

Part II. TurbulenceNOTES
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§1 §17. Kolmogorov's 1941 Dimensional Theory

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\nabla p + \nu \nabla^2 \vec{u} + \vec{f}$$

external forcing

all main results
derivable
dimensionally!

(p=1, incompressible)

Parameters: characteristic velocity, U
 characteristic scale, L
 viscosity, ν

Dimensionless # : $Re = \frac{UL}{\nu}$ (only one parameter!)

- When $Re \lesssim 1$, viscous flow, regular motion on system scales (except possibly boundary layers etc.).
- When $Re > Re_c$, flow becomes destabilised and chaotic. There is a fairly complex process of transition to chaos (period doubling, strange attractors).

depends on
the flow

We shall not dwell on what happens near criticality.

- ~~become~~ Jump right away to the case of $Re \gg Re_c$, which is of most physical interest.

This is the regime of fully developed turbulence.

This means that $\vec{u}$ is very irregular in space and time - fluctuating at each point around its mean value $\vec{U}$ ~~in space~~ and varying rapidly in space on ~~small~~ small scales.

$$\text{So, } \vec{u}(t, \vec{x}) = \vec{U} + \delta \vec{u}$$

mean

fluctuating

What happens to energy $\mathcal{E} = \frac{1}{2} \int d^3x |\vec{u}|^2$?

$$\frac{dE}{dt} = \underbrace{-\gamma \int d^3x |\nabla \vec{u}|^2}_{\text{dissipation}} + \underbrace{\int d^3x \vec{u} \cdot \vec{f}}_{\text{injection}}$$

Energy injection can be more complicated than just a forcip term : energy can come from background gradients, shear etc. (converted from some external field: e.g. gravitational). But it is usually the feature of the overall global dynamics.

- So we define the system scale (outer scale) as the scale at which energy is injected (energy-containing scale). ~~energy-containing scale~~

Then

$$Re \sim \frac{\rho u_{\infty} L}{\mu}$$

change of velocity field
across scale L

At this scale, $\delta U_L \sim \delta U$ (fluctuating part of the field ~~has the same~~ is the same order as the change in the mean flow - U itself does not matter because of Galilean invariance) (The asst)

In our simple model $\vec{f}$ has scale L .

The associated time scale is $\sim L / \epsilon \sigma_L$

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[illegible]

~~So the total amount of injected power is determined only by a function of β alone.~~

- What is the total amount of power comp into the system? This is determined only by large-scale quantities. Viscosity cannot matter:

$$\frac{|\nabla^2 \vec{u}|}{|\vec{u} \cdot \nabla \vec{u}|} \sim \frac{\sqrt{\delta u_L} / L^2}{\delta u_L^2 / L} \sim \frac{1}{Re} \ll 1$$

So, dimensionally, we can only have

Indeed, $\epsilon \sim \frac{\delta u_L^3}{L}$

In our model with forcing,

$$\frac{\partial \vec{u}}{\partial t} \sim \delta u_L \frac{\delta u_L}{L}$$

$$\overset{\substack{\uparrow \\ \text{per unit} \\ \text{Volume}}}{V} \epsilon = \int d^3x \vec{u} \cdot \vec{f} \overset{\substack{\uparrow \\ \text{varies on} \\ \text{scale } L}}{\sim} V \delta u_L f \sim V \frac{\delta u_L^3}{L}$$

- Stationary Situation: $\frac{d\epsilon}{dt} = 0$ on the average (true)

Then we must have

$$\underset{\substack{\uparrow \\ \text{finite}}}{\langle \epsilon \rangle} = \underset{\substack{\uparrow \\ \text{small}}}{\sqrt{\frac{1}{V}}} \int d^3x \underset{\substack{\uparrow \\ \text{must be large!}}}{\langle |\nabla \vec{u}|^2 \rangle}$$

The only way we can get a finite # on the rhs is if

$\langle |\nabla \vec{u}|^2 \rangle$ is dominated by large gradients (small scales).

