§4 ~~§20~~. Intermittency

Lecture 18. 4.03.05

MOS: Lecture 6 19.10.05

We have discussed 2-order statistics (dimensionally)

$$\delta u \sim (\epsilon l)^{1/3} \iff E(k) \sim \epsilon^{2/3} k^{-5/3} \quad \text{K. spectrum}$$

and 3-order statistics (exactly)

$$S_{LLL}(y) = -\frac{4}{5}\epsilon y \quad \text{K.'s } \tfrac{4}{5} \text{ law}$$

Now we'd like to know n-th order statistics:

define structure functions (longitudinal)

$$S_n(y) = \langle |\delta u_L|^n \rangle \sim y^{\zeta_n} \quad \text{for } l_\nu \ll y \ll L$$

expect scaling behavior in the inertial range (scaling invariance!)

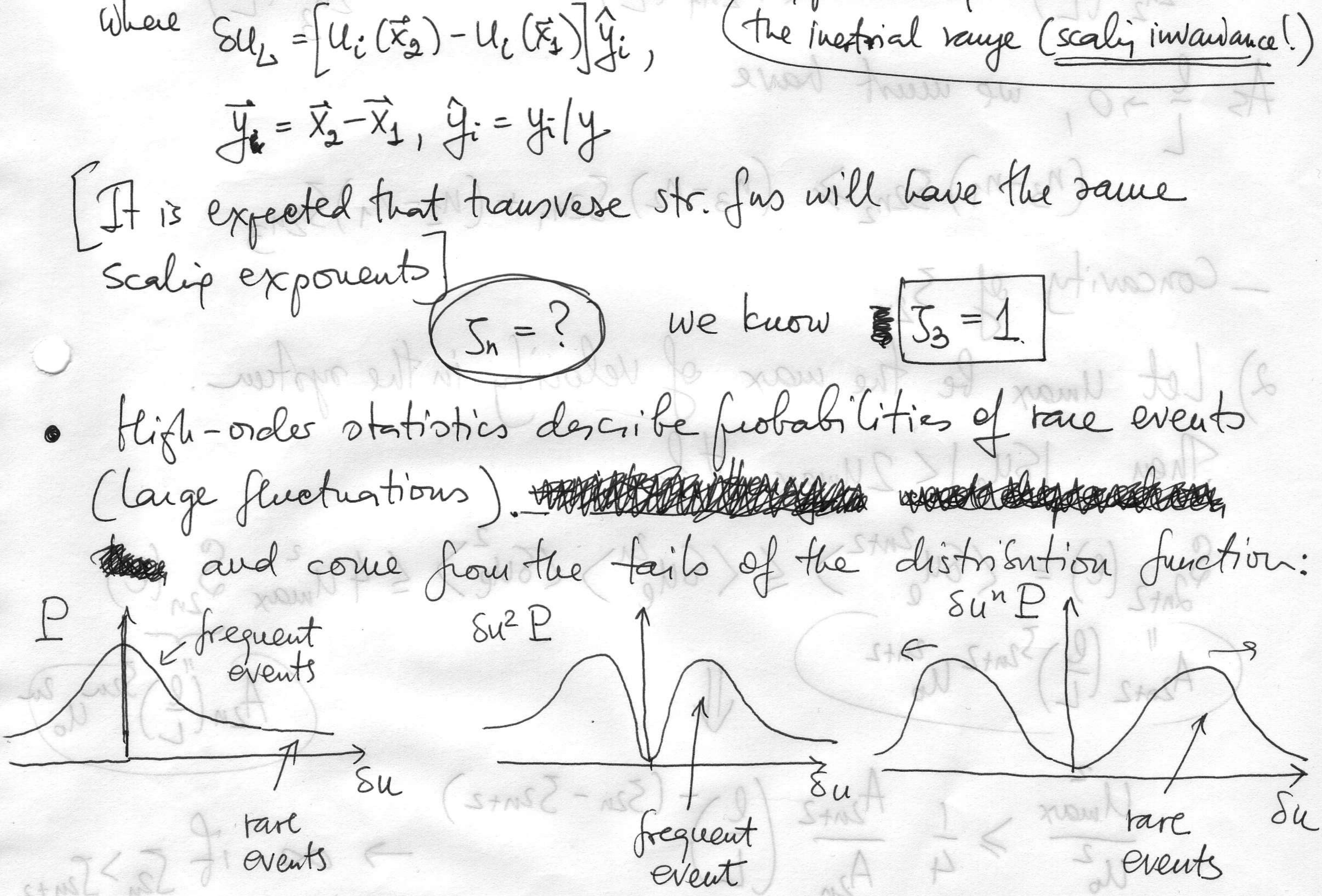
where $\delta u_L = [u_i(\vec{x}_2) - u_i(\vec{x}_1)]\hat{y}_i$,

