NOTES DISTRIBUTED

Lecture 17. 2.03.05

§3 ~~§19.~~ Kolmogorov's $\frac{4}{5}$ Law.

Now let us try to derive an equation for S_{ik} or C_{ik}.

NS Equ:

$$\partial_t u_i = -u_l \frac{\partial u_i}{\partial x_l} - \frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i + f_i$$

$$\rightarrow \partial_t C_{ik} = \partial_t \langle u_{1i} u_{2k} \rangle = \langle u_{1i} \partial_t u_{2k} \rangle + \langle u_{2k} \partial_t u_{1i} \rangle =$$

$$= \langle -u_{1i} u_{2l} \frac{\partial u_{2k}}{\partial x_{2l}} - u_{1i} \frac{\partial p_2}{\partial x_{2k}} + \nu u_{1i} \nabla_2^2 u_{2k} + u_{1i} f_{2k}$$

$$- u_{2k} u_{1l} \frac{\partial u_{1i}}{\partial x_{1l}} - u_{2k} \frac{\partial p_1}{\partial x_{1i}} + \nu u_{2k} \nabla_1^2 u_{1i} + u_{2k} f_{1i} \rangle$$

$$\rightarrow = -\frac{\partial}{\partial x_{1l}} \langle u_{1i} u_{1l} u_{2k} \rangle - \frac{\partial}{\partial x_{2l}} \langle u_{1i} u_{2l} u_{2k} \rangle -$$

$$- \frac{\partial}{\partial x_{1i}} \langle p_1 u_{2k} \rangle - \frac{\partial}{\partial x_{2k}} \langle p_2 u_{1i} \rangle$$

$$\underbrace{+ \nu \nabla_1^2 \underbrace{\langle u_{1i} u_{2k} \rangle}_{= C_{ik}} + \nu \nabla_2^2 \underbrace{\langle u_{1i} u_{2k} \rangle}_{= C_{ik}}}_{= 2\nu \nabla^2 C_{ik}(\vec{y})} + \underbrace{\langle u_{1i} f_{2k} \rangle + \langle u_{2k} f_{1i} \rangle}_{\equiv \epsilon_{ik}(\vec{y})}$$

- Forcing. For a general form of the forcing, $\epsilon_{ik}(\vec{y})$ depends on the solution. However, we expect that the character of the forcing is not going to be crucial at scales $l \ll L$. So for convenience, we chose $f_i(t, \vec{x})$ to be a Gaussian white noise:

$$\langle f_i(t, \vec{x}_1) f_k(t', \vec{x}_2) \rangle = \delta(t-t') \epsilon_{ik}(\vec{y})$$

isotropic, incompressible

This is a convenient choice because

$$\langle u_{2k} f_{1i} \rangle(t) = \langle f_{1i}(t) \int^t dt' \Big[\underbrace{-u_{2l} \frac{\partial u_{2k}}{\partial x_{2l}} - \frac{\partial p_2}{\partial x_{2k}} + \nu \nabla_2^2 u_{2k}}_{\substack{\text{all of these are at } t'<t \\ \text{and their correlation with } f_{1i}(t) \\ \text{is } \phi}} + f_{2k} \Big](t') \rangle =$$

$$= \int^t dt' \langle f_{1i}(t) f_{2k}(t') \rangle = \frac{1}{2} \epsilon_{ik}(\vec{y}) \text{, so}$$

$$\langle u_{1i} f_{2k} \rangle + \langle u_{2k} f_{1i} \rangle = \epsilon_{ik}(\vec{y})$$

Now $\epsilon_{ik} = \epsilon_{TT}(y)(\delta_{ik} - \hat{y}_i \hat{y}_k) + \epsilon_{LL}(y) \hat{y}_i \hat{y}_k$

$\longrightarrow \quad \frac{1}{2} \epsilon_{ii}(0) = \frac{1}{2}\left[(d-1)\epsilon_{TT}(0) + \epsilon_{LL}(0)\right] = \frac{d}{2} \epsilon_{LL}(0) = \epsilon$ Kolmogorov flux (energy input)

Regardless of the forcing nature, this will be

$$\epsilon = \nu \langle |\nabla \vec{u}|^2 \rangle = \text{const in st.-state}$$

White noise forcing allows one to fix ϵ as an input parameter of the system (useful in simulations)

$\boxed{\text{M05: Lecture 5 \quad 17.10.05}}$

- Pressure terms vanish.

