§22. Small-Scale Dynamo.

$$\begin{cases} \dfrac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\nabla p + \vec{B}\cdot\nabla\vec{B} + \nu\nabla^2\vec{u} + \vec{f}, \quad \nabla\cdot\vec{u}=0 \\ \dfrac{\partial \vec{B}}{\partial t} + \vec{u}\cdot\nabla\vec{B} = \underset{①}{\vec{B}\cdot\nabla\vec{u}} + \underset{②}{\eta\nabla^2\vec{B}} \end{cases}$$

Consider now a situation with no imposed field (completely isotropic). Let us set a turbulence and introduce an arbitrarily weak seed field.

- Will this seed field grow? ($\langle B^2\rangle \uparrow$ mean $\exists$ s.s. dynamo)
- What is the saturated state? (iso. MHD turbulence)

We have an extra diff. coeff.: η.

ISM 10^{14} cl. 10^{29} pl 10^{-5} st. $10^{-7...-4}$ discs 10^{-8}

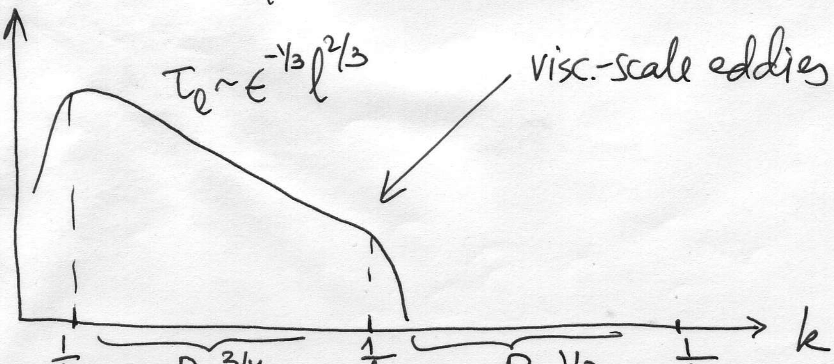
Ohmic diffusion is important if ① ~ ②, or

$$\frac{\eta}{l^2} \sim \frac{\delta u_l}{l} \qquad // \qquad \frac{Rm}{Re} \sim 2.6\cdot 10^{-5}\frac{T^4}{n} \text{ ionized} \qquad 1.7\cdot10^{7}\frac{T^2}{n} \text{ with neutrals}$$

Consider the case of $Pm = \dfrac{\nu}{\eta} \gg 1$. In ~~that~~ this case we are about to see that $l_\eta \ll l_\nu$, so use viscous scaling for the velocity field:

$$\frac{\eta}{l^2} \sim \left(\frac{\epsilon}{\nu}\right)^{1/2} \cancel{l}\,\frac{1}{\cancel{l}} \sim \frac{1}{\tau_\nu}, \quad \tau_\nu \sim \frac{l_\nu}{\delta u_{l_\nu}} \text{ turnover time at visc. scale}$$

Then $l_\eta \sim (\eta\tau_\nu)^{1/2} \sim Pm^{-1/2}\, l_\nu \ll l_\nu$ as assumed.



visc.-scale eddies stretch the field the fastest!

NB: For non-turb. chaotic velocities, $Re \sim 1$, $Pm \sim Rm$, all the same

So let us study magnetic fluctuations at subviscous scales: the velocity field of the visc. eddies looks to these fluctuations as a linear shear (velocity expanded in a Taylor series):

use upper & lower indices for conv.

$$u^i(t,\vec{x}) = \sigma^i_m(t)\, x^m + \dots$$

(we are expanding around some reference point $\vec{x}=0$ — let us go to the frame that moves with that velocity, so $u^i(t,0)=0$). Induction equation:

$$\partial_t B^i = -u^l \frac{\partial B^i}{\partial x^l} + B^l \frac{\partial u^i}{\partial x^l} + \eta \Delta B^i$$

$$\partial_t B^i = -\sigma^l_m x^m \frac{\partial B^i}{\partial x^l} + B^m \sigma^i_m + \eta \Delta B^i \qquad (1)$$

Let us seek the solution to this equation as a sum of "plane waves"

$$B^i(t,\vec{x}) = \int \frac{d^d k_0}{(2\pi)^d}\, \tilde{B}^i(t,\vec{k}_0)\, e^{i\tilde{\vec{k}}(t,\vec{k}_0)\cdot\vec{x}} \qquad (2)$$

such that $\tilde{\vec{k}}(0,\vec{k}_0) = \vec{k}_0$, i.e. at $t=0$ this is just the Fourier expansion of the initial condition.