$\vec{y} = \vec{x}_2 - \vec{x}_1$, $\hat{y}_i = y_i / y$

[It is expected that transverse str. fns will have the same scaling exponents]

$\zeta_n = ?$ we know $\boxed{\zeta_3 = 1}$

- High-order statistics describe probabilities of rare events (large fluctuations). and come from the tails of the distribution function:



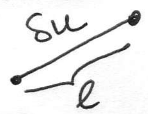
NB: Hard to measure (takes a long time / high resolution).

rare in time (time averages) (spatial averages) rare in space

Note: It is not obvious that $S_n(y)$ ~~are~~ exist ($<\infty$) for all n: if $P(\delta u)$ has a power tail, $S_n(y)$ will exist only up to some finite n. However, experiments show that $P(\delta u)$ is probably ~ exponential (stretched) ($\sim e^{-\delta u_\ell^\alpha}$, $\alpha<1$)

- What can we say about ζ_n?

This depends on the assumptions we make.
In K41 theory, self-similarity of the velocity differences in the inertial range is assumed:
mathematically, this means



$$\delta u_{\lambda \ell} = \lambda^h \delta u_\ell$$

so $S_n(\lambda\ell) = \langle |\delta u_{\lambda\ell}|^n \rangle = \lambda^{hn} \langle |\delta u_\ell|^n \rangle = \lambda^{hn} S_n(\ell)$

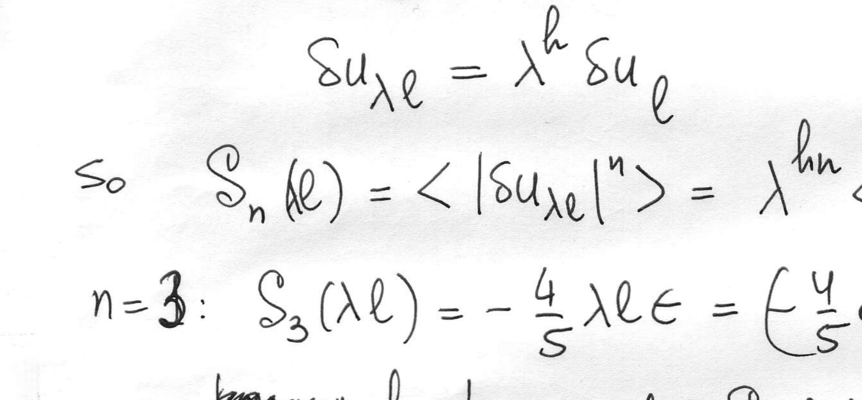
$n=3$: $S_3(\lambda\ell) = -\frac{4}{5}\lambda\ell\epsilon = \left(-\frac{4}{5}\epsilon\ell\right)\lambda = \lambda S_3(\ell)$ exact

so $h=\frac{1}{3}$ and $S_n(\lambda\ell) = \lambda^{\frac{n}{3}} S_n(\ell)$

This means that $\boxed{\zeta_n = \frac{n}{3}}$ K41

Measurements show that

deviations increase with n.



Rigourously, one can prove that
1) ζ_n vs. n is concave
2) ζ_n vs. n is non-decreasing
(Frisch § 8.4)

Proof: 1) From Hölder inequality,

$$S_{2n_2}(l)^{2n_3-2n_1} = \langle \delta u_l^{2n_2} \rangle^{2n_3-2n_1} \le$$

$$\le \langle \delta u_l^{2n_1} \rangle^{2n_3-2n_2} \langle \delta u_l^{2n_3} \rangle^{2n_2-2n_1} =$$

$$= S_{2n_1}(l)^{2n_3-2n_2} S_{2n_3}(l)^{2n_2-2n_1}$$

$$\left(\text{NB: } 2n_1(\cancel{2n_3}-2n_2) + 2n_3(2n_2-\cancel{2n_1}) = 2n_2(2n_3-2n_1) \right)$$

If $S_{2n} = A_{2n}\left(\frac{l}{L}\right)^{\zeta_{2n}} \underbrace{u_0^{2n}}_{\text{some velocity}}$ (in the inertial range, $Re \to \infty$)

$$A_{2n_2}\left(\frac{l}{L}\right)^{(2n_3-2n_1)\zeta_{2n_2}} \le A_{2n_1}\left(\frac{l}{L}\right)^{(2n_3-2n_2)\zeta_{2n_1}} A_{2n_3}\left(\frac{l}{L}\right)^{(2n_2-2n_1)\zeta_{2n_3}}$$

As $\frac{l}{L} \to 0$, we must have

$$(n_3-n_1)\zeta_{2n_2} \ge (n_3-n_2)\zeta_{2n_1} + (n_2-n_1)\zeta_{2n_3}$$

— Concavity of ζ_{2n}

2) Let u_{max} be the max of velocity in the system.

