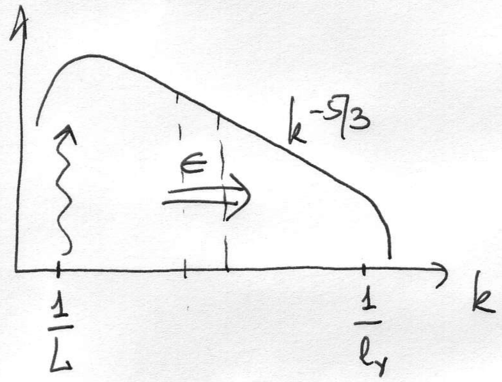
§21. ~~Anisotropic~~ Anisotropic MHD Turbulence: An Overview of Theoretical Uncertainties.

Recall that in pure hydro, we have $+\vec{B}\cdot\nabla\vec{B}$ for MHD

$$\partial_t \vec{u} + \vec{u}\cdot\nabla\vec{u} = -\nabla p + \nu\nabla^2\vec{u} + \vec{f}\,, \quad \nabla\cdot\vec{u} = 0$$



- Scale invariance (I)
- Locality of energy transfer in k space (II)

$$\Downarrow$$

$$\epsilon \sim \delta u_l^2 \frac{1}{\tau_l} \sim \text{const}$$

τ_l ← cascade time

Dimensionally, only one time scale associated with each spatial scale: "eddy-turnover time"

$$\tau_l \sim \frac{l}{\delta u_l} \Rightarrow \epsilon \sim \frac{\delta u_l^3}{l} \sim \text{const} \Rightarrow \boxed{\delta u_l \sim (\epsilon l)^{1/3}}$$

K41

Kolmogorov spectrum was a dimensional result!

Now bring in the magnetic field:

$$\partial_t \vec{B} + \vec{u}\cdot\nabla\vec{B} = \vec{B}\cdot\nabla\vec{u} + \eta\nabla^2\vec{B}$$

- Let us consider first a situation, where there is a strong externally imposed field $\vec{B}_0$. (III) anisotropy

— Recall that in such MHD, finite-amp. Alfvén waves, or Alfvén-wave packets are exact solutions, ~~...~~ and only counterpropagating packets ~~...~~ contribute to the interaction term:

$$\partial_t \vec{Z}_\pm \mp V_A \nabla_{\parallel} \vec{Z}_\pm + \vec{Z}_\mp \cdot \nabla \vec{Z}_\pm = -\nabla p + \frac{\nu+\eta}{2} \nabla^2 \vec{Z}_\pm + \frac{\nu-\eta}{2} \nabla^2 \vec{Z}_\mp$$

$$\vec{Z}_\pm = \vec{u} \pm \delta\vec{B} \ , \ \delta\vec{B} = \vec{B} - \vec{B}_0 \ , \ V_A = \frac{B_0}{\sqrt{4\pi\rho}} \ , \ \nabla_\parallel = \frac{\partial}{\partial z} \ , \ \vec{B}_0 = B_0 \hat{z}$$

$$\left.\begin{array}{l} \vec{Z}_- = 0 \ , \ \vec{Z}_+ = \vec{f}(x, y, z + V_A t) \\ \vec{Z}_+ = 0 \ , \ \vec{Z}_- = \vec{g}(x, y, z - V_A t) \end{array}\right\} \text{exact solutions.}$$

- Based on that, assume that MHD turbulence consists of Alfvén-wave packets, with interactions due to the counterpropagating packets.

So, everything is in an Alfvénic state: $\delta u_\ell \sim \delta B_\ell$ scale-by-scale (IV) $\Rightarrow$ same spectra for $\vec{u}$ and $\vec{B}$.

Can we repeat the K41 argument?

$$\epsilon \sim \delta u_\ell^2 \frac{1}{\tau_\ell} \sim \text{const}$$

but we can't fix τ_ℓ because two time scales are available:

- "eddy-turnover time": $\tau_{eddy} \sim \frac{l_\perp}{\delta u_\ell}$ (or interaction time)
- Alfvén time: $\tau_A \sim \frac{l_\parallel}{V_A}$

So we can't fix scalings solely by dimensional analysis!

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1) <u>Iroshnikov (1964) – Kraichnan (1965) Theory</u>.

