{mn} = \delta^{ij}\delta_{mn} + a\left(\delta^i_m \delta^j_n + \delta^i_n \delta^j_m\right)$$

subject to symmetries wrt $i \leftrightarrow j$, $m \leftrightarrow n$, and $T^{ij}_{in} = 0$ (incompressibility)

$$\Rightarrow \quad a = -\frac{1}{d+1}$$

"Physical" interpretation:

the Kazantsev field is a result of the following limiting procedure:



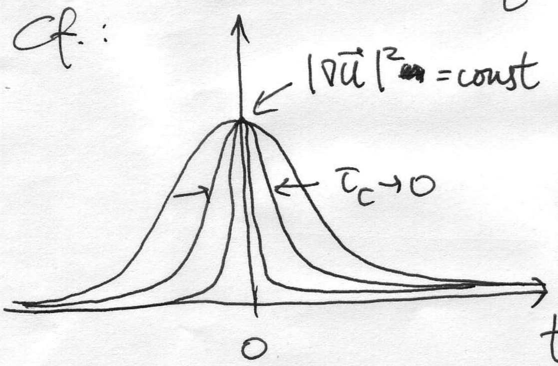
$$\underbrace{\langle \sigma^i_m(t)\, \sigma^j_n(0)\rangle}_{} = \delta(t)\, \varkappa^{ij}_{mn}$$

$$\int_{-\infty}^{+\infty} \langle \dots \rangle\, dt = \varkappa^{ij}_{mn} \text{ - fixed}$$

so $\tau_c \to 0$ but $|\nabla\vec{u}|^2 \to \infty$
- infinite shear acts for a zero time giving finite effect

$$\varkappa^{ij}_{mn} \sim |\nabla\vec{u}|^2 \cdot \tau_c \sim |\nabla\vec{u}|$$
turnover time

Cf.:



Keeping $|\nabla\vec{u}|^2 = \text{const}$ and taking $\tau_c \to 0$, makes the effect vanish (finite shear for a zero time — nothing happens!)

Zero corr. time helps deal with the closure problem and get closed equations [see my paper w/ Kulsrud 2000 — on the course page].

For this velocity field, the distribution of the FTLEs can be computed [Balkovsky & Fouxon 1999, Boldyrev & Schekochihin 2000 — e.g.] — it's Gaussian and then $\tilde{B}^2$ can be averaged over this distribution [Chertkov et al 1999].

However, here, I will show you a somewhat different approach, which gets us the spectrum of the m. field in a quicker way and which also introduces you to a powerful method for computing things.

Lecture 23 — 14.03.05

Go back to our equations:

$$\begin{cases} (1)\quad \partial_t \tilde{B}^i = \sigma^i_m \tilde{B}^m - \eta k^2 \tilde{B}^i \\ (2)\quad \partial_t \tilde{k}_m = -\sigma^l_m \tilde{k}_l \end{cases} \quad \text{for each } \vec{k}_0, \quad \tilde{B}^i(0) = B^i(0,\vec{k}_0), \quad \tilde{k}_m(0) = k_{0m}$$

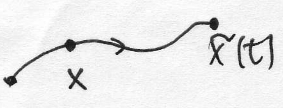
Introduce the joint pdf of $\tilde{B}^i$ and $\tilde{k}_m$:

$$P(\underbrace{B^i, k_m}_{\text{variables}}; \vec{k}_0, t) = \underbrace{\langle}_{\text{over the Kazantsev ensemble!}} \underbrace{\delta(B^i - \tilde{B}^i(t))\,\delta(k_m - \tilde{k}_m(t))}_{\equiv \tilde{P}} \rangle$$

random fields! ($\tilde{B}^i$, $\tilde{k}_m$)

This is because

$$\langle \delta(\underset{\text{variable}}{x} - \underset{\text{random process}}{\tilde{x}(t)}) \rangle = \int \underset{\text{integral over all paths}}{\mathcal{D}\tilde{x}(t)}\, \delta(x - \tilde{x}(t))\, \underset{\text{pdf of path } \tilde{x}(t)}{P[\tilde{x}(t)]} = P(x) \text{ — pdf of } \tilde{x}(t) = x \text{ at time } t.$$



Now we derive an equation for $\mathcal{P}$.

First write an equation for the ~~a~~unaveraged quantity $\widetilde{\mathcal{P}}$ (it's like the Klimontovich pdf in kinetics):

$$\partial_t \widetilde{\mathcal{P}} = \partial_t \left[\delta(B^i - \widetilde{B}^i(t))\, \delta(k_m - \widetilde{k}_m(t)) \right] =$$

$$= \delta'(B^i - \widetilde{B}^i(t)) \left[-\partial_t \widetilde{B}^i(t) \right] \delta(k_m - \widetilde{k}_m(t)) +$$

$$+ \delta(B^i - \widetilde{B}^i(t)]\, \delta'(k_m - \widetilde{k}_m(t)) \left[-\partial_t \widetilde{k}_m(t) \right] =$$

$$= \frac{\partial \widetilde{\mathcal{P}}}{\partial B^i} \underbrace{\left[-\partial_t \widetilde{B}^i(t) \right]}_{(1)\rightarrow\ =\ -\sigma^i_m \widetilde{B}^m(t) + \eta \widetilde{k}^2(t) \widetilde{B}^i(t)} + \frac{\partial \widetilde{\mathcal{P}}}{\partial k_m} \underbrace{\left[-\partial_t \widetilde{k}_m(t) \right]}_{=\ \sigma^i_m \widetilde{k}_i(t)\ \leftarrow (2)} =$$

$$= \frac{\partial}{\partial B^i} \left\{ \widetilde{\mathcal{P}} \left[-\sigma^i_m \widetilde{B}^m(t) + \eta \widetilde{k}^2(t) \widetilde{B}^i(t) \right] \right\} +$$

$$+ \frac{\partial}{\partial k_m} \left\{ \widetilde{\mathcal{P}} \left[\sigma^i_m \widetilde{k}_i(t) \right] \right\} =$$

↑ $\delta(k_m - \widetilde{k}_m(t))\, \delta(B^i - \widetilde{B}^i(t))$, so we can replace $\widetilde{B}^i \to B^i$ and $\widetilde{k}_i \to k_i$ inside $\{\ldots\}$

$$= \underbrace{\left(-\frac{\partial}{\partial B^i} B^m + \frac{\partial}{\partial k_m} k_i \right)}_{\equiv\ \hat{\mathcal{L}}^m_i\ -\ \text{linear operator}} \sigma^i_m \widetilde{\mathcal{P}} + \eta k^2 \frac{\partial}{\partial B^i} B^i \widetilde{\mathcal{P}} \qquad (3)$$

Now average:

$$\partial_t \mathcal{P} = \hat{\mathcal{L}}^m_i \underbrace{\langle \sigma^i_m \widetilde{\mathcal{P}} \rangle} + \eta k^2 \frac{\partial}{\partial B^i} B^i \mathcal{P} \qquad (4)$$

↗ We must calculate this average.