Then $|\delta u_l| < 2u_{max} \quad \forall l$

$$\underbrace{S_{2n+2}(l)}_{= A_{2n+2}\left(\frac{l}{L}\right)^{\zeta_{2n+2}} u_0^{2n+2}} = \langle \delta u_l^{2n+2} \rangle \le \langle \delta u_l^{2n} \rangle \langle \delta u_l^{2} \rangle \le 4 u_{max}^2 \underbrace{S_{2n}(l)}_{= A_{2n}\left(\frac{l}{L}\right)^{\zeta_{2n}} u_0^{2n}}$$

$$\Downarrow$$

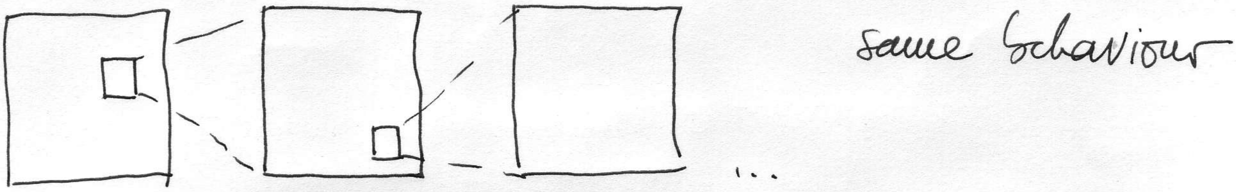
$$\frac{u_{max}^2}{u_0^2} \ge \frac{1}{4} \frac{A_{2n+2}}{A_{2n}} \left(\frac{l}{L}\right)^{-(\zeta_{2n}-\zeta_{2n+2})} \to \infty \text{ if } \zeta_{2n} > \zeta_{2n+2}$$

Thus, if ζ_{2n} is not monotonically increasing, u_{max} blows up.

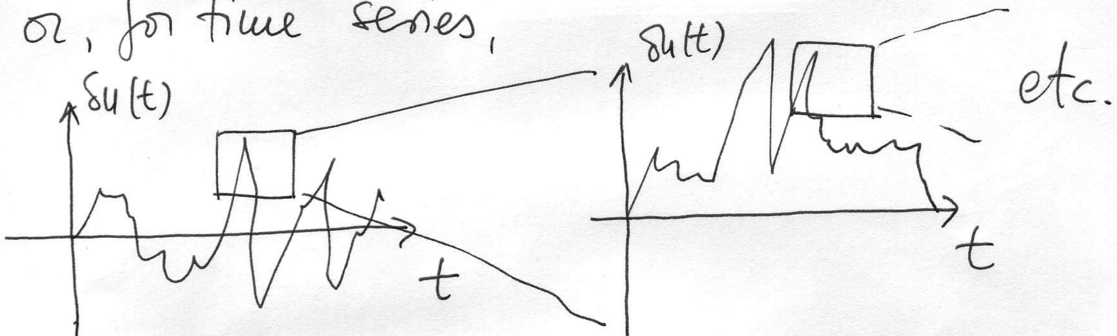
$\Rightarrow \zeta_{2n}$ should be monotonically increasing.

- So something is wrong with the assumption of self-similarity.

Self-similarity means that if we blow up small regions of the system, we see the same picture:



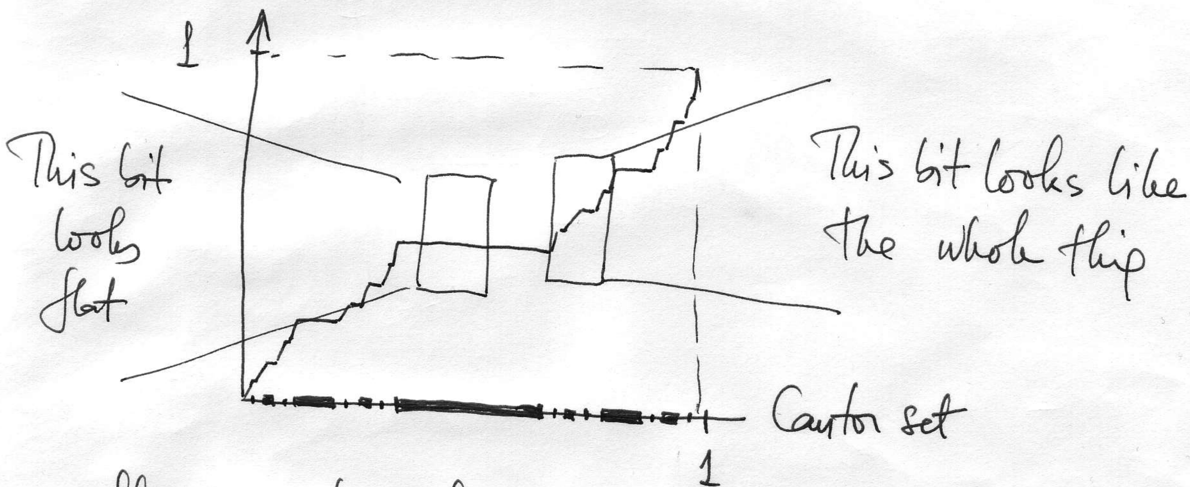
or, for time series,



This is true, e.g., for a random walk.