Pf. By isotropy, $\langle p_1 u_{2k} \rangle = f(y) \hat{y}_k$ the only first-rank tensor available!

Incompressibility: $\frac{\partial}{\partial x_{2k}} \langle p_1 u_{2k} \rangle = \langle p_1 \frac{\partial u_{2k}}{\partial x_{2k}} \rangle = 0 =$

$$= \frac{\partial}{\partial y_k} f(y) \frac{y_k}{y} = f'(y) + f(y)\frac{d}{y} - f(y) \frac{y_k}{y^2} \frac{y_k}{y}$$

$$f'(y) + \frac{d-1}{y} f(y) = 0 \Rightarrow f(y) = \text{const}\, y^{-(d-1)}$$

$$f(y \to 0) < \infty \Rightarrow \text{const} = 0 \Rightarrow f(y) = 0 \quad \text{q.e.d.}$$

So, we have

$$\partial_t C_{ik} = -\frac{\partial}{\partial x_{1l}}\langle u_{1i}u_{1l}u_{2k}\rangle - \frac{\partial}{\partial x_{2l}}\langle u_{1i}u_{2l}u_{2k}\rangle$$

$$+\, 2\nu\nabla^2 C_{ik}(\vec{y}) + \epsilon_{ik}(\vec{y})$$

2-order corr. functions in terms of 3d-order ones:

$$C_{ik,l} = \langle u_i(\vec{x}_1)\, u_k(\vec{x}_1)\, u_l(\vec{x}_2)\rangle$$

– function only of $\vec{y} = \vec{x}_2 - \vec{x}_1$ because of homogeneity.

Since it is a 3d-order tensor, $C_{ik,l}(\vec{y}) = -C_{ik,l}(-\vec{y})$

$$\uparrow\ \vec{x}_2 \leftrightarrow \vec{x}_1$$

Thus,

$$\boxed{\partial_t C_{ik} = \frac{\partial}{\partial y_l}\left(C_{il,k} + C_{kl,i}\right) + 2\nu\nabla^2 C_{ik} + \epsilon_{ik}(\vec{y})} \quad (1)$$

Remark

3-order ones will be expressed in terms of 4th-order ones, etc. $\Rightarrow$ closure problem as usual.

Technically, the hierarchy of moment equations cannot be truncated because there is no small parameter in which to expand (like in kinetics).

Nevertheless, a large fraction of turbulence research has been devoted to the question of ~~correct~~ closures that would be at least qualitatively correct.

Roughly, the idea is to write

EITHER

$$\partial_t\langle uu\rangle \sim \langle uuu\rangle$$

$$\partial_t\langle uuu\rangle \sim \langle uuuu\rangle \sim \langle uu\rangle\langle uu\rangle$$

Millionshchikov hypothesis (Chandrasekhar)

Most important closures: DIA (Kraichnan)
EDQNM (Orszag)

RNG (Yakhot & Orszag) falls into the same group.

See McComb's book for a full review.

OR $\quad \partial_t \langle uu\rangle \sim \langle uuu\rangle \sim -\tau^{-1}\langle uu\rangle$ "τ-approximation"

suitably chosen corr. time
(function of y or k)

- amounts to the same thing as moment splitting, except τ^{-1} is supplied instead of being calculated from the $\langle uuu\rangle$ equation. This approach has been popular in dynamo theories (Rädler, Kleeorin, Rogachevskii, Blackman, Brandenburg...)

I will not cover either type of theories for lack of time...

• Let us go back to our exact equation (1).

I want to work out what $C_{ik,l}$ looks like, given isotropy and incompressibility.

$$C_{ik,l}(\vec y) = A(y)\,\delta_{ik}\,\hat y_l + B(y)\,\delta_{il}\,\hat y_k + C(y)\,\delta_{lk}\,\hat y_i + F(y)\,\hat y_i\,\hat y_k\,\hat y_l$$

