Lecture 15 25.02.05 NB: L&L, Frisch, Davidson, Batchelor

M05: Lecture 2 10.10.05

NOTES DISTRIBUTED

Part II. Turbulence

§1 ~~§17.~~ Kolmogorov's 1941 Dimensional Theory

(all main results derivable dimensionally!)

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\nabla p + \nu\nabla^2\vec{u} + \vec{f}$$

($\vec{f}$ ← external forcing)

($\rho = 1$, incompressible)

Parameters: characteristic velocity, U
characteristic scale, L
viscosity, ν

Dimensionless #: $Re = \frac{UL}{\nu}$ (only one parameter!)

- When $Re \lesssim 1$, viscous flow, regular motion on system scales (except possibly boundary layers etc.).
- When $Re > Re_c$, flow becomes destabilised and chaotic. There is a fairly complex process of transition to chaos (period doubling, strange attractors).
  (Re_c depends on the flow)

We shall not dwell on what happens near criticality.

- Jump right away to the case of $Re \gg Re_c$, which is of most physical interest.

This is the regime of fully developed turbulence.

This means that $\vec{u}$ is very irregular in space and time — fluctuating at each point around its mean value $\vec{U}$ and varying rapidly in space on small scales.

So, $\vec{u}(t,\vec{x}) = \vec{U} + \delta\vec{u}$

($\vec{U}$ ← mean, $\delta\vec{u}$ ← fluctuating)

What happens to energy $\mathcal{E} = \frac{1}{2}\int d^3x |\vec{u}|^2$?

$$\frac{d\mathcal{E}}{dt} = -\nu \int d^3x\, |\nabla \vec{u}|^2 + \int d^3x\, \vec{u}\cdot\vec{f}$$

dissipation — injection

Energy injection can be more complicated than just a forcing term: energy can come from background gradients, shear etc. (converted from some external field: e.g. gravitational). But it is usually the feature of the ~~overall~~ global dynamics.

(disks) (convection)

- So we define the system scale (<u>outer scale</u>) as the scale at which energy is injected (<u>energy-containing scale</u>). ~~$L \sim (\nabla_z T)^{-1}$ for convection~~ ~~$L \sim U^{-1}$ for disks.~~

Then $\boxed{Re \sim \frac{\delta u_L L}{\nu}}$ — change of velocity field across scale L

At this scale, $\delta u_L \sim \delta U$ (fluctuating part of the field ~~has the same~~ is the same order as the <u>change</u> in the mean flow — U itself does not matter because of Galilean invariance)

In our simple model $\vec{f}$ has scale L.

The associated time scale is $\sim L/\delta U_L$

~~• At scale L, dissipation cannot matter:~~

~~$$\frac{|\nu \nabla^2 \vec{u}|}{|\vec{u}\cdot\nabla\vec{u}|} \sim \frac{\nu\, \delta u_L L}{L^2\, \delta u_L^2} \sim \frac{1}{Re} \ll 1$$~~

~~So the total amount of injected power is determined only by quantities at scale L.~~

- What is the total amount of power coming into the system? This is determined only by large-scale quantities. Viscosity cannot matter:

$$\frac{|\nu \nabla^2 \vec{u}|}{|\vec{u}\cdot\nabla\vec{u}|} \sim \frac{\nu\, \delta u_L / L^2}{\delta u_L^2 / L} \sim \frac{1}{Re} \ll 1 .$$

So, dimensionally, we can only have

$$\boxed{\epsilon \sim \frac{\delta u_L^3}{L}}$$

Indeed, in our model with forcing,

$$V\epsilon = \int d^3x\, \vec{u}\cdot\vec{f} \sim V\,\delta u_L\, f \sim V\,\frac{\delta u_L^3}{L}$$

($V\epsilon$: per unit volume; $\vec{f}$: varies on scale L; $f \sim \frac{\partial \vec{u}}{\partial t} \sim \delta u_L \frac{\delta u_L}{L}$)

- Stationary situation: $\frac{d\mathcal{E}}{dt} = 0$ on the average (time)

Then we must have

$$\langle \epsilon \rangle = \nu \frac{1}{V}\int d^3x \langle |\nabla\vec{u}|^2 \rangle$$

($\langle\epsilon\rangle$: finite; ν: small; $\langle|\nabla\vec{u}|^2\rangle$: must be large!)

The only way we can get a finite # on the rhs is if $\langle |\nabla\vec{u}|^2\rangle$ is dominated by large gradients (small scales).