- Assume weak interactions:

$$\left| \vec{Z}_\mp \cdot \nabla \vec{Z}_\pm \right| \ll \left| V_A \nabla_\parallel \vec{Z}_\pm \right| \qquad (\underline{V})$$

$$\frac{\delta u_\ell}{l_\perp} \ll \frac{V_A}{l_\parallel} \quad \text{or} \quad \tau_{eddy} \gg \tau_A$$

interaction time   Alfvén time

This means each interaction decorrelates the wave only a bit (they pass through each other much faster than the time needed to decorrelate them completely):

- time to pass $\Delta t \sim \dfrac{l_\parallel}{V_A} \sim \tau_A$
- amplitude gets a small kick:

$$\Delta \delta u_l \sim \frac{\delta u_l^2}{l}\Delta t \sim \delta u_l \frac{\delta u_l}{l}\frac{l_\parallel}{V_A} \sim \delta u_l \frac{\tau_A}{\tau_{eddy}}$$

- the kicks are random ($\pm$), so they add up as a random walk:

$$\sum^t \Delta \delta u_l \sim \delta u_l \frac{\tau_A}{\tau_{eddy}} \underbrace{\sqrt{\frac{t}{\tau_A}}}_{\sqrt{N} \leftarrow \#\text{ of kicks in time } t}$$

- cascade time is s.t. the sum of kicks ~ amplitude itself:

$$t \sim \tau_l \iff \sum^t \Delta \delta u_l \sim \delta u_l$$

$$\delta u_l \frac{\tau_A}{\tau_{eddy}}\sqrt{\frac{\tau_l}{\tau_A}} \sim \delta u_l \implies \boxed{\tau_l \sim \frac{\tau_{eddy}^2}{\tau_A}} \sim \frac{l_\perp^2 V_A}{\delta u_l^2 l_\parallel}$$

So

$$\epsilon \sim \delta u_l^2 \cdot \frac{\delta u_l^2 l_\parallel}{V_A l_\perp^2} \sim \text{const} \implies \boxed{\delta u_l \sim (\epsilon V_A)^{1/4} l_\perp^{1/2} l_\parallel^{-1/4}} \quad (*)$$

- Finally, IK assumed that at $l \ll L$, fluctuations are <u>isotropic</u> VI : $l_\parallel \sim l_\perp$. This gives

$$\boxed{\delta u_l \sim (\epsilon V_A)^{1/4} l^{1/4}} \quad \text{or} \quad \boxed{E(k) \sim (\epsilon V_A)^{1/2} k^{-3/2}}$$

<u>IK spectrum</u>.

The weak-interaction assumption: $\quad \dfrac{(\epsilon v_A)^{1/4}}{v_A}\, l_\perp^{-1/2} l_\parallel^{3/4}$

$$\frac{\tau_A}{\tau_{eddy}} \sim \frac{l_\parallel}{v_A}\frac{\delta u_l}{l_\perp} \sim \frac{l_\parallel}{v_A}(\epsilon v_A)^{1/4} l_\perp^{-1/2} l_\parallel^{-1/4} \sim$$

$\delta u_L \sim (\epsilon v_A)^{1/4} L^{1/4}$, so

$$\frac{\tau_A}{\tau_{eddy}} \sim \frac{\delta u_L}{v_A}\left(\frac{L}{l_\perp}\right)^{1/2}\left(\frac{l_\parallel}{L}\right)^{3/4} \underset{(**)}{\overset{l_\parallel \sim l_\perp}{\sim}} \frac{\delta u_L}{v_A}\left(\frac{l}{L}\right)^{1/4}$$

small for $l \ll L$ provided $\delta u_L < v_A$ (and gets better at smaller scales!)

So, everything appears self-consistent, assuming

(I) scale inv (II) locality (III) strong field (IV) $\delta u_l \sim \delta B_l$ (VI) $l_\perp \sim l_\parallel$

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2) This was thought to be the right theory until astro observations started showing that spectra are closer to $k^{-5/3}$ than to $k^{-3/2}$ (NB: $\frac{5}{3} = 1.66\ldots$, $\frac{3}{2} = 1.5$ – tough to tell apart!)

Numerical simulations show that $l_\perp \sim l_\parallel$ is <u>not</u> satisfied, so we must reexamine isotropy.