Since $C_{ik,l} = C_{ki,l}$, $\boxed{B(y) = C(y)}$

Incompressibility: $\dfrac{\partial C_{ik,l}}{\partial x_{2l}} = \langle u_{1i}\,u_{1k}\,\nabla_2\cdot\vec u_2\rangle = 0 =$

$$= \frac{\partial C_{ik,l}}{\partial y_l} = \delta_{ik}\left(A'\frac{y_l}{y}\frac{y_l}{y} + A\frac{d}{y} - A\frac{y_l}{y^2}\frac{y_l}{y}\right) +$$

$$+ \delta_{il}\left(B'\frac{y_l}{y}\frac{y_k}{y} + B\frac{\delta_{kl}}{y} - B\frac{y_k}{y^2}\frac{y_l}{y}\right) +$$

$$+ \delta_{lk}\left(B'\frac{y_l}{y}\frac{y_i}{y} + B\frac{\delta_{il}}{y} - B\frac{y_i}{y^2}\frac{y_l}{y}\right) +$$

$$+ F'\frac{y_l}{y}\frac{y_i}{y}\frac{y_k}{y}\frac{y_l}{y} + F\frac{\delta_{il}y_ky_l + \delta_{kl}y_iy_l + d\,y_iy_k}{y^3} - 3F\frac{y_iy_ky_l}{y^4}\frac{y_l}{y} =$$

$$= \delta_{ik}\left(A' + A\frac{d}{y} - \frac{A}{y} + \frac{B}{y} + \frac{B}{y}\right) +$$

$$\hat y_i\hat y_k\left(B' - \frac{B}{y} + B' - \frac{B}{y} + F' + \frac{F}{y} + \frac{F}{y} + \frac{d}{y}F - \frac{3F}{y}\right) =$$

$$\rightarrow \; = \delta_{ik}\underbrace{\left(A' + \frac{d-1}{y}A + \frac{2}{y}B\right)}_{=0} + \hat{y}_i\hat{y}_k\underbrace{\left(2B' - \frac{2}{y}B + F' + \frac{d-1}{y}F\right)}_{=0} = 0$$

$$(2)\quad \underbrace{A' + \frac{d-1}{y}A = \frac{1}{y^{d-1}}\left(y^{d-1}A\right)' = -\frac{2}{y}B}$$

$$\frac{1}{y^{d-1}}\left(y^{d-1}F\right)' + 2\frac{1}{y^{d-1}}\left(y^{d-1}B\right)' - 2\frac{d-1}{y}B - \frac{2}{y}B = 0$$

$$\underbrace{\frac{1}{y^{d-1}}\left[y^{d-1}(F+2B)\right]' = \frac{2d}{y}B} \qquad (3)$$

add (2) and (3)

$$\left[y^{d-1}(F+2B+dA)\right]' = 0$$

Integrate: $\quad dA + 2B + F = \dfrac{const}{y^{d-1}}$

Since $C_{ik,l}(\vec{y}) = -C_{ik,l}(-\vec{y})$, $C_{ik,l}(0) = 0$ (no 3d-rank isotropic tensor that depends on nothing!)

Then $\quad A(0) = B(0) = F(0) = 0 \Rightarrow const = 0$

So $\quad \boxed{dA + 2B + F = 0} \qquad (4)$

From (2), $\boxed{B(y) = -\dfrac{1}{2}\dfrac{1}{y^{d-2}}\left(y^{d-1}A\right)'} = -\dfrac{1}{2}\left[yA' + (d-1)A\right] \qquad (5)$

From (4), $\boxed{F(y) = -dA + yA' + (d-1)A = -A + yA'} \qquad (6)$

Thus,

$$\rightarrow \boxed{\begin{aligned} C_{ik,l}(\vec{y}) = {} & A(y)\delta_{ik}\hat{y}_l - \frac{1}{2}\left[yA' + (d-1)A\right]\left(\delta_{il}\hat{y}_k + \delta_{lk}\hat{y}_i\right) + \\ & + \left(yA' - A\right)\hat{y}_i\hat{y}_k\hat{y}_l \end{aligned}} \qquad (7)$$

All coefficients of the 3-order corr. function depend one <u>one</u> scalar function.

Introduce the 3-order <u>structure function</u>:

$$\rightarrow \quad S_{ikl}(\vec y) = \langle \delta u_i\, \delta u_k\, \delta u_l \rangle, \quad \delta u_i = u_i(\vec x_2) - u_i(\vec x_1)$$