But this is not the only possible fluctuating behaviour.

E.g.: <u>Devil's staircase</u>



The smaller are the fragments we are blowing up, the less the probability that it won't be flat!

Such non-self-similar behaviour is called <u>intermittency</u>.

- Intermittency is related to the volume (or time) filling property of the field: in a crude way, suppose that $\delta u_\ell \neq 0$ only in a fraction χ_ℓ of the volume (or during fraction χ_ℓ of the time) — in general, this fraction may be different for different scales. Then

$$S_n(\ell) \sim \langle |\delta u_\ell|^n \rangle \sim \delta u_\ell^n \cdot \chi_\ell^{(n)} \sim (\epsilon \ell)^{n/3} \chi_\ell^{(n)}$$

NB: $\chi_\ell^{(n)}$ depends on n!

But (scale invariance!) $\chi_\ell^{(n)} \sim \left(\frac{\ell}{L}\right)^{\mu_{n/3}}$, so

$\mu_{n/3}$ — some exponent (must depend also on n, otherwise $\zeta_3 = 1 \Rightarrow \mu = 0$)

$$S_n(\ell) \sim \epsilon^{n/3} L^{-\mu_{n/3}} \ell^{\frac{n}{3} + \mu_{n/3}} \sim \ell^{\zeta_n},$$

so $\boxed{\zeta_n = \frac{n}{3} + \mu_{n/3}}$ and we have to find $\mu_{n/3}$.

- Where does this lack of volume filling come from? Oboukhov and Kolmogorov (1962) proposed to consider the following quantity:

$$\epsilon_\ell(\vec{x}) = \frac{1}{\underbrace{\frac{4}{3}\pi \ell^3}_{\text{vol. of the sphere of radius } \ell}} \int_{|\vec{x}' - \vec{x}| < \ell} d^3x' \, \nu \underbrace{|\nabla \vec{u}|^2}_{\text{or, more generally, } \frac{1}{2}\hat{S}:\hat{S},\ \hat{S} = \nabla\vec{u} + (\nabla\vec{u})^T} (\vec{x}')$$

— indep. of $\vec{x}$ (homogeneity)

Clearly,

Batchelor & Townsend 1947

$$\langle \epsilon_\ell(\vec{x}) \rangle = \epsilon \quad \text{Kolmogorov flux.}$$

NB: "Landau's objection" 1944

Experimentally, $\epsilon_\ell(\vec{x})$ is very intermittent (dissipation occurs in vortex filaments, in a small fraction of the volume):

Frisch §6.4

Then, dimensionally, at scale l,

$\delta u_l \sim (\epsilon_l l)^{1/3}$ as we argued before, but now ϵ_l is average dissipation at scale l (within a volume of radius l).

From scale invariance, we must have

$$\langle \epsilon_l^m \rangle = \epsilon^m \left(\frac{l}{L}\right)^{\mu_m}$$ (i.e., roughly, dissipation is occurring in a fraction $l^{(m)}$ of the volume)

where μ_m is some set of exponents and $\mu_1 = 0$ to ensure $\langle \epsilon_l \rangle = \epsilon$.

Then

$$S_n(l) \sim \langle \delta u_l^n \rangle \sim \langle (\epsilon_l l)^{n/3} \rangle = \langle \epsilon_l^{n/3} \rangle l^{n/3} =$$
$$= \epsilon^{n/3} L^{-\mu_{n/3}} l^{\frac{n}{3} + \mu_{n/3}}$$

So, again we have

$$\boxed{\zeta_n = \frac{n}{3} + \mu_{n/3}}$$

where $\mu_{n/3}$ are scaling exponent of ϵ_l.

This is called refined similarity hypothesis.

- Now we shall try to construct some model for the dissipation field, so we can calculate $\mu_{n/3}$. For this we need to revisit our understanding of the turbulent cascade.

- Take a box the size of the system L.
  The mean rate of energy dissipation there is $\langle \epsilon_L \rangle = \epsilon$.
- Now let's divide it into smaller boxes of size αL $(\alpha < 1)$. The mean dissipation rate in each of these boxes varies (intermittency!). Let us model this variation by saying that

$$\epsilon_{\alpha L} = \epsilon W_1 \qquad \rightarrow \langle \epsilon_{\alpha L} \rangle = \epsilon$$

where W_1 is a random variable $W_1 \geq 0$, $\langle W_1 \rangle = 1$.
- Divide these boxes in turn: $\alpha^2 L$ and let

$$\epsilon_{\alpha^2 L} = \epsilon_{\alpha L} W_2 = \epsilon W_1 W_2 \quad , \quad \langle \epsilon_{\alpha^2 L} \rangle = \epsilon$$

where W_2 is indep. of W_1 and distributed the same way.
- Iterate this procedure, so for boxes of size $l = \alpha^k L$

$$\epsilon_l = \epsilon W_1 W_2 \dots W_k \quad , \quad \langle \epsilon_l \rangle = \epsilon$$