Mathematical digression. 1 Functional Derivatives

Consider a function $F(\varphi_1, \varphi_2, \ldots, \varphi_N)$.

Change the argument infinitesimally: $\varphi_i \to \varphi_i + \delta\varphi_i$.

Then
$$\delta F = \sum_{i=1}^{N} \frac{\partial F}{\partial \varphi_i} \delta\varphi_i$$

Now generalize to case of continuous indices: $i \to q$ (can be multi-dim., etc.)

and F a functional of $\varphi(q)$. Then

$$\delta F = \int dq \, K(q) \, \delta\varphi(q) \quad \text{– linear functional.}$$

By definition,

$$K(q) = \frac{\delta F[\varphi]}{\delta\varphi(q)} \quad \text{– functional derivative}$$

NB: $$\frac{\delta\varphi(q)}{\delta\varphi(q')} = \delta(q-q')$$

and all other properties of differentiation apply.

2 The Gaussian integration / Furutsu-Novikov formula

If $\varphi(q)$ is a Gaussian random field,

$$\boxed{\langle \varphi(q) F[\varphi] \rangle = \int dq' \langle \varphi(q)\varphi(q') \rangle \left\langle \frac{\delta F[\varphi]}{\delta\varphi(q')} \right\rangle}$$

q: all the variables, indices, etc. that the field depends on.

This is just a generalization of the Gaussian splitting property
$$\langle \varphi_1\varphi_2\varphi_3\varphi_4 \rangle = \langle \varphi_1\varphi_2\rangle\langle\varphi_3\varphi_4\rangle + \langle\varphi_1\varphi_3\rangle\langle\varphi_2\varphi_4\rangle + \langle\varphi_1\varphi_4\rangle\langle\varphi_2\varphi_3\rangle$$

Use this formula to write

$$\langle \sigma^i_m(t)\,\tilde{P}\rangle = \int dt' \underbrace{\langle \sigma^i_m(t)\,\sigma^j_n(t')\rangle}_{=\,\delta(t-t')\,\varkappa_2 T^{ij}_{mn}} \left\langle \frac{\delta \tilde{P}(t)}{\delta \sigma^j_n(t')}\right\rangle =$$

$\uparrow$ functional of $\sigma^i_m(t)$

$$= \varkappa_2 T^{ij}_{mn} \underbrace{\left\langle \frac{\delta \tilde{P}(t)}{\delta \sigma^j_n(t)}\right\rangle}$$

$\leftarrow$ calculate this as follows:

Integrate eq. (3) formally:

$$\tilde{P}(t) = \int^t dt' \left[\hat{\mathcal{L}}^m_i\, \sigma^i_m(t')\,\tilde{P}(t') + \eta k^2 \frac{\partial}{\partial B^i} B^i \tilde{P}(t')\right]$$

$$\frac{\delta \tilde{P}(t)}{\delta \sigma^j_n(t)} = \int^t dt' \Big[\hat{\mathcal{L}}^m_i \underbrace{\frac{\delta \sigma^i_m(t')}{\delta \sigma^j_m(t)}}_{=\,\delta^i_j \delta^n_m \delta(t'-t)} \tilde{P}(t') + \hat{\mathcal{L}}^m_i \sigma^i_m(t') \cancel{\frac{\delta \tilde{P}(t')}{\delta \sigma^j_n(t)}} + \eta k^2 \frac{\partial}{\partial B^i} B^i \cancel{\frac{\delta \tilde{P}(t')}{\delta \sigma^j_n(t)}} \Big]$$

$\to 0$ by causality because $t' < t$

($\tilde{P}(t')$ can't depend on future time)

$$= \frac{1}{2} \hat{\mathcal{L}}^n_j \tilde{P}(t),$$ averaging which, we get

$$\langle \sigma^i_m(t)\,\tilde{P}\rangle = \frac{\varkappa_2}{2} T^{ij}_{mn} \hat{\mathcal{L}}^n_j P(t) \rightarrow \text{substitute into (4):}$$

$$\boxed{\partial_t P = \frac{\varkappa_2}{2} T^{ij}_{mn} \hat{\mathcal{L}}^m_i \hat{\mathcal{L}}^n_j P + \eta k^2 \frac{\partial}{\partial B^i} B^i P} \qquad (5)$$

Eureka! Closed equation!

But we must do a bit of work to turn it into a useful form.

Can restrict the form of $P(\vec B, \vec k) \overset{\text{isotropy}}{=} P(B, k, \xi)$, $\xi = \frac{\vec k\cdot\vec B}{kB}$

$$P(\vec B, \vec k) = \delta\left(\frac{\vec k\cdot\vec B}{kB}\right) p(k,B)$$ — must factorise!

because $\vec k \perp \vec B$ from incompressibility.