Go back to interactions of counterpropagating waves:

$$\omega(\vec{k}) = \pm k_\parallel v_A$$

Energy: $\underbrace{\omega(\vec{k}_1)}_{+\text{wave}} + \underbrace{\omega(\vec{k}_2)}_{-\text{wave}} = \omega(\vec{k}_3) \Rightarrow k_{\parallel 1} - k_{\parallel 2} = \pm k_{\parallel 3}$

Momentum: $\vec{k}_1 + \vec{k}_2 = \vec{k}_3 \qquad \Rightarrow k_{\parallel 1} + k_{\parallel 2} = k_{\parallel 3}$

$\Rightarrow$ $\overset{\ominus}{k_{\parallel 1}}$ or $\overset{\oplus}{k_{\parallel 2}} = 0$ say $k_{\parallel 2} = 0$ $\Downarrow$ $k_{\parallel 1} = k_{\parallel 3}$

Thus, actually,

- interaction mediated by $k_\parallel = 0$ modes
- $k_\parallel$ does not change in the interaction (no cascade in $k_\parallel$!)

Weak Turbulence.

Let's quickly implement this: instead of (VI) $l_\perp \sim l_\parallel$, let's assume (VI') $l_\parallel \sim L$ no cascade. Then

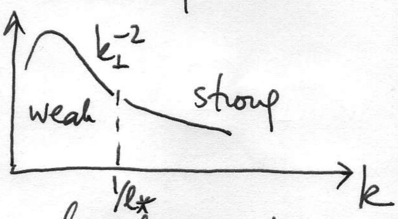
NB

from (*): $\boxed{\delta u_l \sim \left(\frac{\epsilon v_A}{L}\right)^{1/4} l_\perp^{1/2}}$ or $\boxed{E(k_\perp) \sim \left(\frac{\epsilon v_A}{L}\right)^{1/2} k_\perp^{-2}}$

These spectra can, in fact be obtained via a somewhat more formal procedure of expanding in small amplitudes. But is the weak-interaction assumption still self-consistent?

From (**), $\frac{\tau_A}{\tau_{eddy}} \sim \frac{\delta u_L}{v_A}\left(\frac{L}{l_\perp}\right)^{1/2}$ grows with decreasing $l_\perp$!

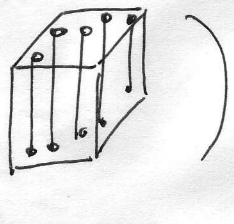
$\frac{\tau_A}{\tau_{eddy}} \sim 1$ when $l_\perp \sim l_* = L\left(\frac{\delta u_L}{v_A}\right)^2 \sim L^{3/2}\epsilon^{1/2} v_A^{-3/2}$



Thus, if $\delta u_L \ll v_A$, weak turbulence might work down to scale l_*, but then it breaks down because $\tau_A \sim \tau_{eddy}$.

Footnote 2a). Sridhar–Goldreich (1994) Theory.

What if 3-wave interactions are not allowed (empty) – e.g. because the boundary conditions such that $k_\parallel = 0$ modes do not exist (e.g. field lines are nailed down to the wall:



) Still sticking to the weak-interaction assumption, consider 4-wave interactions:

$\omega(\vec{k}_1) + \omega(\vec{k}_2) = \omega(\vec{k}_3) + \omega(\vec{k}_4) \Rightarrow k_{\parallel 1} - k_{\parallel 2} = k_{\parallel 3} - k_{\parallel 4}$

$\vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4 \Rightarrow k_{\parallel 1} + k_{\parallel 2} = k_{\parallel 3} + k_{\parallel 4}$

NB: if $k_{\parallel 3} + k_{\parallel 4}$, then $k_{\parallel 2} = 0$; if $-k_{\parallel 3} - k_{\parallel 4}$, then $k_{\parallel 1} = 0$

This gives $k_{\parallel 1} = k_{\parallel 3}$, $k_{\parallel 2} = k_{\parallel 4}$ – again no cascade in $k_\parallel$, parallel momentum is simply exchanged (elastic collisions between "waves" of equal "mass"). We must redo the dimensional derivation:

– kick in amplitude in one interaction:

$$\Delta \delta u_\ell \sim \frac{d^2 \delta u_\ell}{dt^2} \Delta t^2 \sim \frac{d}{dt}\left(\frac{\delta u_\ell^2}{\ell}\right) \cdot \tau_A^2 \sim \frac{\delta u_\ell}{\ell}\left(\frac{d}{dt}\delta u_\ell\right)\tau_A^2 \sim$$

$$\sim \frac{\delta u_\ell}{\ell}\frac{\delta u_\ell^2}{\ell}\tau_A^2 \sim \delta u_\ell \left(\frac{\tau_A}{\tau_{eddy}}\right)^2$$ – second order in $\frac{\tau_A}{\tau_{eddy}}$ instead of first

[in weak turbulence, in fact, n-wave interactions give amplitude changes $\frac{\Delta \delta u_\ell}{\delta u_\ell} \sim \left(\frac{\tau_A}{\tau_{eddy}}\right)^n$].

