NOTES DISTRIBUTED

Lecture 16 28.02.05

~~§2~~ §18. Correlation Functions

What does it mean "to solve turbulence"?

Well, we might describe turbulence in terms of statistics of velocity differences:

$$\delta u_i = u_i(\vec{x}_2) - u_i(\vec{x}_1)$$

- Because of homogeneity, the stat. will only depend on $\vec{y} = \vec{x}_2 - \vec{x}_1$.

→ Correlation function (2 order): $C_{ik}(\vec{y}) = \langle u_i(\vec{x}_1) u_k(\vec{x}_2) \rangle$

Structure function:

$$\to S_{ik}(\vec{y}) = \langle \delta u_i \delta u_k \rangle = \langle (u_{2i} - u_{1i})(u_{2k} - u_{1k}) \rangle =$$

$$= \underbrace{\langle u_{2i} u_{2k} \rangle}_{= C_{ik}(0)} + \underbrace{\langle u_{1i} u_{1k} \rangle}_{= C_{ik}(0)} - \underbrace{\langle u_{1i} u_{2k} \rangle}_{= C_{ik}(\vec{y})} - \underbrace{\langle u_{1k} u_{2i} \rangle}_{= C_{ki}(\vec{y})}$$

- Isotropy and (also assumed) mirror symmetry mean that any 2-rank tensor depending on $\vec{y}$ can be written as

$$C_{ik}(\vec{y}) = C_1(y)\delta_{ik} + C_2(y)\hat{y}_i \hat{y}_k \qquad y = |\vec{y}|, \quad \hat{y}_i = y_i / y$$

(if parity is broken, $\ldots + \underset{\uparrow \text{helicity}}{C_H(y)}\epsilon_{ikl}\hat{y}_l$)

$$\equiv \underset{\uparrow \text{transverse}}{C_{TT}(y)}(\delta_{ik} - \hat{y}_i\hat{y}_k) + \underset{\uparrow \text{longitudinal}}{C_{LL}(y)}\hat{y}_i\hat{y}_k$$

Any 2-rank tensor depending on nothing is

$$C_{ik}(0) = \text{const}\, \delta_{ik} \Rightarrow \text{const} = \frac{C_{ii}(0)}{d} = \frac{\langle u^2 \rangle}{d} = \frac{2}{d}\mathcal{E}$$

Thus

$$S_{ik}(\vec{y}) = \frac{2}{d}\langle u^2\rangle \delta_{ik} - 2C_{ik}(\vec{y}) =$$

$$= S_{TT}(y)(\delta_{ik} - \hat{y}_i\hat{y}_k) + S_{LL}(y)\hat{y}_i\hat{y}_k,$$

where $S_{TT}(y) = \frac{2}{d}\langle u^2\rangle - 2C_{TT}(y)$, $S_{LL}(y) = \frac{2}{d}\langle u^2\rangle - 2C_{LL}(y)$

NB: $\langle u^2\rangle = C_{ii}(0) = C_{TT}(0)(d-1) + C_{LL}(0)$

and $S_{ik}(0) = 0$

- Incompressibility imposes a constraint.

$$\to \frac{\partial C_{ik}}{\partial x_{2k}} = +\frac{\partial C_{ik}}{\partial y_k} = 0 =$$

NB: $\frac{\partial}{\partial y_k} = \frac{y_k}{y}\frac{\partial}{\partial y}$

$$= C'_{TT}\frac{y_k}{y}\underbrace{\cancel{\left(\delta_{ik} - \frac{y_i y_k}{y^2}\right)}}_{\to 0} + C'_{LL}\frac{y_k}{y}\frac{y_i y_k}{y^2} +$$

$$+ (C_{LL} - C_{TT})\left(\frac{\delta_{ik} y_k}{y^2} + \frac{y_i \cdot d}{y^2} - 2\frac{y_i y_k}{y^3}\frac{y_k}{y}\right) =$$

$$= \hat{y}_i\left[C'_{LL} + \frac{d-1}{y}(C_{LL} - C_{TT})\right] = 0$$

$$\frac{1}{y^{d-1}}\frac{\partial}{\partial y} y^{d-1} C_{LL} = \frac{d-1}{y} C_{TT}$$

von Kármán condition

$$\boxed{C_{TT}(y) = \frac{1}{(d-1)y^{d-2}}\left(y^{d-1}C_{LL}\right)' = \left(1 + \frac{1}{d-1}y\frac{\partial}{\partial y}\right)C_{LL}} \quad (1)$$