Let us substitute this into eq. (5):

- $$\hat{\mathcal{L}}^n_j P = \left(-\cancel{\delta^n_j}\,B^n\frac{\partial}{\partial B^j} + \cancel{\delta^n_j} + k_j\frac{\partial}{\partial k_n}\right)\delta\left(\frac{\vec k\cdot\vec B}{kB}\right)p(k,B) =$$

$$= \delta\left(\frac{\vec k\cdot\vec B}{kB}\right)\left(-\frac{B^nB^j}{B}\frac{\partial}{\partial B} + \frac{k_jk_n}{k}\frac{\partial}{\partial k}\right)p \; +$$

$$+\, p(k,B)\underbrace{\delta'\left(\frac{\vec k\cdot\vec B}{kB}\right)\left(-\cancel{\frac{B^nk_j}{kB}} + \vec k\cdot\vec B\,\frac{B^nB^j}{kB}\frac{1}{B^2} + \cancel{\frac{k_jB^n}{kB}} - \vec k\cdot\vec B\,\frac{k_jk_n}{Bk}\frac{1}{k^2}\right)}_{=\;\underbrace{\frac{\vec k\cdot\vec B}{kB}\,\delta'\left(\frac{\vec k\cdot\vec B}{kB}\right)}_{=\;-\delta\left(\frac{\vec k\cdot\vec B}{kB}\right)}\left(\frac{B^nB^j}{B^2} - \frac{k_nk_j}{k^2}\right)}$$

because $0 = [x\delta(x)]' = x\delta'(x) + \delta(x)$

$$= \underbrace{\delta\left(\frac{\vec k\cdot\vec B}{kB}\right)\left(-\frac{B^nB^j}{B}\frac{\partial}{\partial B} + \frac{k_nk_j}{k}\frac{\partial}{\partial k} - \frac{B^nB^j}{B^2} + \frac{k_nk_j}{k^2}\right)p(k,B)}$$

- $$T^{ij}_{mn}\hat{\mathcal{L}}^m_i\hat{\mathcal{L}}^n_j P = T^{ij}_{mn}\hat{\mathcal{L}}^m_i[\ldots] =$$

$$= \delta\left(\frac{\vec k\cdot\vec B}{kB}\right)T^{ij}_{mn}\hat{\mathcal{L}}^m_i(\ldots)^n_j\,p + T^{ij}_{mn}(\ldots)^n_j\,p\,\underbrace{\hat{\mathcal{L}}^m_i\,\delta\left(\frac{\vec k\cdot\vec B}{kB}\right)}_{=\;-\delta\left(\frac{\vec k\cdot\vec B}{kB}\right)\left(\frac{B^mB^i}{B^2} - \frac{k_mk_i}{k^2}\right)} =$$

$$= \underbrace{\delta\left(\frac{\vec k\cdot\vec B}{kB}\right)}T^{ij}_{mn}\left[\hat{\mathcal{L}}^m_i(\ldots)^n_j - \left(\frac{B^mB^i}{B^2} - \frac{k_mk_i}{k^2}\right)(\ldots)^n_j\right]p(B,k) =$$

δ function has factored out as it should have.

$$= \delta\left(\frac{\vec{k}\cdot\vec{B}}{kB}\right) T^{ij}_{mn}\left[\overbrace{-B^m\frac{\partial}{\partial B^i}+k_i\frac{\partial}{\partial k_m}}^{\mathcal{D}^m_i}-\frac{B^mB^i}{B^2}+\frac{k_mk_i}{k^2}\right]\times$$

$$\times\underbrace{\left[-\frac{B^nB^j}{B^2}\left(1+B\frac{\partial}{\partial B}\right)+\frac{k_nk_j}{k^2}\left(1+k\frac{\partial}{\partial k}\right)\right]}_{(\dots)^n_j} p(B,k) =$$

$$= \delta\left(\frac{\vec{k}\cdot\vec{B}}{kB}\right) T^{ij}_{mn}\Bigg[\Bigg(\frac{B^mB^iB^nB^j}{B^4}B\frac{\partial}{\partial B} - \cancel{2}\frac{B^mB^iB^nB^j}{B^4} +$$

$$+ \frac{B^mB^n\delta_{ij}+B^mB^j\delta^n_i}{B^2} - \frac{k_ik_mB^nB^j}{k^2B^2}k\frac{\partial}{\partial k}$$

$$+ \cancel{\frac{B^mB^iB^nB^j}{B^4}} - \frac{k_mk_iB^nB^j}{k^2B^2}\Bigg)\left(1+B\frac{\partial}{\partial B}\right) +$$

$$+ \Bigg(-\frac{B^mB^ik_nk_j}{k^2B^2}B\frac{\partial}{\partial B} + \frac{k_ik_mk_nk_j}{k^4}k\frac{\partial}{\partial k} - \cancel{2}\frac{k_ik_mk_nk_j}{k^4} +$$

$$+ \frac{k_ik_j\delta^{mn}+k_ik_n\delta^m_j}{k^2} - \frac{B^mB^ik_nk_j}{k^2} + \cancel{\frac{k_ik_mk_nk_j}{k^4}}\Bigg)\left(1+k\frac{\partial}{\partial k}\right)\Bigg]p \quad (*)$$

Now use

$$T^{ij}_{mn}\frac{B^mB^iB^nB^j}{B^4} = 1+2a = \frac{d-1}{d+1} \qquad \left(a=-\frac{1}{d+1}\right)$$

$$T^{ij}_{mn}\frac{B^mB^n}{B^2}\delta_{ij} = d+2a = \frac{d^2+d-2}{d+1} = \frac{(d-1)(d+2)}{d+1}$$

$$T^{ij}_{mn}\frac{B^mB^j}{B^2}\delta^n_i = 0 \text{ because } T^{ij}_{mi}=0$$

$$T^{ij}_{mn}\frac{k_mk_iB^nB^j}{k^2B^2} = \cancel{\frac{(\vec{k}\cdot\vec{B})^2}{k^2B^2}}(1+a)+a = -\frac{1}{d+1}$$

vanishes because of $\delta\left(\frac{\vec{k}\cdot\vec{B}}{k^2B^2}\right)$

$$T^{ij}_{mn}\frac{B^mB^ik_nk_j}{k^2B^2} = -\frac{1}{d+1} \text{ same way}$$

$$T^{ij}_{mn}\frac{k_ik_mk_nk_j}{k^4} = 1+2a = \frac{d-1}{d+1}$$

$$T^{ij}_{mn} \frac{k_i k_j \delta^{mn}}{k^2} = d+2a = \frac{(d-1)(d+2)}{d+1}$$

$$T^{ij}_{mn} \frac{k_i k_n \delta^{m}_{j}}{k^2} = 0 \quad \text{because } T^{ij}_{jn} = 0$$