Dimensionally, we can work out the viscous scale:

The only scale that can be cooked up from ϵ and ν

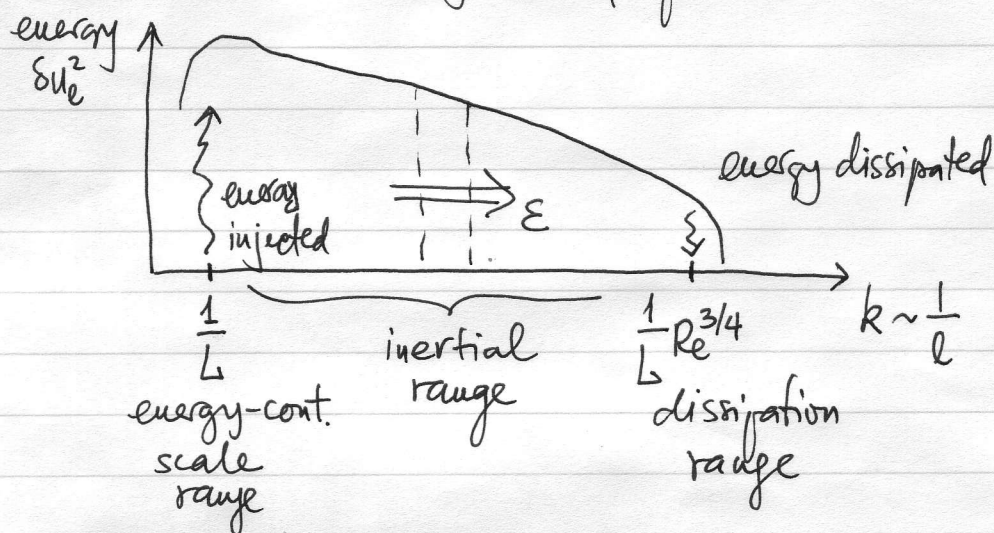
is $l_v \sim \left(\frac{\nu^3}{\epsilon} \right)^{1/4} \sim \left(\frac{\nu^3 L^4}{\delta u_L^3 L^3} \right)^{1/4} \sim L Re^{-3/4} \ll L$

Thus, $l_v \sim \frac{L}{Re^{3/4}}$

inner scale
(Kolmogorov scale)

NB: Velocity gradient at these scales will dominate only if $\frac{\delta u_v}{l_v} \gg \frac{\delta u_L}{L}$

- Thus, we have the following picture:



So what happens at the intermediate scales?

$$L \gg l \gg l_v$$

no special points

- Assumptions:
- 1) homogeneity: ~~all scales are treated equally~~
 - 2) isotropy: no special directions
 - 3) scale invariance: ~~no special scales~~
 - 4) locality of interactions (in k space) (only velocities at comparable scales interact)

(Richardson) Cascade picture

$$L \rightarrow \frac{L}{2} \rightarrow \frac{L}{4} \rightarrow \text{etc}$$

Energy flux in and out of each scale $\sim \epsilon$
(energy cannot pile up anywhere in the inertial range because no scales are special.)

~~Picture of cascade~~

- Energy flux through scale l :

$$\epsilon \sim \delta u_l^2 T_e^{-1}$$

$\uparrow$ energy $\uparrow$ cascade time

But we can (dimensionally) only cook one time scale out of l and u_l :

$$T_e \sim \frac{l}{\delta u_l}$$

Think of "eddies" with velocity $\sim \delta u_l$, size $\sim l$, turnover $\sim \frac{l}{u_l}$

So: $\epsilon \sim \delta u_l^3 \frac{1}{l}$

$$\boxed{\delta u_l \sim (\epsilon l)^{1/3}}$$

NB: dimensional result!
Kolmogorov-Osbykhov law.

- Spectral form: $E(k)dk$ - energy in $(k, k+dk)$

Energy at scales $< l$:

$$\delta u_l^2 \sim \int_{k=1/l}^{\infty} E(k) dk \sim \frac{\epsilon^{2/3}}{k^{2/3}}$$

$$\boxed{E(k) \sim \epsilon^{2/3} k^{-5/3}}$$

(Eddies of size $< l$ contribute to δu_l^2 , those $> l$ do not because their velocity does not vary over l)

Kolmogorov spectrum.