Then

$$\langle \epsilon_l^m \rangle = \epsilon^m \underbrace{\langle W^m \rangle^k}_{\alpha^{k \log_\alpha \langle W^m \rangle} = \left(\frac{l}{L}\right)^{\log_\alpha \langle W^m \rangle}} \quad , \text{ where } k = \log_\alpha \frac{l}{L}$$

So, $$\langle \epsilon_l^m \rangle = \epsilon^m \left(\frac{l}{L}\right)^{\mu_m} \quad , \quad \boxed{\mu_m = \log_\alpha \langle W^m \rangle = \frac{\ln \langle W^m \rangle}{\ln \alpha}}$$

Note that the K41/Richardson cascade corresponded to $W = 1$ (non-random). To model intermittency we must think of how W should be distributed.

- Note that our scheme for dividing the boxes (determined by the parameter α) should not influence the physics. Thus, if we took α^2 or α^r instead of α, everything should work the same way. I.e. products $W_1 W_2$ or $W_1 \dots W_r$ should have the same distribution as W_1.

Let us define $w_i = \ln W_i$, so that

$$\epsilon_\ell = \epsilon\, e^{\sum_{i=1}^{k} w_i} \longleftarrow \text{indep., identically distributed}$$

Legitimate choices for w_i are ~~random~~ random variables that have the property of <u>infinite divisibility</u>, i.e. a sum of w_i's has the same distribution as w_i with $\left\langle \sum_{i=1}^{k} w_i \right\rangle = k\langle w_i\rangle = k\overline{w}$.

- <u>The lognormal model</u>. An obvious candidate is a Gaussian distribution:

$$P(w) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(w-\overline{w})^2}{2\sigma^2}}$$

so that the sum $\sum_{i=1}^{k} w_i$ is also Gaussian with mean $= k\overline{w}$ and variance $k\sigma^2$.

(<u>NB</u>: This is <u>not</u> an application of the Central Limit Theorem, which says that $\dfrac{\sum_{i=1}^{k} w_i - k\overline{w}}{\sqrt{k}}$ is Gaussian as $k\to\infty$ for arbitrarily distributed w_i, but <u>not</u> $\sum_{i=1}^{k} w_i$!)

Thus, ϵ_ℓ is lognormal (proposed by Oboukhov 1962).

Let's compute moments:

$$\langle W^m\rangle = \int dw \, \frac{1}{\sqrt{2\pi\sigma^2}} \, e^{\underbrace{mw - \frac{(w-\bar{w})^2}{2\sigma^2}}} = e^{m\bar{w} + \sigma^2 m^2/2}$$

$$\frac{2\sigma^2 wm - w^2 + 2w\bar{w} - \bar{w}^2}{2\sigma^2} =$$

$$= \frac{-w^2 + 2w(\bar{w}+\sigma^2 m) - (\bar{w}+\sigma^2 m)^2 + 2\sigma^2 m\bar{w} + \sigma^4 m^2}{2\sigma^2} =$$

$$= -\frac{(w-\bar{w}-2\sigma^2 m)^2}{2\sigma^2} + m\bar{w} + \sigma^2 m^2 \frac{1}{2}$$

Requirement: $\langle W\rangle = 1$, so $\bar{w} + \sigma^2/2 = 0$

$$\Downarrow$$

$$\bar{w} = -\sigma^2/2$$

$\mu > 0$ because $\bar{w} = -\frac{\sigma^2}{2} < 0$ and $\ln\alpha < 0, \alpha < 1$

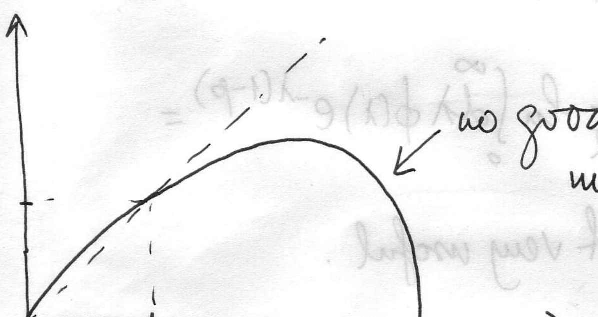
So, $\langle W^m\rangle = e^{\bar{w} m(1-m)}$, whence

$$\mu_m = \frac{\bar{w}\, m(1-m)}{\ln\alpha} \equiv +\frac{\mu}{2} m(1-m), \text{ where } \mu = 2\frac{\bar{w}}{\ln\alpha}$$

is a free parameter.