Substitute into (*):

$$(*) = \delta\left(\frac{\vec{k}\cdot\vec{B}}{kB}\right)\left\{\left[\frac{d-1}{d+1}\left(B\frac{\partial}{\partial B}-1\right) + \frac{(d-1)(d+2)}{d+1} + \frac{1}{d+1}\left(k\frac{\partial}{\partial k}+1\right)\right]\left(1+B\frac{\partial}{\partial B}\right)\right.$$

$$\left. + \left[\frac{1}{d+1}\left(B\frac{\partial}{\partial B}+1\right) + \frac{d-1}{d+1}\left(k\frac{\partial}{\partial k}-1\right) + \frac{(d-1)(d+2)}{d+1}\right]\left(1+k\frac{\partial}{\partial k}\right)\right\} p =$$

$$= \delta\left(\frac{\vec{k}\cdot\vec{B}}{kB}\right)\frac{1}{d+1}\left\{(d-1)\,B\frac{\partial}{\partial B}B\frac{\partial}{\partial B} - (d-1) + (d-1)(d+2) + (d-1)(d+2)B\frac{\partial}{\partial B}\right.$$

$$+ 2 + 2k\frac{\partial}{\partial k} + 2B\frac{\partial}{\partial B} + 2k\frac{\partial}{\partial k}B\frac{\partial}{\partial B} + (d-1)k\frac{\partial}{\partial k}k\frac{\partial}{\partial k} - (d-1) +$$

$$\left. + (d-1)(d+2) + (d-1)(d+2)k\frac{\partial}{\partial k}\right\} =$$

$$= \delta\left(\frac{\vec{k}\cdot\vec{B}}{kB}\right)\frac{1}{d+1}\left\{(d-1)B\frac{\partial}{\partial B}B\frac{\partial}{\partial B} + (d-1)k\frac{\partial}{\partial k}k\frac{\partial}{\partial k} + 2k\frac{\partial}{\partial k}B\frac{\partial}{\partial B}\right.$$

$$\left. + d(d+1)B\frac{\partial}{\partial B} + d(d+1)k\frac{\partial}{\partial k} + 2d^2\right\} p =$$

$$= \delta\left(\frac{\vec{k}\cdot\vec{B}}{kB}\right)\frac{1}{d+1}\left[(d-1)\left(B^2\frac{\partial^2}{\partial B^2} + k^2\frac{\partial^2}{\partial k^2}\right) + 2B\frac{\partial}{\partial B}k\frac{\partial}{\partial k} + \right.$$

$$\left. + (d^2+2d-1)\left(B\frac{\partial}{\partial B} + k\frac{\partial}{\partial k}\right) + 2d^2\right] p$$

— this is the first term on rhs of (5).

Now do the second term:

$$\frac{\partial}{\partial B^i} B^i \delta\left(\frac{\vec{k}\cdot\vec{B}}{kB}\right) p(B,k) = \delta\left(\frac{\vec{k}\cdot\vec{B}}{kB}\right)\left(d + B\frac{\partial}{\partial B}\right)p +$$

$$+ p(B,k)\,\delta'\left(\frac{\vec{k}\cdot\vec{B}}{kB}\right)\underbrace{B^i\frac{\partial}{\partial B^i}\frac{\vec{k}\cdot\vec{B}}{kB}}_{=\frac{B^i k_i}{kB} - B\frac{\vec{k}\cdot\vec{B}}{k}\frac{1}{B^2} = 0} = \delta\left(\frac{\vec{k}\cdot\vec{B}}{kB}\right)\left(d + B\frac{\partial}{\partial B}\right)p$$

Eq. (5) becomes:

$$\boxed{\begin{aligned}\partial_t p = \frac{\varkappa_2}{2(d+1)}\Big[&(d-1)\Big(B^2\frac{\partial^2}{\partial B^2} + k^2\frac{\partial}{\partial k^2}\Big) + 2B\frac{\partial}{\partial B}\,k\frac{\partial}{\partial k} + \\ &+ (d^2+2d-1)\Big(B\frac{\partial}{\partial B} + k\frac{\partial}{\partial k}\Big) + 2d^2\Big]p + \\ &+ \eta k^2\Big(d + B\frac{\partial}{\partial B}\Big)p\end{aligned}} \qquad (6)$$

What is the normalisation condition for this pdf?

$$1 = \int d^dB\, d^dk \underbrace{P(\vec{B},\vec{k})}_{=\, p(B,k)\,\delta\left(\frac{\vec{k}\cdot\vec{B}}{kB}\right)} = S_d \int_0^\infty dB\, B^{d-1}\int_0^\infty dk\, k^{d-1}\, p(B,k)\int d\Omega_d\, \delta(\underbrace{\cos\theta}_{\equiv\, \xi})$$

$$3D: \int d\Omega_d\, \delta(\cos\theta) = \int_0^{2\pi} d\varphi \underbrace{\int_0^{\pi} d\theta \sin\theta\, \delta(\cos\theta)}_{=\int_{-1}^{1} d\xi\, \delta(\xi) = 1} = 2\pi$$

in d dimensions:

$$\int d\Omega_d\, \delta(\cos\theta) = \int d\Omega_d \frac{1}{2\pi}\int_{-\infty}^{+\infty} dx\, e^{ix\cos\theta} = \frac{1}{2\pi}(2\pi)^{d/2}\int_{-\infty}^{+\infty} dx\, \frac{J_{d/2-1}(x)}{x^{d/2-1}} =$$

$= S_{d-1}$ (area of unit sphere in $d-1$ dimensions) — see lecture on corr. functions. (§18)

So:

$$\boxed{1 = S_d S_{d-1}\int_0^\infty dB\, B^{d-1}\int_0^\infty dk\, k^{d-1}\, p(B,k)} \qquad (7)$$