- Energies: $\delta u_l \sim \epsilon^{1/3} l^{-2/3}$ dominated by large scales

Gradients (turnover times): $\frac{\delta u_l}{l} \sim \epsilon^{1/3} l^{-5/3}$ dominated by small scales

Viscous cutoff: $\vec{u} \cdot \nabla \vec{u} \sim \sqrt{\nabla^2 \vec{u}}$

$$\frac{\delta u_{l_v}}{l_v} \sim \sqrt{\frac{\delta u_l}{l_v}} \Rightarrow l_v \sim \left(\frac{\nu^3}{\epsilon} \right)^{1/4} \sim \frac{L}{Re^{3/4}}$$

Velocity at viscous scale:

$$\delta u_{l_v} \sim \epsilon^{1/3} \frac{L^{1/3}}{Re^{1/4}} \sim \delta u_L Re^{-1/4}$$

as we estimated before.

- Let us check that these results are consistent with local-interaction assumption:
take some scale l_0 in the inertial range.

Influence of scales $l \ll l_0$: like diffusion:

turbulent diffusion with $v_e \sim u_l \sim \epsilon^{1/3} l^{1/3}$

$$\frac{\partial u_{l_0}}{\partial t} \sim v_e \nabla^2 u_{l_0} \sim (\underbrace{\epsilon^{1/3} l^{1/3}}_{\text{largest for } l \sim l_0}) \frac{u_{l_0}}{l_0^2}$$

Influence of scales $l \gg l_0$: shearing (stretching) of the eddy

$$\frac{\partial u_{l_0}}{\partial t} \sim \frac{u_e}{l} u_{l_0} \sim \underbrace{\left(\frac{\epsilon^{1/3}}{l^{2/3}} \right)}_{\text{largest for } l \sim l_0} u_{l_0}$$

~~What happens at $l \ll l_v$ (dissipation range)?~~

- What happens at $l \ll l_v$ (dissipation range)?

$$\epsilon \sim \nu |\nabla u|^2 \sim \nu \frac{u_e^2}{l^2}$$

$$u_e \sim \left(\frac{\epsilon}{\nu} \right)^{1/2} l \sim \underbrace{\left(\frac{u_{l_v}^2}{l_v} \right)}_{\text{turnover rate at } l_v} l \quad \Rightarrow \quad E(k) \sim \frac{\epsilon}{\nu} k^{-3}$$

[So effectively this means that $\epsilon \sim u_e^2 \tau_e^{-1}$ energy being dissipated.]
 $\tau_e \sim \frac{l^2}{\nu}$ viscous time

NB: $l \gg l_v$ $u_e \sim (\epsilon l)^{1/3}$ non-smooth/irregular

$l \ll l_v$ $u_e \sim \left(\frac{\epsilon}{\nu} \right)^{1/2} l$ smooth (Taylor-expandable)

Appendix:

Turbulent diffusion: heuristic derivation ($l \ll l_0$)

$$\frac{\partial \vec{u}_l}{\partial t} \sim - \vec{u}_l \cdot \nabla \vec{u}_l \sim \vec{u}_l(t) \cdot \nabla \int_0^t dt' \vec{u}_l(t') \cdot \nabla \vec{u}_l(t') \quad \text{---} \quad \vec{u}_l \cdot \nabla \vec{u}_l =$$

$$= \nabla \cdot \left[\int_0^t dt' \vec{u}_l(t) \vec{u}_l(t') \right] \cdot \nabla \vec{u}_l(t') - \vec{u}_l \cdot \nabla \vec{u}_l$$

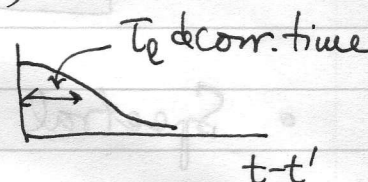
vanishes under averaging

Average over ~~small~~ scales $\ll l_0$ (including l):

$$\langle \vec{u}_l \rangle = \vec{u}_l, \text{ so}$$

$$\frac{\partial \vec{u}_l}{\partial t} \sim \nabla \cdot \int_0^t dt' \langle \vec{u}_l(t) \vec{u}_l(t') \rangle \cdot \nabla \vec{u}_l(t')$$

$$\frac{1}{3} \mathbb{I} \langle \vec{u}_l(t) \cdot \vec{u}_l(t') \rangle$$



Assume that $T_e \ll$ char. time of change of $\vec{u}_l$.

$$\text{Then } \vec{u}_l(t') \sim \vec{u}_l(t) + (t'-t) \frac{\partial \vec{u}_l}{\partial t} \sim \vec{u}_l(t)$$