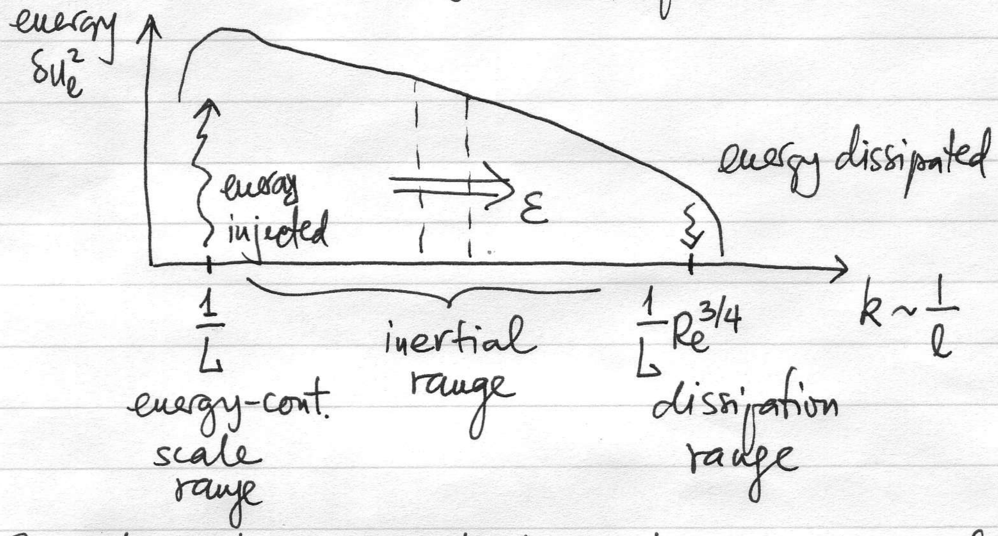
Dimensionally, we can work out the viscous scale: the only scale that can be cooked up from ϵ and ν is

$$l_\nu \sim \left(\frac{\nu^3}{\epsilon}\right)^{1/4} \sim \left(\frac{\nu^3 L^4}{\delta u_L^3 L^3}\right)^{1/4} \sim L\, Re^{-3/4} \ll L$$

Thus, $\boxed{l_\nu \sim \frac{L}{Re^{3/4}}}$ inner scale (Kolmogorov scale)

NB: Velocity gradient at these scales will dominate only if $\frac{\delta u_{\ell_\nu}}{\ell_\nu} \gg \frac{\delta u_L}{L}$

- Thus, we have the following picture:



So what happens at the intermediate scales?

$$L \gg \ell \gg \ell_\nu$$

<u>Assumptions</u>:
1) homogeneity: no special points
2) isotropy: no special directions
3) scale invariance: no special scales
4) locality of interactions (in k space) (only velocities at comparable scales interact)

↓

(Richardson) Cascade picture

$$L \to \frac{L}{2} \to \frac{L}{4} \to \text{etc}$$

Energy flux in and out of each scale $\sim \epsilon$
(energy cannot pile up anywhere in the inertial range because no scales are special.

- Energy flux through scale l:

$$\epsilon \sim \delta u_l^2 \, \tau_l^{-1}$$

(arrows: δu_l^2 — energy; τ_l^{-1} — cascade time)

But we can (dimensionally) only cook one time scale out of l and u_l:

$$\tau_l \sim \frac{l}{\delta u_l}$$

Think of "eddies" with velocity $\sim \delta u_l$, size $\sim l$, turnover time $\sim \frac{l}{u_l}$

So: $\epsilon \sim \delta u_l^3 \frac{1}{l}$

$$\Downarrow$$

$$\boxed{\delta u_l \sim (\epsilon l)^{1/3}}$$

NB: dimensional result!

Kolmogorov-Obukhov law.

- Spectral form: $E(k)dk$ – energy in $(k, k+dk)$

Energy at scales $< l$:

$$\delta u_l^2 \sim \int_{k=1/l}^{\infty} E(k)\,dk \sim \frac{\epsilon^{2/3}}{k^{2/3}} \Rightarrow \boxed{E(k) \sim \epsilon^{2/3} k^{-5/3}}$$

(Eddies of size $< l$ contribute to δu_l^2, those $> l$ do not because their velocity does not vary over l)

Kolmogorov spectrum.