Define the pdf of B and k by $F(B,k) = S_d S_{d-1} B^{d-1} k^{d-1} p(B,k)$, so that $\int dB \int dk\, F(B,k) = 1$ $\Rightarrow$ can rewrite equation (6) in terms of $F(B,k)$ if we want to be solving an equation for a conserved quantity: so, substitute

$$p = \frac{F(B,k)}{S_d S_{d-1} B^{d-1} k^{d-1}}$$ into Eq. (6)

We will need the following expressions:

$$\frac{\partial}{\partial k}\frac{F}{k^{d-1}} = \frac{1}{k^{d-1}}\frac{\partial F}{\partial k} - \frac{d-1}{k^{d}}F$$

$$k\frac{\partial}{\partial k}\frac{F}{k^{d-1}} = \frac{1}{k^{d-1}}\left[k\frac{\partial F}{\partial k}-(d-1)F\right] = \frac{1}{k^{d-1}}\left[\frac{\partial}{\partial k}kF - dF\right]$$

$$k^{2}\frac{\partial^{2}}{\partial k^{2}}\frac{F}{k^{d-1}} = k^{2}\left[\frac{1}{k^{d-1}}\frac{\partial^{2} F}{\partial k^{2}} - 2\frac{d-1}{k^{d}}\frac{\partial F}{\partial k} + \frac{d(d-1)}{k^{d+1}}F\right]=$$

$$= \frac{1}{k^{d-1}}\left[k^{2}\frac{\partial^{2}F}{\partial k^{2}} - 2(d-1)k\frac{\partial F}{\partial k} + d(d-1)F\right]=$$

$$=\frac{1}{k^{d-1}}\left[\frac{\partial}{\partial k}k^{2}\frac{\partial F}{\partial k} - \cancel{2k\frac{\partial F}{\partial k}} - 2(d-\cancel{1})k\frac{\partial F}{\partial k} + d(d-1)F\right]=$$

$$=\frac{1}{k^{d-1}}\left[\frac{\partial}{\partial k}k^{2}\frac{\partial F}{\partial k} - 2d\frac{\partial}{\partial k}kF + d(d+1)F\right]$$

and analogous expressions for $B\frac{\partial}{\partial B}\frac{F}{B^{d-1}}$, $B^{2}\frac{\partial^{2}}{\partial B^{2}}\frac{F}{B^{d-1}}$,
also,

$$B\frac{\partial}{\partial B}k\frac{\partial}{\partial k}\frac{F}{k^{d-1}B^{d-1}} = B\frac{\partial}{\partial B}\frac{1}{k^{d-1}B^{d-1}}\left(\frac{\partial}{\partial k}kF-dF\right)=$$

$$=\frac{1}{k^{d-1}B^{d-1}}\left(\frac{\partial}{\partial B}B-d\right)\left(\frac{\partial}{\partial k}k-d\right)F$$

Using all that in Eq. (6), get

$$\partial_t F = \frac{\varkappa_2}{2(d+1)}\Big\{(d-1)\left[\frac{\partial}{\partial B}B^{2}\frac{\partial}{\partial B} - 2d\frac{\partial}{\partial B}B + d(d+1) + \frac{\partial}{\partial k}k^{2}\frac{\partial}{\partial k} - 2d\frac{\partial}{\partial k}k + d(d+1)\right]$$
$$+2\left[\frac{\partial}{\partial B}B\frac{\partial}{\partial k}k + d^{2} - d\frac{\partial}{\partial k}k - d\frac{\partial}{\partial B}B\right] +$$
$$+(d^{2}+2d-1)\left[\frac{\partial}{\partial B}B - d + \frac{\partial}{\partial k}k - d\right] + 2d^{2}\Big\}F + \eta k^{2}\frac{\partial}{\partial B}BF =$$

$$= \frac{\varkappa_2}{2(d+1)}\Big\{(d-1)\Big[\frac{\partial}{\partial B}B^2\frac{\partial}{\partial B}+\frac{\partial}{\partial k}k^2\frac{\partial}{\partial k}\Big]+2\frac{\partial}{\partial B}B\frac{\partial}{\partial k}k$$

$$-2d^2+\cancel{2d}-\cancel{2d}+d^2+2d-1=-d^2+2d-1=-(d-1)^2$$

$$+\Big[-2d(d-1)-2d+d^2+2d-1\Big]\Big(\frac{\partial}{\partial B}B+\frac{\partial}{\partial k}k\Big)+$$

$$+\underbrace{2d(d+1)(d-1)+2d^2-2d(d^2+2d-1)+2d^2}_{\parallel}\Big\}F+\eta k^2\frac{\partial}{\partial B}BF$$

$2d(\cancel{d^2}-\cancel{1}+\cancel{d}-\cancel{d^2}-\cancel{2d}+\cancel{1}+\cancel{d})=0$ – the term with no derivatives had to disappear because the equation F had to satisfy

$$\partial_t\int_0^\infty dB\int_0^\infty dk\,F=0\ ,\ \text{since}\ \int_0^\infty dB\int_0^\infty dk\,F=1$$

So, we get an equation that looks much nicer:

$$\boxed{\frac{\partial F}{\partial t}=\frac{1}{2}\frac{d-1}{d+1}\varkappa_2\Big[\frac{\partial}{\partial B}B^2\frac{\partial F}{\partial B}+\frac{\partial}{\partial k}k^2\frac{\partial F}{\partial k}+\frac{2}{d-1}\frac{\partial}{\partial B}B\frac{\partial}{\partial k}kF-(d-1)\Big(\frac{\partial}{\partial B}BF+\frac{\partial}{\partial k}kF\Big)\Big]+\eta k^2\frac{\partial}{\partial B}BF} \quad (8)$$

This equation will allow us to find the spectrum of the magnetic field. Here is why.