Eq. (1) is linear in B^i, so if we make sure that each plane wave in (2) is a solution, so will be their sum.

OK: $$\partial_t \tilde{B}^i e^{i\tilde{\vec{k}}\cdot\vec{x}} = e^{i\tilde{\vec{k}}\cdot\vec{x}}\left[\partial_t \tilde{B}^i + \underbrace{\tilde{B}^i\, i x^m \partial_t \tilde{k}_m}\right] =$$

$$= e^{i\tilde{\vec{k}}\cdot\vec{x}}\left[\underbrace{-\sigma^l_m x^m\, i\tilde{k}_l \tilde{B}^i} + \tilde{B}^m \sigma^i_m - \eta \tilde{k}^2 \tilde{B}^i\right]$$

F.tr.

$$\boxed{\begin{array}{ll} \vec{x}=0: & \partial_t \tilde{B}^i = \sigma^i_m \tilde{B}^m - \eta \tilde{k}^2 \tilde{B}^i \\ \text{the rest}: & \partial_t \tilde{k}_m = -\sigma^l_m \tilde{k}_l \end{array}}$$

$$\tilde{B}^i(0) = B^i(0,\vec{k}_0) \equiv B^i_0 \qquad (3)$$

$$k_m(0) = k_{0m} \qquad (4)$$

Note: $\tilde{k}_i \tilde{B}^i = 0$ at all times if $k_{0i} B^i_0 = 0$

NB: ordinary diff. equations!

Let us now relate this to the Lagrangian MHD.

Recall that if the fluid particle at $\vec{x}$ at time t came from $\vec{x}_0$ at time 0, we have

$$\frac{\partial x^i(t,\vec{x}_0)}{\partial t} = u^i(t,\vec{x}(t,\vec{x}_0)) = \sigma^i_l(t)\, x^l(t,\vec{x}_0)$$

Then $\dfrac{\partial}{\partial t}\dfrac{\partial x^i}{\partial x_0^m} = \sigma^i_l \dfrac{\partial x^l}{\partial x_0^m}$ strain tensor (5)

we (will) also need the evolution of $\dfrac{\partial x_0^m}{\partial x^j}$:

$$0 = \frac{\partial}{\partial t} \underbrace{\frac{\partial x^i}{\partial x_0^m}\frac{\partial x_0^m}{\partial x^j}}_{= \delta^i_j = \text{const}} = \frac{\partial x_0^m}{\partial x^j}\frac{\partial}{\partial t}\frac{\partial x^i}{\partial x_0^m} + \frac{\partial x^i}{\partial x_0^m}\frac{\partial}{\partial t}\frac{\partial x_0^m}{\partial x^j}, \text{ so}$$

$$\frac{\partial x_0^r}{\partial x^i}\,\Bigg|\; \frac{\partial x^i}{\partial x_0^m}\frac{\partial}{\partial t}\frac{\partial x_0^m}{\partial x^j} = -\frac{\partial x_0^m}{\partial x^j}\frac{\partial}{\partial t}\frac{\partial x^i}{\partial x_0^m} = -\frac{\partial x_0^m}{\partial x^j}\sigma^i_l \frac{\partial x^l}{\partial x_0^m} = -\sigma^i_l\,\delta^l_j = -\sigma^i_j$$

$$\delta^r_m \frac{\partial}{\partial t}\frac{\partial x_0^m}{\partial x^j} = -\sigma^i_j \frac{\partial x_0^r}{\partial x^i}$$

or $\dfrac{\partial}{\partial t}\dfrac{\partial x_0^r}{\partial x^j} = -\sigma^i_j \dfrac{\partial x_0^r}{\partial x^i}$ inverse strain tensor. (6)

Now we show that the solution of equations (3-4) is

$$\tilde{k}_m(t) = \frac{\partial x_0^r}{\partial x^m} k_{0r} \qquad (7)$$

and

$$\tilde{B}^i(t) = \frac{\partial x^i}{\partial x_0^m} B_0^m \, e^{-\eta \int_0^t dt' \tilde{k}^2(t')} \qquad (8)$$

(Ex.)