- Energies: $\delta u_l^2 \sim \epsilon^{2/3} l^{2/3}$ dominated by large scales

Gradients (turnover times): $\frac{\delta u_l}{l} \sim \epsilon^{1/3} l^{-2/3}$ dominated by small scales

Viscous cutoff: $\vec{u}\cdot\nabla\vec{u} \sim \nu \nabla^2 \vec{u}$

$$\frac{\delta u_{l_\nu}^2}{l_\nu} \sim \nu \frac{\delta u_l}{l_\nu^2} \Rightarrow l_\nu \sim \left(\frac{\nu^3}{\epsilon}\right)^{1/4} \sim \frac{L}{Re^{3/4}}$$

Velocity at viscous scale:

$$\delta u_{l_\nu} \sim \epsilon^{1/3} \frac{L^{1/3}}{Re^{1/4}} \sim \delta u_L \, Re^{-1/4}$$

as we estimated before.

- Let us check that these results are consistent with local-interaction assumption:

take some scale l_0 in the inertial range.

Influence of scales $l \ll l_0$: like diffusion:

turbulent diffusion with $\nu_l \sim \delta u_l \, l \sim \epsilon^{1/3} l^{4/3}$

$$\frac{\partial u_{l_0}}{\partial t} \sim \nu_l \nabla^2 u_{l_0} \sim \underbrace{(\epsilon^{1/3} l^{4/3})}_{\text{largest for } l \sim l_0} \frac{u_{l_0}}{l_0^2}$$

Influence of scales $l \gg l_0$: shearing/stretching of the eddy

$$\frac{\partial u_{l_0}}{\partial t} \sim \frac{u_l}{l} u_{l_0} \sim \underbrace{\left(\frac{\epsilon^{1/3}}{l^{2/3}}\right)}_{\text{largest for } l \sim l_0} u_{l_0}$$

~~[illegible]~~

- What happens at $l \ll l_\nu$ (dissipation range)?

$$\epsilon \sim \nu |\nabla u|^2 \sim \nu \frac{\delta u_l^2}{l^2}$$

$$\delta u_l \sim \left(\frac{\epsilon}{\nu}\right)^{1/2} l \sim \left(\frac{\delta u_{l_\nu}}{l_\nu}\right) l \quad \Rightarrow E(k) \sim \frac{\epsilon}{\nu} k^{-3}$$

($\frac{\delta u_{l_\nu}}{l_\nu}$ ← turnover rate at l_ν)

> So effectively this means that
>
> $$\epsilon \sim \delta u_l^2 \, \tau_l^{-1}$$ energy being dissipated.
>
> (τ_l → ~~[illegible]~~ viscous time $\sim \frac{l^2}{\nu}$)

NB: $l \gg l_\nu$ $\quad \delta u_l \sim (\epsilon l)^{1/3}$ non-smooth/irregular

$l \ll l_\nu$ $\quad \delta u_l \sim \left(\frac{\epsilon}{\nu}\right)^{1/2} l$ smooth (Taylor-expandable)

Appendix:

Turbulent diffusion: heuristic derivation ($l \ll l_0$)

$$\frac{\partial \vec{u}_{l_0}}{\partial t} \sim -\vec{u}_l \cdot \nabla \vec{u}_{l_0} \simeq \vec{u}_l(t)\cdot\nabla \int_0^t dt'\, \vec{u}_l(t')\cdot\nabla \vec{u}_{l_0}(t') =$$

(annotations: $\vec{u}_{l_0} \approx -\int^t dt'\, \vec{u}_l\cdot\nabla\vec{u}_{l_0}(t')$, with $-\vec{u}_{l_0}\cdot\nabla\vec{u}_l$ terms)

$$= \nabla\cdot\left[\int_0^t dt'\, \vec{u}_l(t)\vec{u}_l(t')\right]\cdot\nabla\vec{u}_{l_0}(t') - \vec{u}_{l_0}\cdot\nabla\vec{u}_l$$

(the last term: vanishes under averaging)

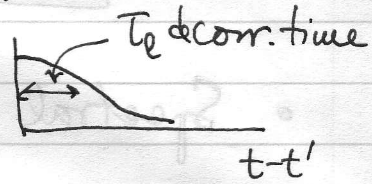
[illegible]

Average over small scales $\ll l_0$ (including l):

$\langle \vec{u}_{l_0}\rangle = \vec{u}_{l_0}$, so

$$\frac{\partial \vec{u}_{l_0}}{\partial t} \sim \nabla\cdot\int_0^t dt' \langle \vec{u}_l(t)\vec{u}_l(t')\rangle \cdot\nabla\vec{u}_{l_0}(t')$$

$$\langle \vec{u}_l(t)\vec{u}_l(t')\rangle = \frac{1}{3}\mathbb{1}\langle \vec{u}_l(t)\cdot\vec{u}_l(t')\rangle$$



Assume that $\tau_l \ll$ char. time of change of $\vec{u}_{l_0}$.