So,

$$\frac{\partial \vec{u}_l}{\partial t} \sim \frac{1}{3} \nabla \cdot \hat{D}_l \cdot \nabla \vec{u}_l(t), \quad \hat{D}_l = \frac{1}{3} \mathbb{I} \int_0^t dt' \langle \vec{u}_l(t) \vec{u}_l(t') \rangle$$

$$\text{Now } D_l \sim u_l^2 \cdot T_e \sim u_l^2 \frac{l}{u_l} \sim u_l l \text{ g.e.d.}$$

Stretching: $\frac{\partial \vec{u}_l}{\partial t} \sim - \vec{u}_l \cdot \nabla \vec{u}_l - \vec{u}_l \cdot \nabla \vec{u}_l$ sweeping, Galilean transform to frame moving with $\vec{u}_l$

$$(l \ll l) \sim - \vec{u}_l \cdot \nabla \vec{u}_l$$

$$\text{linear matrix} \sim \frac{u_l}{l}$$

- How do trajectories of fluid particles separate?

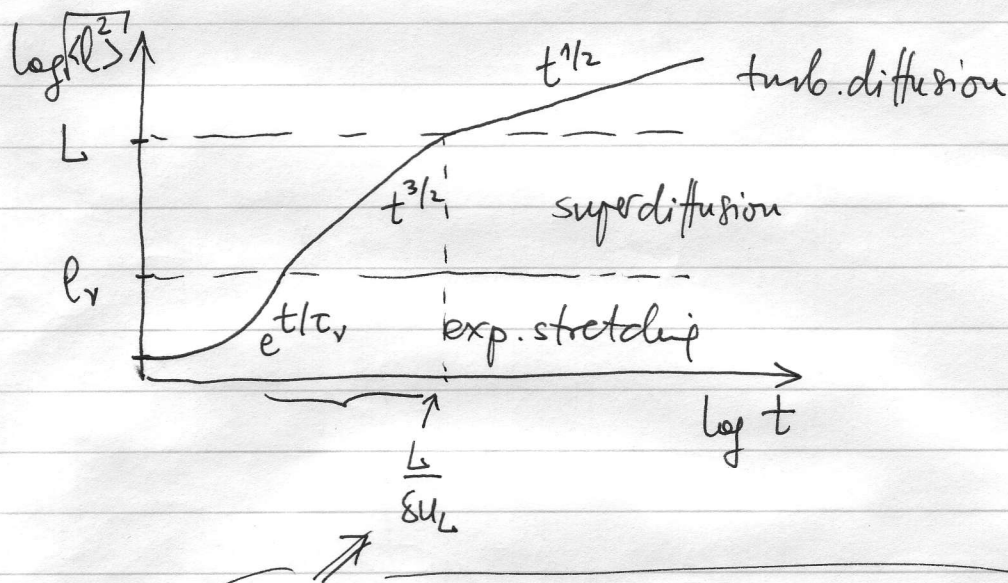
$$\frac{dl}{dt} \sim \delta u_l$$

$l \gg l_v$: $\frac{dl}{dt} \sim (\epsilon l)^{1/3} \Rightarrow l \sim \epsilon^{1/2} t^{3/2}$ power law
(Richardson law)

$l \ll l_v$: $\frac{dl}{dt} \sim \left(\frac{\epsilon}{\nu}\right)^{1/2} l \Rightarrow l \sim l(0) e^{t/\tau_v}$ exp. separation
(Orszag)
 $\tau_v = \frac{\nu}{\epsilon} - \text{visc. eddy turnover time}$

$l \gg L$: turbulence is like diffusion (eddy viscosity)

random walk $l \sim \sqrt{\nu_L t} \sim \sqrt{\delta u_L L} t^{1/2}$



$$\epsilon^{1/2} t^{3/2} \sim L \Rightarrow t \sim \frac{L^{2/3}}{\epsilon^{1/3}} \sim \frac{L^{2/3}}{(\delta u_L^3/L)^{1/3}} \sim \frac{L}{\delta u_L}$$

$$\text{or } \sqrt{\delta u_L L} t^{1/2} \sim L \Rightarrow t \sim \frac{L^2}{\delta u_L L} \sim \frac{L}{\delta u_L}$$

Ex.