Pf. $\partial_t \tilde{k}_m(t) \overset{(7)}{=} k_{0r}\, \partial_t \frac{\partial x_0^r}{\partial x^m} \overset{(6)}{=} -k_{0r}\, \sigma_m^l \frac{\partial x_0^r}{\partial x^l} \overset{(7)}{=} -\sigma_m^l k_l$

– (7) satisfies (4) q.e.d.

$$\partial_t \tilde{B}^i(t) \overset{(8)}{=} B_0^m \underbrace{\left[\partial_t \left(\frac{\partial x^i}{\partial x_0^m}\right)\right]}_{\overset{(5)}{=}\ \sigma_l^i \frac{\partial x^l}{\partial x_0^m}} e^{-\eta \int_0^t dt' k^2(t')} + \underbrace{\frac{\partial x^i}{\partial x_0^m} B_0^m e^{-\eta \int_0^t dt' k^2(t')}}_{=\ \tilde{B}^i(t)} (-\eta k^2)$$

$$\sigma_l^i \tilde{B}^l \overset{(8)}{=} \ldots$$

$$= \sigma_l^i \tilde{B}^l - \eta k^2 \tilde{B}^i$$

– so (8) satisfies (3) q.e.d.

Let us find the m. energy of the field (2):

$$\langle B^2 \rangle = \int d^d x \int \frac{d^d k_0}{(2\pi)^d} \int \frac{d^d k_0'}{(2\pi)^d} \tilde{B}^i(t, \vec{k}_0) \tilde{B}^i(t, \vec{k}_0')\, e^{i[\vec{\tilde{k}}(t,\vec{k}_0) + \vec{\tilde{k}}(t,\vec{k}_0')]\cdot \vec{x}} =$$

$$= \int \frac{d^d k_0}{(2\pi)^d} \int \frac{d^d k_0'}{(2\pi)^d} \tilde{B}^i(t, \vec{k}_0) \tilde{B}^i(t, \vec{k}_0') (2\pi)^d \underbrace{\delta\left(\vec{\tilde{k}}(t,\vec{k}_0) + \vec{\tilde{k}}(t,\vec{k}_0')\right)}$$

$$\delta\left(\frac{\partial x_0^r}{\partial x^m}(k_{0r} + k_{0r}')\right) = \delta(\vec{k}_0 + \vec{k}_0') \left[\det \frac{\partial x_0^r}{\partial x^m}\right]^{-1} = \delta(\vec{k}_0 + \vec{k}_0') \quad \text{(by } J = 1\text{)}$$

vol. average (scales $\ll l_v$)

Thus,

$$\langle B^2\rangle = \int\frac{d^dk_0}{(2\pi)^d}\,\tilde B^i(t,\vec k_0)\,\tilde B^i(t,-\vec k_0) = \int\frac{d^dk_0}{(2\pi)^d}\,|\tilde{\vec B}(t,\vec k_0)|^2 \qquad (9)$$

$\left(B^i(t,\vec x) \text{ is real} \Leftrightarrow \tilde B^i(t,\vec k_0) = \tilde B^{i*}(t,-\vec k_0) \right)$

Thus, magnetic energy is a sum of the energies of all the plane waves. Let's calculate them individually:

$$\tilde B^2(t) \overset{(8)}{=} \frac{\partial x^i}{\partial x_0^m}\frac{\partial x^i}{\partial x_0^n}\,B_0^m B_0^n\, e^{-2\eta\int_0^t dt'\,k^2(t')} \qquad (10)$$

and
$$k^2(t) = \frac{\partial x_0^r}{\partial x^l}\frac{\partial x_0^s}{\partial x^l}\,k_{0r}k_{0s} \qquad (11)$$