Then $\vec{u}_{l_0}(t') \sim \vec{u}_{l_0}(t) + (t'-t)\frac{\partial \vec{u}_{l_0}}{\partial t} \sim \vec{u}_{l_0}(t)$

(where $t'-t \lesssim \tau_l$, $\frac{\partial \vec{u}_{l_0}}{\partial t} \sim \frac{u_{l_0}}{t_{l_0}}$: small)

So,

$$\frac{\partial \vec{u}_{l_0}}{\partial t} \sim \nabla\cdot\hat{D}_l\cdot\nabla\vec{u}_{l_0}(t), \quad \hat{D}_l = \frac{1}{3}\mathbb{1}\int_0^t dt' \langle \vec{u}_l(t)\vec{u}_l(t')\rangle$$

Now $D_l \sim u_l^2\cdot\tau_l \sim u_l^2 \frac{l}{u_l} \sim u_l l$ q.e.d.

Stretching: ($l_0 \ll l$)

$$\frac{\partial \vec{u}_{l_0}}{\partial t} \sim -\vec{u}_{l_0}\cdot\nabla\vec{u}_l - \vec{u}_l\cdot\nabla\vec{u}_{l_0}$$

(last term struck out: sweeping, Galilean-transform to frame moving with $\vec{u}_l$)

$$\sim -\vec{u}_{l_0}\cdot\nabla\vec{u}_l$$

($\nabla\vec{u}_l$: linear matrix $\sim \frac{u_l}{l}$.)

- **How do trajectories of fluid particles separate?**

$$\frac{dl}{dt} \sim \delta u_l$$

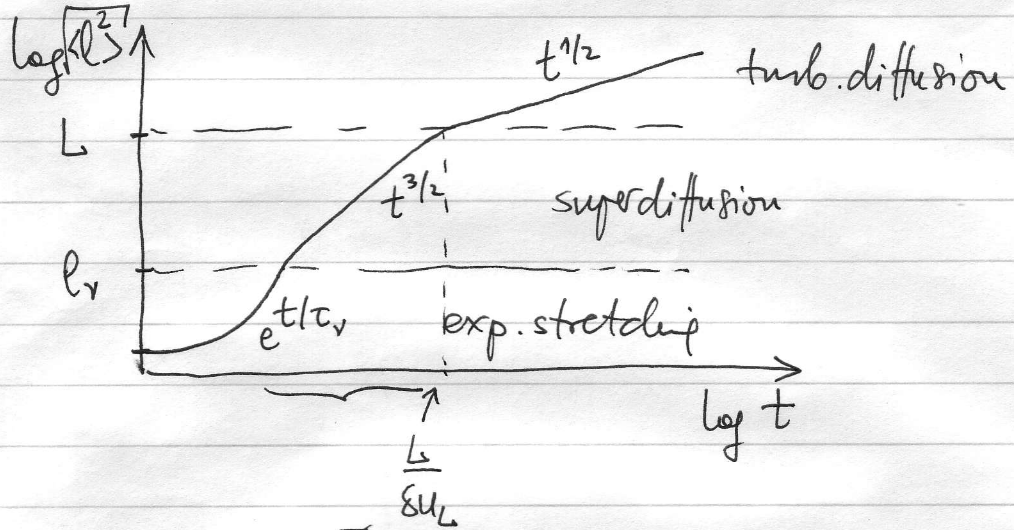
$l \gg l_\nu$: $\frac{dl}{dt} \sim (\epsilon l)^{1/3} \Rightarrow l \sim \epsilon^{1/2} t^{3/2}$ power law (Richardson law)

$l \ll l_\nu$: $\frac{dl}{dt} \sim \underbrace{\left(\frac{\epsilon}{\nu}\right)^{1/2}}_{\tau_{l_\nu}^{-1}} l \Rightarrow l \sim l(0)\, e^{t/\tau_{l_\nu}}$ exp. separation (Orszag)

$\tau_{l_\nu}^{-1}$ - visc. eddy turnover time

$l \gg L$: turbulence is like diffusion (eddy viscosity)

random walk $l \sim \sqrt{\nu_L t} \sim \sqrt{\delta u_L L}\, t^{1/2}$



$$\epsilon^{1/2} t^{3/2} \sim L \Rightarrow t \sim \frac{L^{2/3}}{\epsilon^{1/3}} \sim \frac{L^{2/3}}{(\delta u_L^3 / L)^{1/3}} \sim \frac{L}{\delta u_L}$$ Ex.

or $\sqrt{\delta u_L L}\, t^{1/2} \sim L \Rightarrow t \sim \frac{L^2}{\delta u_L L} \sim \frac{L}{\delta u_L}$