NB:
$C_{TT}(0) = C_{LL}(0)$
$\varepsilon = \frac{1}{2}C_{ii}(0) =$
$= \frac{1}{2}[(d-1)C_{TT}(0) + C_{LL}(0)]$
$= \frac{d}{2}C_{LL}(0)$

and analogous expression for S_{TT} vs. S_{LL}

Thus, second-order (2-pt) stats. ~~only~~ all contained in one scalar function: say,

$$S_{LL}(y)$$

- There is an equivalent description in k space.

$$u_i(\vec{k}) = \int d^dx\, e^{-i\vec{k}\cdot\vec{x}}\, u_i(\vec{x})$$

periodic discrete case

$$u_i(\vec{x}) = \int \frac{d^dk}{(2\pi)^d} e^{i\vec{k}\cdot\vec{x}}\, u_i(\vec{k}) \Rightarrow \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} u_i(\vec{k}) \frac{1}{L^d}$$

$$\langle u_i(\vec{k}) u_j(\vec{k}')\rangle = \int d^dx \int d^dx'\, e^{-i\vec{k}\cdot\vec{x} - i\vec{k}'\cdot\vec{x}'} \underbrace{\langle u_i(\vec{x}) u_j(\vec{x}')\rangle}_{= C_{ij}(\vec{y})} =$$

$\vec{y} = \vec{x} - \vec{x}'$

$$= \underbrace{\int d^dy\, e^{-i\vec{k}\cdot\vec{y}} C_{ij}(\vec{y})}_{= C_{ij}(\vec{k})} \underbrace{\int d^dx'\, e^{-i\vec{x}\cdot(\vec{k}+\vec{k}')}}_{= (2\pi)^d \delta(\vec{k}+\vec{k}')}$$

$$(2\pi)^d \delta(\vec{k}+\vec{k}') \Rightarrow L^d \delta_{\vec{k},\vec{k}'} \quad \text{(discr.)}$$

spatial homogeneity ~~in k space~~.

But, from isotropy,

$$C_{ij}(\vec{k}) = C_1(k)\delta_{ij} + C_2(k)\frac{k_ik_j}{k^2}$$

Incompressibility: $k_i C_{ij} = 0 \Rightarrow C_2 = -C_1$, so

$$\boxed{C_{ij}(\vec{k}) = C(k)\left(\delta_{ij} - \frac{k_ik_j}{k^2}\right)}$$

Now
$$C(k) = \frac{1}{d-1} C_{ii}(\vec{k}) = \int d^dy\, e^{-i\vec{k}\cdot\vec{y}}\, C_{ii}(\vec{y}) =$$

$C_{ii}(\vec{y}) = C_{TT}(y)(d-1) + C_{LL}(y)$

$$= \frac{1}{d-1}\int_0^\infty dy\, y^{d-1}\left[C_{TT}(y)(d-1) + C_{LL}(y)\right] \underbrace{\int d\Omega_d\, e^{-i\vec{k}\cdot\vec{y}}}_{= S_d \Phi_d(ky)}$$

$$S_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$ area of unit sphere in d dimensions $\to 2\pi$ 2D, $\to 4\pi$ 3D

$$\Phi_d(z) = \Gamma\left(\frac{d}{2}\right)\frac{J_{\frac{d-2}{2}}(z)}{\left(\frac{z}{2}\right)^{\frac{d-2}{2}}} \to J_0(z) \ \text{2D}, \quad \to \frac{\sin z}{z} \ \text{3D}$$

[Important property of this function:

$$\left(\frac{1}{z}\frac{\partial}{\partial z}\right)^n \Phi_d(z) = \left(-\frac{1}{2}\right)^n \frac{\Gamma(d/2)}{\Gamma(n+\frac{d}{2})}\,\Phi_{d+2n}(z)$$]