Let $M_{mn} = \dfrac{\partial x^i}{\partial x_0^m}\dfrac{\partial x^i}{\partial x_0^n}$ and $M^{rs} = \dfrac{\partial x_0^r}{\partial x^l}\dfrac{\partial x_0^s}{\partial x^l}$

$$M^{rj}M_{jn} = \frac{\partial x_0^r}{\partial x^l}\underbrace{\frac{\partial x_0^j}{\partial x^l}\frac{\partial x^i}{\partial x_0^j}}_{=\delta^i_l}\frac{\partial x^i}{\partial x_0^n} = \frac{\partial x_0^r}{\partial x^i}\frac{\partial x^i}{\partial x_0^n} = \delta^r_n,$$

So M^{rs} is the inverse of M_{mn}.

M_{mn} and M^{rs} are co- and contravariant metric tensors of transformation $\vec x\to\vec x_0$.

They are symmetric matrices and can, therefore, be diagonalized by an appropriate rotation of the coordinate system:

$$\hat M = \hat R^T\cdot\hat\Lambda\cdot\hat R\,,\quad \hat M^{-1} = \overset{=\hat R^T}{\hat R^{-1}}\cdot\hat\Lambda^{-1}\cdot\overbrace{(\hat R^T)^{-1}}^{=\hat R}$$

Then
$$\vec B_0\cdot\hat M\cdot\vec B_0 = \vec B_0\cdot\hat R^T\cdot\hat\Lambda\cdot\hat R\cdot\vec B_0$$
$$\vec k_0\cdot\hat M^{-1}\cdot\vec k_0 = \vec k_0\cdot\hat R^T\cdot\hat\Lambda^{-1}\cdot\hat R\cdot\vec k_0$$

In principle, $\hat{R}$ depends on time. However, it is possible to prove that it converges exp-lly in time to a constant matrix (Goldhirsch-Sulem-Orszag 1987).

They also proved that $\qquad \hat{R} = (\hat{e}_1 \hat{e}_2 \hat{e}_3)$

$$\hat{\Lambda} = \begin{pmatrix} e^{\zeta_1(t)} & & 0 \\ & e^{\zeta_2(t)} & \\ 0 & & e^{\zeta_3(t)} \end{pmatrix}$$

where $\zeta_i(t) \to 2\lambda_i t$, where λ_i are called <u>Lyapunov exponents</u> of the system (convergence is slow with corrections to λ_i of order $\frac{1}{t}$).

The instantaneous values $\frac{1}{2}\frac{\zeta_i(t)}{t}$ are called <u>finite-time Lyapunov exponents (FTLE's)</u>.

- We are now going to go to the ref. frame rotated so that $\hat{M} = \hat{\Lambda}$ and wait long enough so that $\zeta_i(t) \sim 2\lambda_i t$.
- If you find the above construction hard to ~~understand~~ digest, think of the following <u>model problem</u>:

11.03.05

$$\hat{\sigma} = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix} \Rightarrow \frac{\partial x^i}{\partial x_0^m} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & e^{\lambda_3 t} \end{pmatrix}$$

Lecture 22

$$\frac{\partial x_0^r}{\partial x^l} = \begin{pmatrix} e^{-\lambda_1 t} & & 0 \\ & e^{-\lambda_2 t} & \\ 0 & & e^{-\lambda_3 t} \end{pmatrix}$$

Incompressibility:

$$\mathrm{tr}\,\hat{\sigma} = \nabla\cdot\vec{u} = 0 \Rightarrow \lambda_1 + \lambda_2 + \lambda_3 = 0$$

order $\lambda_1 \geq \lambda_2 \geq \lambda_3$, so if $\underset{\text{stretching}}{\lambda_1 > 0}$, $\underset{\text{compression}}{\lambda_3 < 0}$

[Zeldovich et al. (1986)]