Recall $B^i(t,\vec{x})=\underbrace{\int\frac{d^dk_0}{(2\pi)^d}\tilde{B}^i(t,\vec{k}_0)e^{i\tilde{k}_m(t,\vec{k}_0)x^m}}$ $\quad(2\pi)^d\delta(k_m-\tilde{k}_m(t,\vec{k}_0))$

Fourier-transform:

$$B^i(t,\vec{k})=\int d^dx\,B^i(t,\vec{x})e^{-i\vec{k}\cdot\vec{x}}=\int\frac{d^dk_0}{(2\pi)^d}\tilde{B}^i(t,\vec{k}_0)\underbrace{\int d^dx\,e^{i\tilde{k}_m(t,\vec{k}_0)x^m-ik_mx^m}}$$

$$=\int d^dk_0\,\tilde{B}^i(t,\vec{k}_0)\,\delta(k_m-\tilde{k}_m(t,\vec{k}_0))$$

Spectrum of the magnetic field: $\frac{1}{2}\langle B^2\rangle = \int_0^\infty dk\, M(t,k)$, so

$$M(t,k) = \frac{1}{2}\underbrace{\int d\Omega_d}_{\text{angle integrated}} k^{d-1} \langle |\vec{B}(t,\vec{k})|^2\rangle =$$

$$= \frac{1}{2}\int d\Omega_d\, k^{d-1} \int d^dk_0 \int d^dk_0' \langle \tilde{B}^i(t,\vec{k}_0)\tilde{B}^i(t,\vec{k}_0')\,\delta(k_m - \tilde{k}_m(t,\vec{k}_0))\,\delta(-k_m - \tilde{k}_m(t,\vec{k}_0'))\rangle$$

$$\delta(\tilde{k}_m(t,\vec{k}_0) + \tilde{k}_m(t,\vec{k}_0')) = \delta\left(\frac{\partial x_0^r}{\partial x^m}(k_{0r} + k_{0r}')\right) =$$

$$= \delta(\vec{k}_0 + \vec{k}_0') \text{ because } \det \frac{\partial x_0^r}{\partial x^m} = 1$$

$$= \frac{1}{2}\int d\Omega_d\, k^{d-1} \int d^dk_0 \langle |\tilde{\vec{B}}(t,\vec{k}_0)|^2\, \delta(k_m - \tilde{k}_m(t,\vec{k}_0))\rangle =$$

$$= \frac{1}{2}\int d\Omega_d\, k^{d-1} \int d^dk_0 \int d^dB\, \underbrace{|\vec{B}|^2}_{= B^2} \underbrace{\langle \delta(B^i - \tilde{B}^i(t,\vec{k}_0))\,\delta(k_m - \tilde{k}_m(t,\vec{k}_0))\rangle}_{= P(\vec{B},\vec{k};\vec{k}_0) = \delta(\xi)\, p(B,k)} =$$

$$\xi = \frac{\vec{k}\cdot\vec{B}}{kB}$$

$$= \frac{1}{2}\int d^dk_0\, S_d \int_0^\infty dB\, B^{d-1} B^2 k^{d-1} p(B,k) \underbrace{\int d\Omega_d\, \delta(\xi)}_{= S_{d-1}} =$$

$$= \frac{1}{2}\int d^dk_0 \int_0^\infty dB\, B^2 F(B,k;\vec{k}_0).$$

Thus, the spectrum is a superposition of modes that initially are concentrated at $\vec{k}_0$ and for each of them

$$\boxed{M(t,k) = \frac{1}{2}\int_0^\infty dB\, B^2 F(B,k)} \qquad (9)$$

Now we use eq. (8) to get an equation for $M(t,k)$:

$\int_0^\infty dB\, B^2\,(8)$:

$$\frac{\partial M}{\partial t} = \frac{1}{2}\frac{d-1}{d+1}\varkappa_2\left[6 + \frac{\partial}{\partial k}k^2\frac{\partial}{\partial k} - \underbrace{\frac{4}{d-1}\frac{\partial}{\partial k}k}_{} - (d-1)\left(-2 + \underbrace{\frac{\partial}{\partial k}k}_{}\right)\right]M - 2\eta k^2 M$$

$$= \frac{1}{2}\frac{d-1}{d+1}\varkappa_2\left[\frac{\partial}{\partial k}k^2\frac{\partial}{\partial k} - \frac{d^2-2d+5}{d-1}\frac{\partial}{\partial k}k + 2(d+2)\right]M - 2\eta k^2 M$$

Set $d=3$ (Exercise: do the analysis for $d=2$!)

$$\boxed{\partial_t M = \frac{1}{4}\varkappa_2\frac{\partial}{\partial k}\left(k^2\frac{\partial M}{\partial k} - 4kM\right) + \frac{5}{2}\varkappa_2 M - 2\eta k^2 M} \quad (10)$$

Let $\frac{5}{2}\varkappa_2 = 2\gamma$ (you'll see why in a moment)

Kulsrud-Anderson Equation. 1992 (originally derived by Kazantsev in 1967)

NB: This equation can, in fact, be derived from the Kazantsev model field w/o assuming that the velocity is linear:

- First derive an integrodifferential equation

$$\partial_t M = \int d^3k'\,\underbrace{K(\mathbf{k},\mathbf{k}')}_{\text{depends on } \varkappa^{ij}(\mathbf{k}')}\,M(|\mathbf{k}-\mathbf{k}'|) - 2\eta k^2 M$$

- Expand the integrand ~~[illegible]~~ up to 2 order in $\frac{k'}{k} \ll 1$ (velocity scale >> field scale)

$\Downarrow$

Get Eq. (10)

See papers by Kulsrud & Anderson, Kazantsev, etc.

Lecture 24

16.03.05

Solutions of equation (10):

1) <u>Ideal regime</u>. Initial condition: $M(0,k) = \underset{\swarrow\text{ initial energy}}{\mathcal{E}_0}\,\delta(k-k_0)$

First set $\eta = 0$.