~~Spectrum is defined as follows:~~

Thus, we have

Bochner transform

$$C(k) = \frac{S_d}{d-1}\int_0^\infty dy\, y^{d-1}\Big[\underbrace{d\, C_{LL}(y) + y\, C_{LL}'(y)}_{=\frac{1}{y^{d-1}}\frac{\partial}{\partial y} y^d C_{LL}(y)}\Big]\,\Phi_d(ky)$$

$$\boxed{C(k) = \frac{S_d}{d-1}\int_0^\infty dy\,\Phi_d(ky)\,\frac{\partial}{\partial y}\, y^d C_{LL}(y)}$$

Now we want to define spectrum in such a way that

$$\frac{1}{2}\langle u^2\rangle = \int_0^\infty dk\, E(k)$$

$$\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\int\frac{d^dk'}{(2\pi)^d}\, e^{i(\vec{k}+\vec{k}')\cdot\vec{x}}\,\underbrace{\langle u_i(\vec{k})\,u_i(\vec{k}')\rangle}_{=C_{ii}(\vec{k})\,(2\pi)^d\,\delta(\vec{k}+\vec{k}')} =$$

$$= \frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\,(d-1)\,C(k) =$$

$$= \frac{1}{2}\frac{S_d}{(2\pi)^d}\int_0^\infty dk\, k^{d-1}\, C(k) \;\Rightarrow\; E(k) = \frac{1}{2}\frac{S_d}{(2\pi)^d}\, k^{d-1}\, C(k)$$

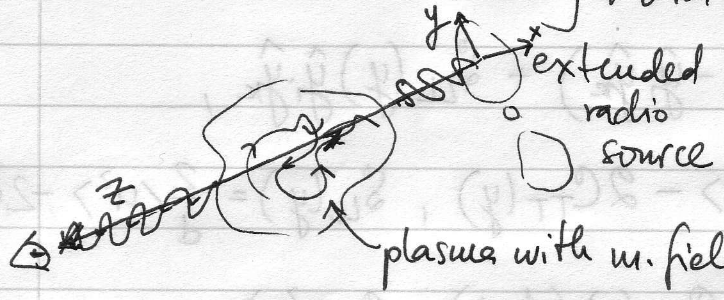
$$\boxed{E(k) = \underbrace{\frac{1}{2}\frac{S_d^2}{(2\pi)^d}}_{=\;\frac{1}{2}\ (2D),\ \frac{2}{\pi}\ (3D)} k^{d-1}\int_0^\infty dy\,\Phi_d(ky)\,\frac{\partial}{\partial y}\, y^d C_{LL}(y)} \qquad (2)$$

This is the relation between $E(k)$ and $C_{LL}(y)$

Exam

Observational Example: Magnetic Field in Clusters.

Observable: Faraday Rotation measure



Polarization rotates by angle:

$$\Delta\phi = \lambda^2 \, RM(x,y)$$

assume const

$$RM(x,y) = \underbrace{\frac{e^3}{2\pi m_e^2 c^4}}_{= a_0} \int_{\text{source}}^{\text{observer}} dz \, n_e B_z$$

Correlation function of RM: $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$C_{RM} = a_0^2 n_e^2 \int dz_1 \int dz_2 \langle B_z(\vec{r}_1) B_z(\vec{r}_2) \rangle =$$

$$= a_0^2 n_e^2 \int dz_1 \int dz_2 \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} e^{i\vec{k}_1\cdot\vec{r}_1 + i\vec{k}_2\cdot\vec{r}_2} \underbrace{\langle B_z(\vec{k}_1) B_z(\vec{k}_2)\rangle}_{= (2\pi)^3 \delta(\vec{k}_1+\vec{k}_2)\, C_{zz}(\vec{k}_1)}$$

$$= a_0^2 n_e^2 \int \frac{d^3k}{(2\pi)^3} \underbrace{\int dz_1 \int dz_2 \, e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)}}_{= L_z e^{i\vec{k}\cdot(\vec{r}_{1\perp}-\vec{r}_{2\perp})} (2\pi)\delta(k_z)} C_{zz}(\vec{k}) =$$

$$\vec{k}_\perp = \begin{pmatrix} k_x \\ k_y \\ 0 \end{pmatrix}$$

$$= a_0^2 n_e^2 L_z \int \frac{d^2k_\perp}{(2\pi)^2} e^{i\vec{k}_\perp\cdot(\vec{r}_{1\perp}-\vec{r}_{2\perp})} C_{zz}(k_x, k_y, 0)$$