Then

$$\hat{M} = \frac{\partial x^i}{\partial x_0^m}\frac{\partial x^i}{\partial x_0^n} = \begin{pmatrix} e^{2\lambda_1 t} & & 0 \\ & e^{2\lambda_2 t} & \\ 0 & & e^{2\lambda_3 t}\end{pmatrix}, \quad \hat{M}^{-1} = \frac{\partial x_0^r}{\partial x^l}\frac{\partial x_0^s}{\partial x^l} = \begin{pmatrix} e^{-2\lambda_1 t} & & 0 \\ & e^{-2\lambda_2 t} & \\ 0 & & e^{-2\lambda_3 t}\end{pmatrix}$$

Substitute this into (10) and (11):

$$\tilde{k}^2(t) = k_{01}^2 e^{-2\lambda_1 t} + k_{02}^2 e^{-2\lambda_2 t} + \underbrace{k_{03}^2 e^{-2\lambda_3 t}}_{\text{fastest growth}}$$

$$\tilde{B}^2(t) = \Big(\underbrace{B_{01}^2 e^{2\lambda_1 t}}_{\text{fastest growth, dominates the rest}} + B_{02}^2 e^{2\lambda_2 t} + B_{03}^2 e^{2\lambda_3 t}\Big) e^{-2\eta\int_0^t dt'\,\tilde{k}^2(t')} \approx$$

$$\approx B_{01}^2 e^{2\lambda_1 t - 2\eta\int_0^t dt'\,\tilde{k}^2(t')} = B_{01}^2 e^{2\underbrace{\left[\lambda_1 - \frac{\eta}{t}\int_0^t dt'\,\tilde{k}^2(t')\right]}_{>0 \text{ for dynamo to work}} t}$$

$$\lambda_1 > \frac{\eta}{t}\int_0^t dt'\,\tilde{k}^2(t') = \frac{\eta}{t}\left[\frac{k_{01}^2}{2\lambda_1}\big(1-\underbrace{e^{-2\lambda_1 t}}_{\to 0}\big) + \frac{k_{02}^2}{2\lambda_2}\big(1-\underbrace{e^{-2\lambda_2 t}}_{\to ?}\big) + \frac{k_{03}^2}{2\lambda_3}\big(1-\underbrace{e^{-2\lambda_3 t}}_{= e^{2|\lambda_3| t} \to \infty}\big)\right]$$

$$\approx \frac{\eta}{t}\left[\frac{k_{01}^2}{2\lambda_1} + \frac{k_{02}^2}{2\lambda_2}\big(1-e^{-2\lambda_2 t}\big) + \frac{k_{03}^2}{2|\lambda_3|\, e^{-2|\lambda_3| t}}\right]$$

~~Suppose $\lambda_2 > 0$ (usually the case in small-scale turbulence). Then~~

~~(12) gives us~~

$$\cancel{k_{03}^2 < 2|\lambda_3|\, e^{-2|\lambda_3| t}\Big[\underbrace{\frac{\lambda_1 t}{\eta}}_{\text{large and grows with time}} - \underbrace{\frac{k_{01}^2}{2\lambda_1} - \frac{k_{02}^2}{2\lambda_2}}_{\text{finite}}\Big]} \quad \cancel{(13)}$$

We can rewrite this so:

$$\frac{k_{01}^2}{2\lambda_1^2 t/\eta} + \frac{k_{02}^2}{2\lambda_1\lambda_2 t\,(1-e^{-2\lambda_2 t})^{-1}/\eta} + \frac{k_{03}^2}{2\lambda_1|\lambda_3|t\, e^{-2|\lambda_3|t}/\eta} \lesssim 1 \quad (12)$$

Recall that

$\langle B^2\rangle = \int\frac{d^3k_0}{(2\pi)^3}\tilde{B}^2$, so (12) gives a volume in k_0 space (space of initial plane waves), which contributes growing modes to this integral.