– cascade time:

$$\delta u_\ell \sim \sum^{\tau_\ell} \Delta \delta u_\ell \sim \delta u_\ell \left(\frac{\tau_A}{\tau_{eddy}}\right)^2 \sqrt{\frac{\tau_\ell}{\tau_A}} \Rightarrow \tau_\ell \sim \frac{\tau_{eddy}^4}{\tau_A^3} \sim \frac{l_\perp^4 V_A^3}{\delta u_\ell^4 l_\parallel^3}$$

So

$$\epsilon \sim \delta u_\ell^2 \frac{\delta u_\ell^4 l_\parallel^3}{l_\perp^4 V_A^3} \sim \text{const} \Rightarrow \delta u_\ell \sim \epsilon^{1/6} V_A^{1/2} l_\perp^{2/3} l_\parallel^{-1/2}$$

No cascade in $k_\parallel$: $l_\parallel \sim L \Rightarrow$ $\boxed{E(k_\perp) \sim \epsilon^{1/3} \frac{V_A}{L} k_\perp^{-7/3}}$

$$\frac{\tau_A}{\tau_{eddy}} \sim \frac{l_\parallel}{V_A} \epsilon^{1/6} V_A^{1/2} l_\perp^{-1/3} l_\parallel^{-1/2} \sim \frac{\delta u_L}{V_A}\left(\frac{L}{l_\perp}\right)^{1/3} \underbrace{\left(\frac{l_\parallel}{L}\right)^{1/2}}_{\lesssim 1}$$ again grows at small $l_\perp$

$\delta u_L \sim \epsilon^{1/6} V_A^{1/2} L^{1/6}$

~ 1 when $l_\perp \sim L\left(\frac{\delta u_L}{V_A}\right)^3$

so weak interaction assumption breaks down eventually

<u>end of footnote</u>

3) Goldreich - Sridhar (1995) Theory.

- Assumption: $\boxed{\tau_{eddy} \sim \tau_A}$ – strong interactions
  critical balance. (VI)

(in other words $\vec{z}_- \cdot \nabla \vec{z}_+ \sim v_A \nabla_\parallel \vec{z}_+$)

This removes the ambiguity in the dimensional theory: again, at each scale, we only have one char. time.
So $\tau_\ell \sim \frac{\ell_\perp}{\delta u_\ell}$ and we are back to the K41 result but for the $\perp$ spectrum:

$$\epsilon \sim \delta u_\ell^2 \frac{\delta u_\ell}{\ell_\perp} \Rightarrow \delta u_\ell \sim (\epsilon \ell_\perp)^{1/3}, \quad \boxed{E(k_\perp) \sim \epsilon^{2/3} k_\perp^{-5/3}}$$

GS spectrum

Back to critical balance:

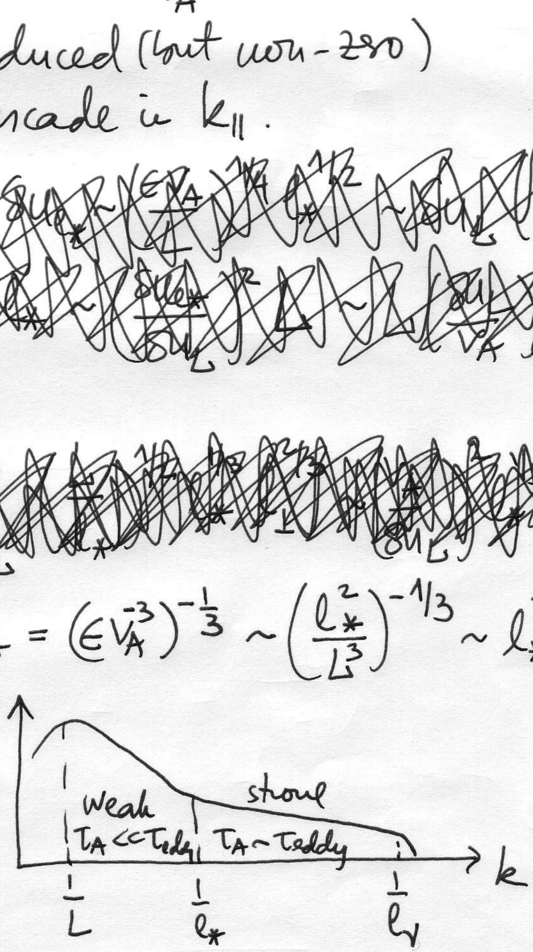
$$\tau_{eddy} \sim \frac{\ell_\perp}{\delta u_\ell} \sim \epsilon^{-1/3} \ell_\perp^{2/3} \sim \tau_A \sim \frac{\ell_\parallel}{v_A}$$