Finally,

$$\boxed{\zeta_n = \frac{n}{3} + \mu_{n/3} = \frac{n}{3} + \frac{\mu}{18} n(3-n)}$$



no good because ζ_n must be monotonic (otherwise δu blows up in the limit $l \to 0$ – see Frisch §8.4

- The She-Lévêque Model. (main papers: see links on the course blog page)

Another possible infinitely divisible distribution is the Poisson distribution: let

$$w = r \ln\beta + w_0 \;, \quad \text{where } r \text{ is an integer}$$

β and w_0 are free parameters

and r is Poisson-distributed:

$$\mathbb{P}(r) = \frac{\lambda^r}{r!} e^{-\lambda} \;, \quad \sum_{r=0}^{\infty} \mathbb{P}(r) = 1 \;, \quad \langle r \rangle = \lambda .$$

Then ϵ_ℓ is log-Poisson. Calculate moments:

$$\langle W^m \rangle = \sum_{r=0}^{\infty} e^{m(r\ln\beta + w_0) - \lambda} \frac{\lambda^r}{r!} = e^{m w_0 - \lambda} \sum_{r=0}^{\infty} \frac{\left(\lambda e^{m \ln\beta}\right)^r}{r!} =$$

$$= e^{m w_0 - \lambda + \lambda e^{m\ln\beta}} = e^{m w_0 - \lambda(1-\beta^m)}$$

$$\langle W \rangle = 1 \Rightarrow w_0 - \lambda(1-\beta) = 0 \Rightarrow \lambda = \frac{w_0}{1-\beta}$$

So $\langle W^m \rangle = e^{w_0\left(m - \frac{1-\beta^m}{1-\beta}\right)}$, whence

$$\mu_m = \frac{w_0}{\ln\alpha}\left(m - \frac{1-\beta^m}{1-\beta}\right) \equiv -x\left(m - \frac{1-\beta^m}{1-\beta}\right) \;,$$

$x = -\frac{w_0}{\ln\alpha}$, and β are free parameters.

$$\boxed{\zeta_n = \frac{n}{3} + \mu_{n/3} = (1-x)\frac{n}{3} + x\,\frac{1-\beta^{n/3}}{1-\beta}} \qquad (1)$$

NB: in order for ζ_n to be monotonic, $\underline{\beta < 1}$.

In order to find x and β, we follow the original construction of She & Lévêque:

introduce
$$\epsilon_\ell^{(m)} = \frac{\langle \epsilon_\ell^{m+1}\rangle}{\langle \epsilon_\ell^{m}\rangle} = \frac{\epsilon^{m+1}(\ell/L)^{\mu_{m+1}}}{\epsilon^{m}(\ell/L)^{\mu_m}} =$$

$$= \epsilon\left(\frac{\ell}{L}\right)^{\mu_{m+1}-\mu_m} = \epsilon\left(\frac{\ell}{L}\right)^{-\cancel{xm}-x+x\frac{1-\beta^{m+1}}{1-\beta}+\cancel{xm}-x\frac{1-\beta^{m}}{1-\beta}} =$$

$$= \epsilon\left(\frac{\ell}{L}\right)^{-x(1-\beta^{m})} \Rightarrow \boxed{\epsilon_\ell^{(\infty)} = \epsilon\left(\frac{\ell}{L}\right)^{-x}} \qquad (2)$$

Physically, $\epsilon_\ell^{(m)}$ are dominated by the tails of the distribution of ϵ_ℓ, with $\epsilon_\ell^{(\infty)}$ representing the most extreme dissipation events.

Footnote: The original SL paper did not assume (log) Poisson distribution, but simply postulated

(I)
$$\epsilon_\ell^{(m+1)} = A_m\left[\epsilon_\ell^{(m)}\right]^{\beta}\left(\epsilon_\ell^{(\infty)}\right)^{1-\beta} \qquad (3)$$

Then Dubrulle (1994) introduced normalised diss. rates

$$\Pi_\ell = \frac{\epsilon_\ell}{\epsilon_\ell^{(\infty)}}$$, for which

$$\langle \Pi_\ell^{m+2}\rangle = \frac{\langle \epsilon_\ell^{m+2}\rangle}{[\epsilon_\ell^{(\infty)}]^{m+2}} \overset{\text{def. of}}{=} \frac{\langle \epsilon_\ell^{m+1}\rangle\, \epsilon_\ell^{(m+1)}}{[\epsilon_\ell^{(\infty)}]^{m+2}} =$$

$$= \frac{\langle \epsilon_\ell^{m+1}\rangle A_m\left[\epsilon_\ell^{(m)}\right]^{\beta}\left(\epsilon_\ell^{(\infty)}\right)^{1-\beta}}{[\epsilon_\ell^{(\infty)}]^{m+2}} \overset{\text{def. of}}{=} A_m\frac{\langle \epsilon_\ell^{m+1}\rangle^{1+\beta}\left[\epsilon_\ell^{(\infty)}\right]^{1-\beta}}{\langle \epsilon_\ell^{m}\rangle^{\beta}\left[\epsilon_\ell^{(\infty)}\right]^{m+2}} =$$

$$= A_m \frac{\langle \Pi_e^{m+1}\rangle^{1+\beta} \left[\epsilon_e^{(\infty)}\right]^{\overbrace{(m+1)(1+\beta)+1-\beta}^{=m+m\beta+1+\beta+1-\beta}}}{\langle \Pi_e^{m}\rangle^{\beta} \left[\epsilon_e^{(\infty)}\right]^{m\beta+m+2}} = A_m \frac{\langle \Pi_e^{m+1}\rangle^{1+\beta}}{\langle \Pi_e^{m}\rangle^{\beta}}$$