Suppose $\lambda_2 > 0$ (it usually is in real turbulence). Then

$$\frac{k_{01}^2}{2\lambda_1^2 t/\eta} + \frac{k_{02}^2}{2\lambda_1\lambda_2 t/\eta} + \frac{k_{03}^2}{2\lambda_1|\lambda_3|t\, e^{-2|\lambda_3|t}/\eta} \lesssim 1$$

This gives

$$\langle B^2\rangle \sim \underbrace{\sqrt{\frac{2\lambda_1^2 t}{\eta}}}_{\text{finite in }k_{01}}\ \underbrace{\sqrt{\frac{2\lambda_1\lambda_2 t}{\eta}}}_{\text{finite in }k_{02}}\ \underbrace{\sqrt{\frac{2\lambda_1|\lambda_3|t}{\eta}}\, e^{-\overbrace{|\lambda_3|}^{\lambda_3=-\lambda_1-\lambda_2}t}}_{\text{shrinking exp-lly in }k_{03}} \cdot B_{01}^2 e^{2\lambda_1 t} =$$

$$= \underbrace{\frac{2^{3/2}\lambda_1^2\sqrt{\lambda_2|\lambda_3|}\,t^{3/2}}{\eta^{3/2}}}_{\text{never mind these factors.}} B_{01}^2 e^{(\lambda_1-\lambda_2)t} \qquad \boxed{\text{grows at rate } \lambda_1-\lambda_2}$$

NB: ~~Incompressibility~~ Solenoidality: $B_{01}k_{01} + B_{02}k_{02} + B_{03}k_{03} = 0$

so $B_{03} = -\dfrac{B_{01}k_{01} + B_{02}k_{02}}{k_{03}} \sim (\dots)\, e^{|\lambda_3|t}$ for the contributing modes

but $B_{03}^2 e^{2\lambda_3 t} \sim (\dots)\, e^{2\lambda_3 t - 2\lambda_3 t} \sim (\dots) \ll B_{01}^2 e^{2\lambda_1 t}$,

So it was OK to neglect B_{03}

For completeness, let us also consider

$\lambda_2 < 0$. Then (12) becomes:

$$\frac{k_{01}^2}{2\lambda_1^2 t/\eta} + \frac{k_{02}^2}{2\lambda_1|\lambda_2| t\, e^{-2|\lambda_2|t}/\eta} + \frac{k_{03}^2}{2\lambda_1|\lambda_3| t\, e^{-2|\lambda_3|t}/\eta} \lesssim 1$$

Solenoidality:

$$B_{01} = -\frac{B_{02}k_{02} + B_{03}k_{03}}{k_{01}} \simeq -B_{02}\sqrt{\frac{|\lambda_2|}{|\lambda_1|}}\, e^{-|\lambda_2|t} \overset{\text{smaller}}{\cancel{- B_{03}\sqrt{\frac{|\lambda_3|}{|\lambda_1|}}\, e^{-|\lambda_3|t}}}$$

(contributing $\vec{k}_0$ is almost $\parallel \vec{e}_1$, but $\vec{B}_0 \perp \vec{k}_0$, so contributing $\vec{B}_0$ have small B_{01}). Therefore

$$\langle B^2\rangle \sim \sqrt{\frac{2\lambda_1^2 t}{\eta}}\sqrt{\frac{2\lambda_1|\lambda_2|t}{\eta}}\sqrt{\frac{2\lambda_1|\lambda_3|t}{\eta}}\, e^{\overbrace{-|\lambda_2|t-|\lambda_3|t}^{=-\lambda_1 t}}\, B_{02}^2\, e^{\overbrace{-2|\lambda_2|t}^{=2\lambda_2 t}}\, e^{2\lambda_1 t} =$$

$$= \frac{2^{3/2}\lambda_1^2\sqrt{|\lambda_2||\lambda_3|}\, t^{3/2}}{\eta^{3/2}}\, B_{02}^2\, e^{\underbrace{(\lambda_1+2\lambda_2)}_{=\lambda_1+\lambda_2+\lambda_2=\lambda_2-\lambda_3=|\lambda_3|-|\lambda_2|>0} t} \qquad \boxed{\text{grows at rate } |\lambda_3|-|\lambda_2|}$$

