Part ~~I~~ II. Magnetohydrodynamics

§5 ~~§1~~ MHD Equations

I will give a quick derivation based on postulating the fluid equations (which should be familiar from undergrad. courses on fluid dynamics)

① Continuity equation (mass conservation)

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0$$

② Navier-Stokes equation (momentum conservation)

$$\underbrace{\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right)}_{\frac{d\vec{u}}{dt}} = -\underbrace{\nabla p}_{\text{pressure}} + \underbrace{\nabla\cdot\hat{\Pi}}_{\text{viscous stress}} + \underbrace{\vec{F}}_{\text{all other forces}} \quad \left(\text{per unit volume}\right)$$

$\frac{d}{dt}$: Lagrangian (conv.) derivative

— Here $\hat{\Pi} = \mu\left(\nabla\vec{u} + (\nabla\vec{u})^T - \frac{2}{3}\nabla\cdot\vec{u}\,\mathbb{1}\right)$

μ: Newtonian viscosity (bulk viscosity = 0)

$$\nabla\cdot\hat{\Pi} = \mu\left(\nabla^2\vec{u} + \frac{1}{3}\nabla\nabla\cdot\vec{u}\right) \equiv \rho\nu\left(\nabla^2\vec{u} + \frac{1}{3}\nabla\nabla\cdot\vec{u}\right)$$

ν: kinematic viscosity

— To get pressure, we need another equation. There are several schemes available for this

- Adiabatic (no heat flow): $\frac{d}{dt}\frac{p}{\rho^\gamma} = 0$ + transport/dissipation terms, $\gamma = \frac{5}{3}$ (ideal gas)
- Isothermal: $p = \frac{T}{m}\rho = \frac{\rho v_{th}^2}{2}$, $T = \text{const}$

- ***Incompressible***: $u \ll c_s = \sqrt{\frac{\gamma p}{\rho}}$ sound speed

$\rho = \text{const}$, so from ①, $\boxed{\nabla\cdot\vec{u} = 0}$

Take divergence of ②:

$$\nabla^2 p = \nabla\cdot\vec{F} - \rho\nabla\cdot(\vec{u}\cdot\nabla\vec{u})$$ equation for pressure

— Now deal with the forces.

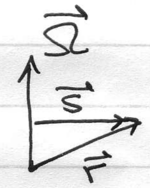
We have a conducting fluid, so

- ***Lorentz force*** $\vec{F}_L = \frac{1}{c}\vec{J}\times\vec{B}$ ($\vec{J}$: current density)

- ***Electrostatic force*** $\vec{F}_E = \rho_e\vec{E}$ (ρ_e: charge density)

Other forces that often occur are

- ***Gravity*** $\vec{F}_g = \rho\vec{g}$
- ***Coriolis*** $\vec{F}_c = -2\rho\,\vec{\Omega}\times\vec{u}$
- ***Centrifugal*** $\vec{F}_{cf} = \rho\,\Omega^2\vec{s}$



} in rotating systems

- etc.

We won't be interested in these. So:

$$\begin{cases} ① & \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0 \\ ② & \rho\frac{d\vec{u}}{dt} = -\nabla p + \rho\nu\left(\nabla^2\vec{u} + \frac{1}{3}\nabla\cdot\vec{u}\right) + \frac{1}{c}\vec{J}\times\vec{B} + \rho_e\vec{E} + \text{etc.} \\ & \text{++ equation for } p \end{cases}$$

Now we need to determine $\vec{E}$ and $\vec{B}$.

Maxwell's equations:

(3) $\nabla\cdot\vec{E} = 4\pi\rho_e$ Gauss

(4) $\nabla\cdot\vec{B} = 0$ solenoidality of $\vec{B}$

(5) $\dfrac{\partial\vec{B}}{\partial t} = -c\,\nabla\times\vec{E}$ Faraday's law (evolves $\vec{B}$)

(6) $\nabla\times\vec{B} = \dfrac{4\pi}{c}\vec{J} + \cancel{\dfrac{1}{c}\dfrac{\partial\vec{E}}{\partial t}}$ Ampère/Maxwell eqn.