Change variables: $z = \ln k \Rightarrow \frac{\partial}{\partial k} = \frac{\partial z}{\partial k}\frac{\partial}{\partial z} = \frac{1}{k}\frac{\partial}{\partial z}$, $k\frac{\partial}{\partial k} = \frac{\partial}{\partial z}$

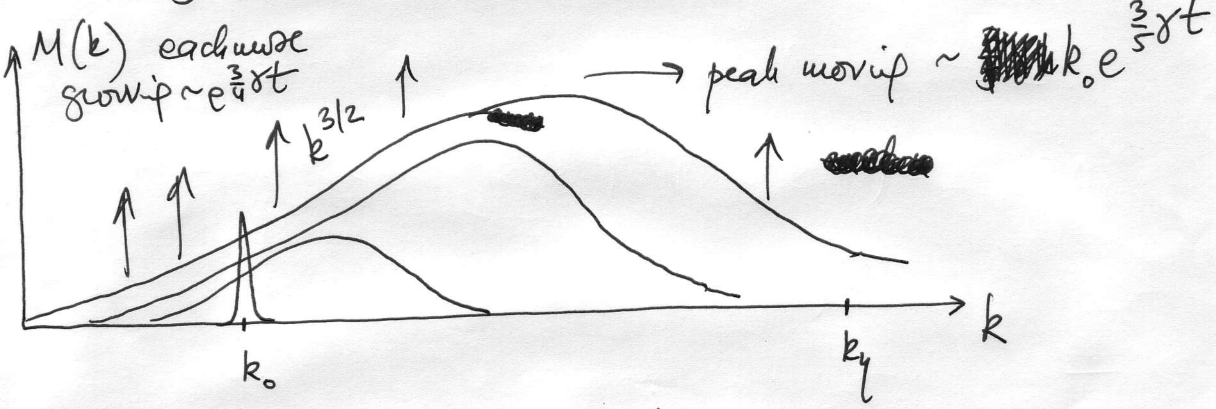
$$\frac{\partial}{\partial k}k = \frac{\partial z}{\partial k}\frac{\partial}{\partial z}e^z = \frac{1}{k}\frac{\partial}{\partial z}e^z = \frac{\partial}{\partial z} + 1$$

So, $\partial_t M = \frac{\gamma}{5}\left[\frac{\partial^2 M}{\partial z^2} \underbrace{- 3\frac{\partial M}{\partial z}}_{\text{drift}} + 6M\right]$ – diffusion equation

Solution: $M(t,z) = \int dz'\, M(0,z')\, \frac{e^{-\frac{[z-z'-\frac{3}{5}\gamma t]^2}{\frac{4}{5}\gamma t}}}{\sqrt{\frac{4}{5}\pi\gamma t}}\, e^{\frac{6}{5}\gamma t}$

$$= e^{\frac{3}{4}\gamma t}\frac{1}{\sqrt{\frac{4}{5}\pi\gamma t}}\int\frac{dk'}{k'}\underbrace{M(0,k')}_{\mathcal{E}_0\delta(k'-k_0)}\left(\frac{k}{k'}\right)^{3/2} e^{-\frac{[\ln k/k']^2}{\frac{4}{5}\gamma t}} =$$

$$= e^{\frac{3}{4}\gamma t}\frac{\mathcal{E}_0}{\sqrt{\frac{4}{5}\pi\gamma t}}\frac{1}{k_0}\left(\frac{k}{k_0}\right)^{3/2} e^{-[\ln k/k_0]^2/\frac{4}{5}\gamma t} \qquad (11)$$



The overall energy $\mathcal{E}(t) = \mathcal{E}_0 e^{2\gamma t}$ – growth comes from

1) each mode growing $\propto e^{\frac{3}{4}\gamma t}$
2) # of excited modes grows exp-lly as well (spectrum is spreading to ever-larger k)

The spectrum is $\sim k^{3/2}$ (Kulsrud–Anderson, or Kazantsev spectrum) after a time $\sim \frac{1}{\gamma}\ln\frac{k_\eta}{k_0} \sim \frac{1}{\gamma}\ln Pm^{1/2}$

2) Resistive regime. Eventually, the spectrum spreads to k large enough that resistivity is important ($\eta k^2 \sim \gamma \Rightarrow k_\eta \sim \sqrt{\frac{\gamma}{\eta}}$)

The Green's function solution (11) can be generalized to this case (see Schekochihin & Boldyrev & Kulsrud 2002), but that is not very edifying. Instead, we seek solutions in the form of an eigenvalue problem:

$$M(t,k) = e^{\lambda\gamma t}\,\Phi(\xi), \text{ where } \xi = \frac{k}{k_\eta}, \quad k_\eta = \sqrt{\frac{\gamma}{10\eta}}$$

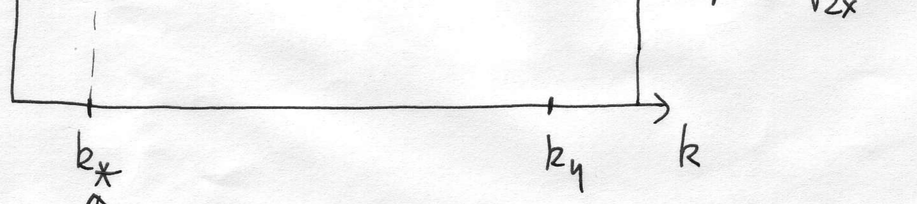
Then $\Phi(\xi)$ satisfies

$$\boxed{\xi^2\Phi'' - 2\xi\Phi' + (6-5\lambda)\Phi - \xi^2\Phi = 0} \qquad (12)$$

Solutions of this equation are $\xi^{3/2}\cdot$ modified Bessel functions. The solution that satisfies $\Phi(\xi\to\infty)\to 0$ (i.e. spectrum does not diverge at ∞) is

$$\boxed{M(t,k) = e^{\lambda\gamma t}\left(\frac{k}{k_\eta}\right)^{3/2} K_{\nu(\lambda)}\left(\frac{k}{k_\eta}\right)}, \qquad (13)$$

where $\nu(\lambda) = \sqrt{5(\lambda - \frac{3}{4})}$



Energy:

$\mathcal{E} \sim e^{\lambda\gamma t}$

and each mode is growing at $e^{\lambda\gamma t}$.

λ is determined from the boundary condition at low k.

Let us fix some k_* and impose the boundary condition so: recall that [Eq.(10)]

$$\frac{\partial M}{\partial t} = -\frac{\partial}{\partial k}\mathcal{F}(k) + 2\gamma M - 2\eta k^2 M$$

where $\mathcal{F}(k) = -\frac{\gamma}{5}\left(k^2 M'(k) - 4kM(k)\right)$ – flux through k.