$\lambda_2 = 0$ (reversible flow). Then in (12), we have

$$\frac{k_{01}^2}{2\lambda_1^2 t/\eta} + \frac{k_{02}^2}{\lambda_1/\eta} + \frac{k_{03}^2}{2\lambda_1|\lambda_3| t\, e^{\underbrace{-2|\lambda_3|t}_{=\lambda_3 t=-\lambda_1 t}}/\eta} \lesssim 1$$

$$\langle B^2\rangle \sim \sqrt{\frac{2\lambda_1^2 t}{\eta}}\sqrt{\frac{\lambda_1}{\eta}}\sqrt{\frac{2\lambda_1|\lambda_3|t}{\eta}}\, e^{-|\lambda_3|t}\, B_{01}^2\, e^{2\lambda_1 t} =$$

$$= \frac{2\lambda_1^2\sqrt{|\lambda_3|}\, t}{\eta^{3/2}}\, B_{01}^2\, e^{\lambda_1 t} \qquad \boxed{\text{grows at rate } \lambda_1 - \overset{=0}{\lambda_2}}$$

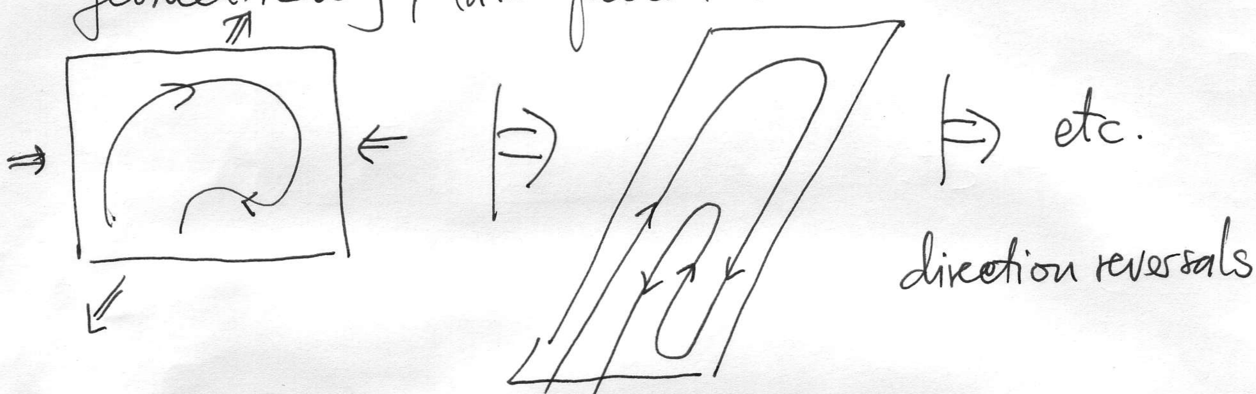
(situation similar to $\lambda_2 > 0$)

Let us analyse what has happened for $\lambda_2 \geq 0$.

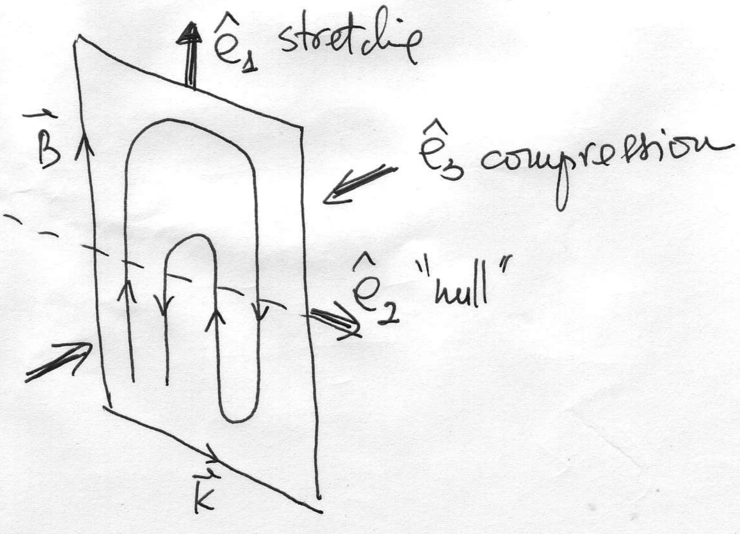
During stretching, magnetic field aligns preferentially with the stretching direction: $\vec{B} \sim \hat{e}_1 B_{01} e^{\lambda_1 t}$.