(completely symmetric wrt permutations of indices)

$$S_{ikl} = \langle (u_{2i}-u_{1i})(u_{2k}-u_{1k})(u_{2l}-u_{1l})\rangle =$$

$$= \langle \cancelto{0}{u_{2i}u_{2k}u_{2l}} - u_{2i}u_{1k}u_{2l} - u_{2i}u_{2k}u_{1l} + u_{2i}u_{1k}u_{1l}$$

$$- u_{1i}u_{2k}u_{2l} + u_{1i}u_{1k}u_{2l} + u_{1i}u_{2k}u_{1l} - \cancelto{0}{u_{1i}u_{1k}u_{1l}} \rangle =$$

$$= 2\left(C_{ik,l} + C_{il,k} + C_{lk,i}\right) \underset{\text{substitute (7)}}{=}$$

$$= 2A\delta_{ik}\hat y_l - [yA' + (d-1)A](\delta_{il}\hat y_k + \delta_{lk}\hat y_i) +$$

$$+ 2(yA' - A)\hat y_i \hat y_k \hat y_l \cdot 3 +$$

$$+ 2A\delta_{il}\hat y_k - [yA' + (d-1)A](\delta_{ik}\hat y_l + \delta_{lk}\hat y_i)$$

$$+ 2A\delta_{lk}\hat y_i - [yA' + (d-1)A](\delta_{il}\hat y_k + \delta_{ik}\hat y_l)$$

$$\rightarrow \quad = -2[yA' + (d-2)A](\delta_{ik}\hat y_l + \delta_{il}\hat y_k + \delta_{lk}\hat y_i) +$$

$$+ 6(yA' - A)\hat y_i \hat y_k \hat y_l \qquad (8)$$

$$S_{ikl}\hat y_i = -2[yA' + (d-2)A](\hat y_k \hat y_l + \hat y_l \hat y_k + \delta_{lk}) + 6(yA' - A)\hat y_k \hat y_l =$$

$$= -2[yA' + (d-2)A]\delta_{lk} + 2\Big[\underbrace{-2yA' - (2d-4)A + 3yA' - 3A}_{\substack{yA' - (2d-1)A = \\ = yA' + (d-2)A - (3d-3)A}}\Big]\hat y_k \hat y_l$$

$$= -2[yA' + (d-2)A](\delta_{lk} - \hat y_k \hat y_l) - 6(d-1)A\,\hat y_k \hat y_l \qquad (9)$$

This means that

$$\to \begin{cases} S_{LLL} \overset{\text{def}}{=} S_{ike}\,\hat{y}_i\hat{y}_k\hat{y}_e = -6(d-1)A(y) \\ S_{LTT} = -2\left[yA' + (d-2)A\right] = \frac{1}{3(d-1)y^{d-3}}\left(y^{d-2}S_{LLL}\right)' \\ S_{LLT} = S_{TTT} = 0 \end{cases}$$

$\hookrightarrow$ We will use $S_{LLL}(y)$ as the main 3-order function.

The correlation function $C_{ik,e}(\vec{y})$ is expressed in terms of $S_{LLL}(y)$ via eq. (7) and

$$A(y) = -\frac{1}{6(d-1)}S_{LLL}(y)$$

Let's go back to dynamics – eq. (1).

Find the longitudinal part of eq. (1) – all we need (all other functions expressible in terms of longitudinal ones) : $(1)_{ik}\cdot\hat{y}_i\hat{y}_k$ gives

$$\to \partial_t C_{LL} - \epsilon_{LL}(y) - 2\nu\underbrace{(\nabla^2 C_{ik})\hat{y}_i\hat{y}_k}_{=\ \frac{1}{y^{d+1}}\frac{\partial}{\partial y}y^{d+1}\frac{\partial C_{LL}}{\partial y}} =$$

Exercise: get this by using the formula from §18 for C_{ik}

$$= \frac{y_iy_k}{y^2}\frac{\partial}{\partial y_e}\left(C_{ie,k} + C_{ke,i}\right) =$$

$$= \frac{1}{y^2}\frac{\partial}{\partial y_e}\underbrace{\left(y_iy_k C_{ie,k} + y_iy_k C_{ke,i}\right)}_{=\ 2C_{LLL}\hat{y}_e y^2 = 2C_{LL}y_e y} - \left(C_{ie,k}+C_{ke,i}\right)\frac{\delta_{ie}y_k + \delta_{ke}y_i}{y^2} =$$

$$= 2C'_{LLL}y^2 + 2C_{LLL}y + 2d\,C_{LLL}y$$

$$= 2y\left[yC'_{LLL} + (d+1)C_{LLL}\right]$$

$$= 2C'_{LLL} + \frac{2(d+1)}{y} C_{LLL} - \frac{2}{y}\left(C_{Li,k}\hat{y}_k + C_{ki,i}\hat{y}_k\right), \qquad (10)$$