So $\boxed{\ell_\parallel \sim \epsilon^{-1/3} v_A \, \ell_\perp^{2/3}}$ – reduced (but non-zero) cascade in $k_\parallel$.

NB: $\ell_* \sim L^{3/2} \epsilon^{1/2} v_A^{-3/2} \Rightarrow \epsilon^{-1/3} v_A = (\epsilon v_A^{-3})^{-1/3} \sim \left(\frac{\ell_*^2}{L^3}\right)^{-1/3} \sim \ell_*^{-2/3} L$

$$\boxed{\ell_\parallel \sim L \left(\frac{\ell_\perp}{\ell_*}\right)^{2/3}}$$

"GS cone" $k_\parallel \sim L^{-1} (\ell_* k_\perp)^{2/3}$



— Let's work out the viscous scale (assuming $\eta \le \nu$, otherwise this is resistive scale):

$$\epsilon \sim \frac{\delta u_\ell^3}{l_\perp} \sim \nu \frac{\delta u_\ell^2}{l_\perp^2} \sim \nu \frac{(\epsilon l_\perp)^{2/3}}{l_\perp^2} \Rightarrow l_\nu \sim \frac{\nu^{3/4}}{\epsilon^{1/4}} \text{ as before}$$

but now $$l_\nu \sim Re^{-3/4} \frac{\delta u_L^{3/4} L^{3/4}}{\epsilon^{1/4}} \sim Re^{-3/4} \left(\frac{v_A}{\delta u_L}\right)^{1/4} L$$

$$\epsilon^{1/4} \sim \frac{\delta u_L}{v_A^{1/4} L^{1/4}}$$

Let's make sure the interval $l_* \gg l_\perp \gg l_\nu$ is non-empty: $l_* \sim L\left(\frac{\delta u_L}{v_A}\right)^2 \gg l_\nu \sim Re^{-3/4}\left(\frac{v_A}{\delta u_L}\right)^{1/4} L$

so $\left(\frac{\delta u_L}{v_A}\right)^{\frac{9}{4}} \gg Re^{-\frac{3}{4}}$ $\qquad$ $\underset{\uparrow \text{ weak non-empty}}{1 \gg} \frac{\delta u_L}{v_A} \underset{\uparrow \text{ GS non-empty}}{\gg Re^{-\frac{1}{3}}}$

In other words, to see strong turbulence, we must have

$$\boxed{Re \gg \left(\frac{v_A}{\delta u_L}\right)^3}$$

so it does matter in which order the limits $Re \to \infty$ and $\frac{v_A}{\delta u_L} \to \infty$ (str. field) are taken!

— Parallel cascade: $\delta u_\ell \sim (\epsilon l_\perp)^{1/3} \sim \epsilon^{1/3} l_\parallel^{1/2} \epsilon^{1/6} v_A^{-1/2}$

$$\boxed{\delta u_\ell \sim \left(\frac{\epsilon}{v_A}\right)^{1/2} l_\parallel^{1/2}} \quad \text{or} \quad \boxed{E(k_\parallel) \sim \frac{\epsilon}{v_A} k_\parallel^{-2}}$$

simply $\epsilon \sim \delta u_\ell^2 \frac{v_A}{l_\parallel}$

This is roughly the state of affairs now. Numerical simulations do not quite confirm GS theory: the spectrum seems closer to $k_\perp^{-3/2}$ but everything is definitely anisotropic.