Energy: $$\frac{\partial \mathcal{E}}{\partial t} = -\underbrace{\mathcal{F}(\infty)}_{0} + \underbrace{\mathcal{F}(k_*)}_{=0} + 2\gamma\mathcal{E} - 2\eta\int_{k_*}^{\infty} dk\, k^2 M(k) \quad (14)$$

set this to vanish, so no energy is flowing "locally" into high-k modes.

So: $\boxed{k_* M'(k_*) - 4M(k_*) = 0}$ – zero-flux bndry condition

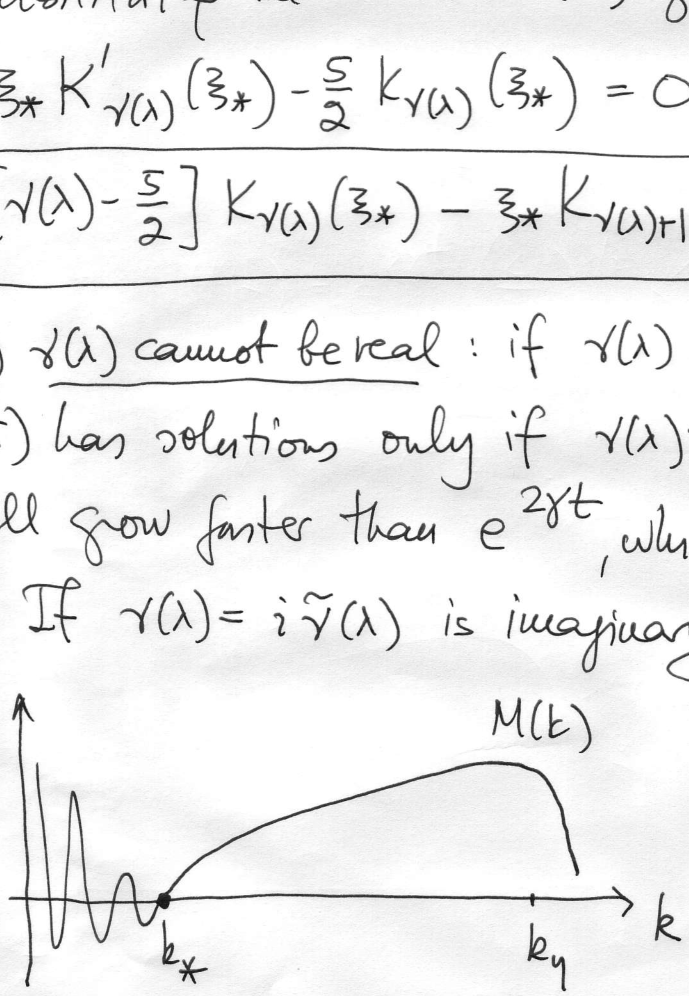
Substituting the solution (13) gives

$$\xi_* K'_{\nu(\lambda)}(\xi_*) - \frac{5}{2}K_{\nu(\lambda)}(\xi_*) = 0, \quad \xi_* \doteq \frac{k_*}{k_\eta}$$

$\boxed{\left[\nu(\lambda) - \frac{5}{2}\right]K_{\nu(\lambda)}(\xi_*) - \xi_* K_{\nu(\lambda)+1}(\xi_*) = 0}$ – equation for λ. (15)

1) $\nu(\lambda)$ cannot be real: if $\nu(\lambda)$ is real $\left(\lambda > \frac{3}{4}\right)$, then (15) has solutions only if $\nu(\lambda) > \frac{5}{2} \Leftrightarrow \lambda > 2$, i.e. energy will grow faster than $e^{2\gamma t}$, which is impossible by Eq.(14)

2) If $\nu(\lambda) = i\tilde{\nu}(\lambda)$ is imaginary, we have an oscillating solution. But spectrum cannot be negative, so we set $k_* \geq$ the rightmost zero of M (i.e., we push all the zeroes outside the interval $[k_*, \infty)$)



~~[illegible]~~ Now $\xi_* = \frac{k_*}{k_\eta} \sim \frac{1}{\sqrt{Pm}} \ll 1$, so all zeroes of $K_{i\tilde{\nu}(\lambda)}(\xi)$ must be $\ll 1$, which means $\tilde{\nu}(\lambda) \ll 1$. In this limit, Eq. (15) becomes

$$K_{i\tilde{\nu}(\lambda)}(\xi_*) \approx -\frac{1}{\tilde{\nu}(\lambda)} \sin\left[\tilde{\nu}(\lambda) \ln \frac{\xi_*}{2}\right] = 0, \quad \tilde{\nu}(\lambda) = \sqrt{5\left(\tfrac{3}{4} - \lambda\right)}$$

The rightmost zero is at

$$\tilde{\nu}(\lambda) \ln \frac{\xi_*}{2} = \pi, \text{ whence}$$

$$\boxed{\lambda \simeq \frac{3}{4} - \frac{\pi^2}{5\left[\ln(k_*/2k_\eta)\right]^2} \simeq \frac{3}{4} - \frac{\pi^2}{5\left[\ln Pm^{1/2}\right]^2}} \qquad (16)$$

so, all modes grow at $e^{\frac{3}{4}\gamma t}$ with a negative correction to the growth rate $\sim \mathcal{O}\left(\frac{1}{\ln^2 Pm}\right)$ [very slow logarithmic convergence]

Note that from (16), we can get a rough estimate for the critical Rm for the dynamo to work:

$$\underset{\substack{\uparrow \\ \text{single-scale flow}}}{Rm_c \sim} Pm_c \sim \exp\left[\frac{4\pi}{\sqrt{15}}\right] \simeq 26 \qquad \left[\begin{array}{l}\text{numerical solution} \\ \text{gives} \sim 53\end{array}\right]$$

Because of $\log^2 Pm$ convergence, the above theory is quite hard to check against DNS, but inasmuch as we can simulate the large-Pm case, it seems to work very well...

[NB: The $k^{3/2}$ spectrum of the kinematic dynamo is the spectrum of direction <u>reversals</u>.

NB: All of the above can also be done in x space ⇒ "Kazantsev quantum mechanics" - see papers posted on the web.