where $C_{LLL} = C_{ik,l}\hat{y}_i\hat{y}_k\hat{y}_l \underset{\substack{\uparrow \\ \text{eq.}(7)}}{=} \cancel{A} - \cancel{yA'} - (d-1)A + \cancel{yA'} - \cancel{A} =$

$$= -(d-1)A = \frac{1}{6} S_{LLL}(y)$$

$$C_{ii,k}\hat{y}_k = d\cancel{A} - \cancel{yA'} - \cancel{(d-1)}A + \cancel{yA'} - \cancel{A} = 0$$

$$C_{ki,i}\hat{y}_k = \cancel{A} - \frac{1}{2}\left[yA' + (d-1)A\right](1+d) + yA' - \cancel{A} =$$

$$= -\frac{d-1}{2} yA' - \frac{(d-1)(d+1)}{2} A = \frac{1}{12}\left[y S'_{LLL} + (d+1)S_{LLL}\right]$$

Substitute these into (10):

$$\to \underbrace{\partial_t C_{LL}}_{\frac{2}{d}\frac{d\mathcal{E}}{dt} - \frac{1}{2}\partial_t S_{LL}} - \epsilon_{LL}(y) - 2\nu \frac{1}{y^{d+1}}\frac{\partial}{\partial y} y^{d+1} \frac{\partial C_{LL}}{\partial y} = \frac{1}{3}S'_{LLL} + \frac{d+1}{3y} S_{LLL} -$$

$$-\frac{1}{6}S'_{LLL} - \frac{d+1}{6y} S_{LLL} = \frac{1}{6}\left[S'_{LLL} + \frac{d+1}{y} S_{LLL}\right] = \frac{1}{6y^{d+1}}\frac{\partial}{\partial y} y^{d+1} S_{LLL}$$

$\frac{\partial C_{LL}}{\partial y} \to -\frac{1}{2}S'_{LL}$

$\frac{2}{d}\frac{d\mathcal{E}}{dt} - \frac{1}{2}\partial_t S_{LL}$ ← to write everything in terms of structure functions

We get the following equation:

$$\to \boxed{\frac{\partial S_{LL}}{\partial t} = \frac{4}{d}\frac{d\mathcal{E}}{dt} - 2\epsilon_{LL}(y) - \frac{1}{3}\frac{1}{y^{d+1}}\frac{\partial}{\partial y} y^{d+1} S_{LLL} + 2\nu \frac{1}{y^{d+1}}\frac{\partial}{\partial y} y^{d+1} \frac{\partial S_{LL}}{\partial y}} \qquad (11)$$

<u>von Kármán–Howarth equation</u>

Consider steady state: $\frac{\partial}{\partial t} = 0$. Then

$$\frac{\partial}{\partial y} y^{d+1}\left[\frac{1}{3}S_{LLL} - 2\nu S'_{LL}\right] = -2\underbrace{\epsilon_{LL}(y)}_{\overset{\shortparallel}{\frac{2}{d}\epsilon + \dots}} y^{d+1} \simeq -\frac{4}{d}\epsilon y^{d+1}$$

for $y \ll L$ (outer scale)

Integrate:

$$\frac{1}{3} S_{LLL} - 2\nu S'_{LL} = -\Bigg[\frac{4\epsilon}{d(d+2)}\, y^{d+2} + \underset{\substack{\parallel \\ 0}}{\text{const}}\Bigg] \frac{1}{y^{d+1}}$$

0 so there is no singularity at $y=0$

$$\boxed{S_{LLL} = -\underbrace{\frac{12}{d(d+2)}}_{\frac{4}{5} \text{ for } d=3} \epsilon y + 6\nu \underbrace{S'_{LL}}_{\text{small when } y \gg l_\nu}} \qquad (12)$$

This gives ~~the~~ Kolmogorov's $\frac{4}{5}$ law in the inertial range:

$$\boxed{S_{LLL} \approx -\frac{4}{5}\epsilon y}$$

Note: Decaying turbulence: expect self-similar decay,

so $\frac{\partial S_{LL}}{\partial t} = 0$, no forcing $\epsilon_{LL}(y) = 0$.

Let $\frac{d\mathcal{E}}{dt} = -\epsilon$ rate of energy dissipation

Then Eq. (11) again leads to (12) and the $\frac{4}{5}$ law (that is Kolmogorov's original formulation)

Thus, we have an exact law (the only exact result for turbulence!) Cf. the dimensional result

$$(S_{LLL} \sim)\ \delta u^3 \sim \epsilon l.$$