But $C_{ij}(\vec{k}) = C(k)\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)$, so

$$C_{zz}(k_x,k_y,0) = C(k)\left(1 - \frac{k_z^2}{k^2}\right)\Big|_{k_z=0} = C(k_\perp)$$

[continues p.11 en regard.]

[Continued from p. 10 en regard:]

We have $C_{RM}(|\vec{r}_{2\perp}-\vec{r}_{1\perp}|) = a_0^2 n_e^2 L_z \int \frac{d^2k_\perp}{(2\pi)^2} e^{-i\vec{k}_\perp\cdot(\vec{r}_{2\perp}-\vec{r}_{1\perp})} \mathcal{C}(k_\perp)$

Then
$$\mathcal{C}(k) = \frac{1}{a_0^2 n_e^2 L_z} \int \underset{\text{2D!}}{d^2 r}\, e^{i\vec{k}\cdot\vec{r}}\, C_{RM}(r) =$$

$$= \frac{1}{a_0^2 n_e^2 L_z} \int_0^\infty dr\, r\, C_{RM}(r) \underbrace{\int d\Omega_2\, e^{i\vec{k}\cdot\vec{r}}}_{=\,2\pi J_0(kr)}$$

Thus,
$$\underset{\text{desired quantity}}{\mathcal{C}(k)} = \frac{2\pi}{a_0^2 n_e^2 L_z} \int_0^\infty dr\, r\, J_0(kr)\, \underset{\text{observable quantity}}{C_{RM}(r)}$$

$E(k) = \frac{1}{2}\frac{2\pi}{(2\pi)^2} k\, \mathcal{C}(k)$, so

spectrum of magnetic field is

$$\boxed{E(k) = \frac{1}{2 a_0^2 n_e^2 L_z}\, k \int_0^\infty dr\, r\, J_0(kr)\, C_{RM}(r)}$$

Ref.: Enßlin & Vogt A&A 401, 835 (2003)

There is an interesting consequence to this formula.

Since
$$\Phi_d(z) = 1 - \frac{z^2}{2d} + \frac{z^4}{8d(d+2)} + \dots \quad \text{as } z \to 0$$
we can work out the Taylor expansion of $E(k)$ at small k ($\ll \frac{1}{L}$ – scales larger than outer scale)

$$E(k) = \frac{1}{2}\frac{S_d^2}{(2\pi)^d} k^{d-1}\Big[\underbrace{\int_0^\infty dy \frac{\partial}{\partial y} y^d C_{LL}(y)}_{=\,0} - \frac{1}{2d}k^2 \underbrace{\int_0^\infty dy\, y^2 \frac{\partial}{\partial y} y^d C_{LL}(y)}_{=\,-2\int_0^\infty dy\, y^{d+1} C_{LL}(y)} + \dots\Big]$$

0 provided $C_{LL}(y\to\infty) \to 0$ faster than $\frac{1}{y^d}$

$-2\int_0^\infty dy\, y^{d+1} C_{LL}(y)$ provided $C_{LL}(y) \to 0$ faster than $\frac{1}{y^{d+2}}$

So,
$$E(k) = \frac{1}{2d}\frac{S_d^2}{(2\pi)^d} k^{d+1} \underbrace{\int_0^\infty dy\, y^{d+1} C_{LL}(y)}_{=\,\Lambda} + \dots$$

Λ Loytsyanskii integral (see next §!)

Thus, $\boxed{E(k) \sim k^{d+1}}$ provided correlations decay suff. fast (faster than $\frac{1}{y^{d+2}}$)

NB: Davidson seminar 21 Oct @ 4pm