This is a recursion relation. Seek solution in the form

$\langle \Pi_e^m \rangle = B_m\, a^{\gamma_m}$. This gives

$$B_{m+2}\, a^{\gamma_{m+2}} = A_m \frac{B_{m+1}^{1+\beta}}{B_m^{\beta}}\, a^{(1+\beta)\gamma_{m+1}-\beta\gamma_m}$$

Fix $\langle \Pi_e \rangle = B_1 a^{\gamma_1} = \dfrac{\langle \epsilon_e \rangle}{\epsilon_e^{(\infty)}} = \dfrac{\epsilon}{\epsilon_e^{(\infty)}}$. Then

$$a = \left(\frac{\langle \Pi_e \rangle}{B_1}\right)^{1/\gamma_1} \quad \text{and}$$

$$\gamma_{m+2} = (1+\beta)\gamma_{m+1} - \beta\gamma_m$$

Solution: $\gamma_m = \gamma_1 \dfrac{1-\beta^m}{1-\beta}$. Check:

$$1-\beta^{m+2} = (1+\beta)(1-\beta^{m+1}) - \beta(1-\beta^m) = 1+\cancel{\beta} - \cancel{\beta^{m+1}} - \beta^{m+2} - \cancel{\beta} + \cancel{\beta^{m+1}}$$
$$= 1-\beta^{m+2} \quad \text{OK}$$

So, $$\langle \Pi_e^m \rangle = B_m \left(\frac{\langle \Pi_e \rangle}{B_1}\right)^{\frac{1-\beta^m}{1-\beta}},$$

where $$\frac{B_{m+2} B_m^{\beta}}{B_{m+1}^{1+\beta}} = A_m$$ the recursion relation for coefficients.

This gives $$\langle \epsilon_e^m \rangle = B_m \left[\epsilon_e^{(\infty)}\right]^m \left[\frac{\overbrace{\langle \epsilon_e \rangle}^{=\epsilon}}{B_1 \epsilon_e^{(\infty)}}\right]^{\frac{1-\beta^m}{1-\beta}} =$$

$$= B_m\left[\frac{\epsilon}{B_1}\right]^{\frac{1-\beta^m}{1-\beta}}\left[\epsilon_\ell^{(\infty)}\right]^{m-\frac{1-\beta^m}{1-\beta}} = B_m B_1^{-\frac{1-\beta^m}{1-\beta}}\epsilon^m\left[\frac{\epsilon_\ell^{(\infty)}}{\epsilon}\right]^{m-\frac{1-\beta^m}{1-\beta}}$$

But, from (1),

from (2)

NB: in the derivation based on (3), one postulates (2) with some x and then recovers the μ_m in (1)

$$\langle\epsilon_\ell^m\rangle = \epsilon^m\left(\frac{\ell}{L}\right)^{-x\left(m-\frac{1-\beta^m}{1-\beta}\right)} = \epsilon^m\left[\frac{\epsilon_\ell^{(\infty)}}{\epsilon}\right]^{m-\frac{1-\beta^m}{1-\beta}},$$

So ~~[illegible]~~ our log-Poisson case is a particular case of Dubrulle's solution with all $B_m = 1$ or of the assumption (3) holding with $A_m = 1$.

[The case of general set of B_m's corresponds to a generalized Poisson distribution in the form

~~[illegible] where $\sum_r$ [illegible]~~

~~[illegible]~~:

$$P(r) = \int ds\,\frac{\lambda^s}{\Gamma(s+1)}e^{-\lambda}G(r-s), \text{ where } \int dr\,G(r)=1,\ G>0$$

an arbitrary pdf.]

end of footnote.