The vector $\vec{k}$ mostly wants to align with the compression direction: $\vec{k} \sim \hat{e}_3 e^{|\lambda_3| t}$, which makes most modes decay superexponentially. The only ones that don't are those that had their initial $\vec{k}_0$ almost $\perp \hat{e}_3$, so that $k_{03} < (\dots) e^{-|\lambda_3| t}$ (exponentially narrowing deviation of angle between $\vec{k}_0$ and $\hat{e}_3$ from $90°$). But $\vec{B}_0 \perp \vec{k}_0$, so the modes that got stretched most had $\vec{k}_0 \sim k_{02} \hat{e}_2$.

Geometrically, this process occurs so:



and the winning configurations are



(Basically, any other alignment means that each time you stretch, you also compress so antiparallel fields annihilate)

This makes it clear why we can't have dynamo in 2D: $\hat{e}_2$ is compression and, as the field is stretched along $\hat{e}_1$, it must reverse direction along $\hat{e}_2$ (perfect cancellation).

For completeness, let's do the Zeldovich et al. calculation in 2D: we have $\lambda_1 + \lambda_2 = 0$, so

$$k^2(t) = k_{01}^2 e^{-2\lambda_1 t} + k_{02}^2 e^{2\lambda_1 t}$$

$$\tilde{B}^2(t) = (\underbrace{B_{01}^2 e^{2\lambda_1 t}}_{\text{dominates}} + B_{02}^2 e^{-2\lambda_1 t}) e^{-2\eta\left[\frac{k_{01}^2}{2\lambda_1}(1-\underbrace{e^{-2\lambda_1 t}}_{\to 0}) + \frac{k_{02}^2}{2\lambda_1}(\underbrace{e^{2\lambda_1 t}}_{\to\infty}-1)\right]}$$

$$\approx B_{01}^2 e^{2\left[\lambda_1 - \frac{\eta}{2\lambda_1 t}\left(k_{01}^2 + \frac{k_{02}^2}{e^{-2\lambda_1 t}}\right)\right]t}$$

$$[\dots] > 0 \text{ if } \frac{k_{01}^2}{2\lambda_1^2 t/\eta} + \frac{k_{02}^2}{2\lambda_1^2 t\, e^{-2\lambda_1 t}/\eta} \lesssim 1$$

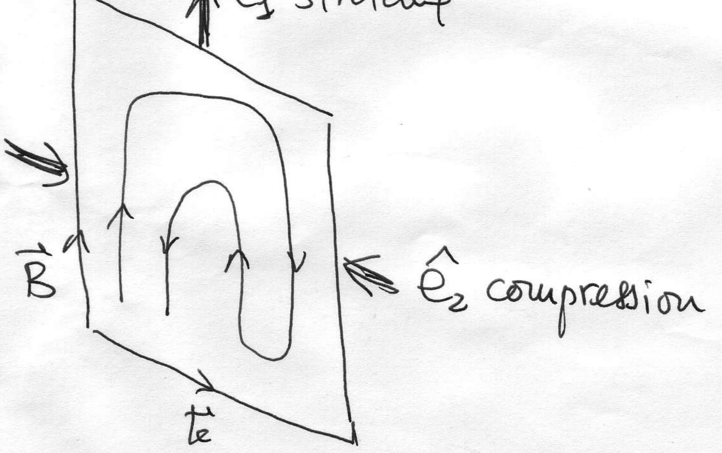
initial fields must be $\perp \vec{k}_0$, which must be almost $\perp \hat{e}_2$

Solenoidality: $B_{01} = -\frac{k_{02}}{k_{01}} B_{02} \sim -e^{-\lambda_1 t} B_{02}$

So,

$$\langle B^2 \rangle \sim \frac{2\lambda_1^2 t}{\eta} e^{-\lambda_1 t} B_{02}^2 \cancel{e^{-2\lambda_1 t}} \cancel{e^{2\lambda_1 t}}$$

$\boxed{\text{decays at rate } \lambda_1}$



See work by Ott et al. (1980's–1990's) for an alternative formalism in terms of cancellation exponents and Lyapunov exponents.