Non-relativistic limit

time scales τ

spatial scales l

velocities $u \sim l/\tau \ll c$

remove Maxwell's contribution (means we filter out EM waves)

(5) $\Rightarrow$ $E \sim \dfrac{1}{c}\dfrac{l}{\tau}B \sim \dfrac{u}{c}B$

in (6): $\dfrac{\left|\frac{1}{c}\frac{\partial\vec{E}}{\partial t}\right|}{|\nabla\times\vec{B}|} \sim \dfrac{\frac{1}{c}\frac{1}{\tau}\frac{u}{c}B}{\frac{1}{l}B} \sim \dfrac{u^2}{c^2} \ll 1$

So we've got a simple equation for $\vec{J}$:

$\vec{J} = \dfrac{c}{4\pi}\nabla\times\vec{B}$ $\Rightarrow$ $\vec{F}_L = \dfrac{(\nabla\times\vec{B})\times\vec{B}}{4\pi}$

Ampère's law

in (2): $\dfrac{|\rho_e\vec{E}|}{\left|\frac{1}{c}\vec{J}\times\vec{B}\right|} \sim \dfrac{\rho_e E}{\frac{1}{l}B^2} \sim \dfrac{\cancel{\frac{1}{l}}E^2}{\cancel{\frac{1}{l}}B^2} \sim \dfrac{u^2}{c^2} \ll 1$

from (3), $\rho_e = \dfrac{\nabla\cdot\vec{E}}{4\pi}$

So electrostatic force is negligible.

see p. 5a

Magnetic forces : from ⑥

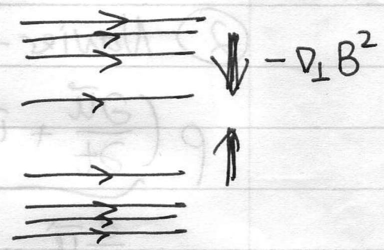
$$\vec F_L = \frac{1}{c}\vec j \times \vec B = \frac{(\nabla\times\vec B)\times\vec B}{4\pi} = \underbrace{\frac{\vec B\cdot\nabla\vec B}{4\pi}}_{\text{magnetic tension}} - \underbrace{\nabla\frac{B^2}{8\pi}}_{\text{magnetic pressure}} =$$

$\hat b = \frac{\vec B}{B}$

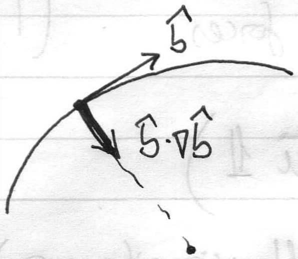
$$\vec B\hat b\cdot\nabla B\hat b = B^2\hat b\cdot\nabla\hat b + \hat b\,\nabla_\parallel\frac{B^2}{2}$$

NB: all forces $\perp \vec B$

$$= \underbrace{\frac{B^2}{4\pi}\hat b\cdot\nabla\hat b}_{\text{curvature force}} - \underbrace{\nabla_\perp\frac{B^2}{8\pi}}_{\text{magnetic pressure}}$$



field wants to be uniform



$\hat b\cdot\nabla\hat b \perp \hat b$

field lines want to straighten

Magnetic field is hard to compress and to bend.

It remains to determine $\vec{B}$: from ⑤, for which we need $\vec{E}$ in terms of known quantities.

Ohm's law: electric field of a fluid element

$$\vec{E}' = \vec{E} + \frac{1}{c}\vec{u}\times\vec{B} = \vec{J}\,\frac{1}{\sigma} \quad \leftarrow \text{conductivity} \tag{7}$$

~~magnetic resistivity~~

Substitute this into ⑤ and use Ampère's law:

$$\frac{\partial \vec{B}}{\partial t} = -c\nabla\times\left[\frac{1}{\sigma}\frac{c}{4\pi}\nabla\times\vec{B} - \frac{1}{c}\vec{u}\times\vec{B}\right] =$$

$$= \nabla\times(\vec{u}\times\vec{B}) - \frac{c^2}{4\pi\sigma}\nabla\times(\nabla\times\vec{B}) = \quad \leftarrow \text{use } \nabla\cdot\vec{B}=0$$

$$\boxed{\frac{\partial \vec{B}}{\partial t} = \nabla\times(\vec{u}\times\vec{B}) + \underbrace{\eta}_{\text{magnetic diffusivity}}\nabla^2\vec{B}} \tag{8}$$

induction equation (Hertz)

Together with ① and ②, we have a closed system.

NB: $\nabla\cdot\vec{B}=0$ must be satisfied initially, then it will always be satisfied (check!)