Now SL ~~[illegible]~~ assumed that (their 2nd assumption)

(II) $$\epsilon_\ell^{(\infty)} \sim \frac{U^2}{\tau_\ell} \sim U^2\epsilon^{1/3}\ell^{-2/3} \sim \epsilon\left(\frac{\ell}{L}\right)^{-2/3},$$

i.e. all the energy dissipates in the most singular structures and the cascade time has K41 dimensional scaling.

$$\Downarrow$$

$$\boxed{x = \frac{2}{3}} \quad (4)$$

Now let's find β: the dissipation rate in the most singular dissipative structures is

$$\epsilon_\ell \sim \epsilon_\ell^{(\infty)} = \epsilon\left(\frac{\ell}{L}\right)^{-x}$$

(diss. of δu_ℓ due to most sing. structure)

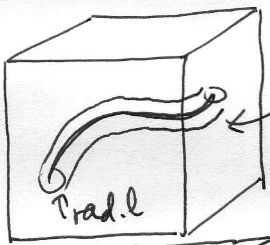
Let us then write for $m \gg 1$ (high-order moments):

$$\langle \epsilon_\ell^m \rangle \sim \left[\epsilon_\ell^{(\infty)}\right]^m \chi_\ell$$

filling fraction of the most singular structures

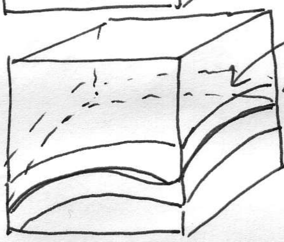
But $\chi_\ell \sim \left(\frac{\ell}{L}\right)^{d-D}$, where d is dim. of space and D is the dimension of the diss. structures. Indeed χ_ℓ is the probability for a ball of radius ℓ to intersect with a diss. structure.

$D=1$: $d=3$



volume $\sim \ell^2 L$, $\chi_\ell \sim \frac{\ell^2 L}{L^3} \sim \left(\frac{\ell}{L}\right)^{2 = 3-1}$

$D=2$: $d=2$



volume $\sim \ell L^2$, $\chi_\ell \sim \frac{\ell L^2}{L^3} \sim \left(\frac{\ell}{L}\right)^{1 = 3-2}$

etc.

Thus, $\langle \epsilon_\ell^m \rangle \sim \left[\epsilon_\ell^{(\infty)}\right]^m \chi_\ell \sim \epsilon^m \left(\frac{\ell}{L}\right)^{-xm + \underline{d-D}}$

compare

On the other hand, we know

$$\langle \epsilon_\ell^m \rangle \sim \epsilon^m \left(\frac{\ell}{L}\right)^{\mu_m}, \quad \mu_m = -xm + x\frac{1-\beta^m}{1-\beta} \approx -xm + \frac{x}{1-\beta}$$

when $m \gg 1$

This gives $\boxed{\dfrac{x}{1-\beta} = d - D \equiv C}$ – codimension of the most singular structures (5)

Combining (1) and (5), we have

$$\boxed{\zeta_n = (1-x)\frac{n}{3} + C\left[1-\left(1-\frac{x}{C}\right)^{\frac{n}{3}}\right]}$$

This is the general form of the SL formula:

- x is the scaling exponent of the cascade time taken from dimensional theory

C is the codimension of the most singular (diss.) structures.

For incompressible turbulence: $x = \frac{2}{3}$ (K41 cascade time) and $C = 2$ ($D = 1$, vortex filaments).

This gives $\boxed{\zeta_n = \dfrac{n}{9} + 2\left[1-\left(\dfrac{2}{3}\right)^{\frac{n}{3}}\right]}$ SL formula

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Note: one can derive analogous results for other kinds of turbulence. E.g. Boldyrev (2002) argued that for supersonic turbulence,

$x = \frac{2}{3}$ (Kolmogorov cascade in the inertial range)

$C = 1$ ($D = 2$, dissipation in shocks),

so he got

$$\zeta_n = \frac{n}{9} + 1 - \left(\frac{1}{3}\right)^{\frac{n}{3}}$$

SL formula has been tremendously successful: it just seems to fit the data almost too well!

Although both mathematical (Novikov's so-called objection) and physical (why log-Poisson?) uncertainties remain.

When measurements (or simulations) are done with not too large Re, so that scaling is not very good, the following empirical property of the structure functions helps:

- Extended Self-Similarity (ESS) (Benzi et al. 1993).

Plot structure functions $S_n(\ell)$ not vs. ℓ, but vs. $S_3(\ell)$ [or, indeed, any other str. fn with fixed n_0].

Clearly, $S_n(\ell) \sim \ell^{\zeta_n}$, $S_3(\ell) \sim \ell$, so, in the inertial range, we should have

$$\ln S_n(\ell) \sim \zeta_n \ln S_3(\ell) + \mathrm{const}_n$$

$$\text{or} \quad \zeta_n = \frac{\partial \ln S_n(\ell)}{\partial \ln \ell} = \frac{\partial \ln S_n(\ell)}{\partial \ln S_3(\ell)}$$

It turns out (experimentally and numerically) that the second equality gives a very good approx. to ζ_n even when Re is not huge. This is because deviations from scaling (arising from finite Re) in $S_n(\ell)$ of different orders n are, in fact, correlated and cancel when $S_n(\ell)$ is measured vs. $S_3(\ell)$. [To my knowledge, there is no theory on why this trick works.]