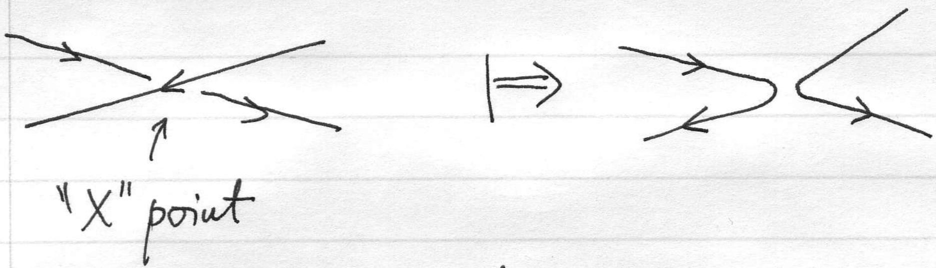
Lecture 14 23.02.05

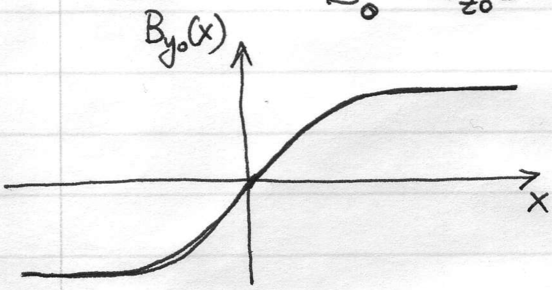
§16. Tearing Mode.

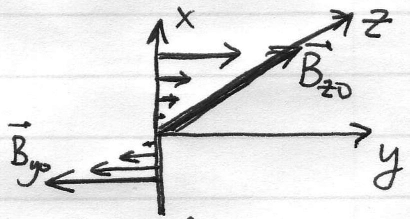
Finally, we consider another type of reconnection: reconnection in a strong external field.
This is relevant in tokamaks (sawtooth crash — see Kadomtsev's review), but it is also useful for (future) understanding of 3D reconnection, where generically field lines are sheared wrt each other:



So, consider the following equilibrium field:

$$\vec{B}_0 = B_{z0}\hat{z} + B_{y0}(x)\hat{y}$$





E.g. $B_y(x) = \bar{B}\tanh(x)$
Harris sheet

At $x=0$, we have antiparallel B_y.

B_{z0} is assumed to be very strong $\Rightarrow$ it effectively two-dimensionalises everything and we have

$$\vec{B} = \vec{B}_0 + \delta\vec{B}_\perp = B_{z0}\hat{z} + \left[B_{y0}(x) + \delta B_y\right]\hat{y} + \delta B_x\hat{x}$$

$$B_{y0}(x) + \delta B_y = \frac{\partial\psi}{\partial x}, \quad \delta B_x = -\frac{\partial\psi}{\partial y}$$

flux function $\psi = -A_\parallel$

$$\vec{u} = \begin{pmatrix} u_x \\ u_y \\ 0 \end{pmatrix}, \quad u_y = \frac{\partial\phi}{\partial x}, \quad u_x = -\frac{\partial\phi}{\partial y}$$

stream function $\phi = \frac{c}{B_{0z}}\varphi$

and $\nabla = \nabla_\perp = \begin{pmatrix} \partial_x \\ \partial_y \\ 0 \end{pmatrix}$

ψ and ϕ satisfy RMHD equations (see ES1 #3):

$$\left\{\begin{aligned}
&\frac{\partial}{\partial t}\underbrace{\nabla_\perp^2\phi}_{\omega_z} + \underbrace{[\phi, \underbrace{\nabla_\perp^2\phi}_{\omega_z}]}_{\vec{u}\cdot\nabla_\perp\omega_z} = \underbrace{[\psi, \underbrace{\nabla_\perp^2\psi}_{j_z}]}_{\vec{B}\cdot\nabla_\perp j_z} + \cancel{\underbrace{\nu\nabla_\perp^4\phi}_{\nu\nabla_\perp^2\omega_z}} \quad \left[\nabla\times \text{mom. equation}\right]_z \quad (1)\\
&\frac{\partial}{\partial t}\psi + \underbrace{[\phi,\psi]}_{=-\vec{B}\cdot\nabla_\perp\phi} = \underbrace{\eta\nabla_\perp^2\psi}_{=\eta j_z} - \eta\psi_0''(x) \quad \text{uncurled ind. equation} \quad (2)
\end{aligned}\right.$$

ignore viscosity

to maintain equilibrium

Here $[\phi,\psi] = \dfrac{\partial\phi}{\partial x}\dfrac{\partial\psi}{\partial y} - \dfrac{\partial\phi}{\partial y}\dfrac{\partial\psi}{\partial x}$ Poisson bracket

Equilibrium: $\phi = 0$, $\psi = \psi_0(x)$

Perturbation: $\delta\phi(t,x,y)$, $\boxed{\delta\psi(t,x,y) = \psi_1(t,x)\cos ky}$

Then $B_z = B_{0z}$

$$B_y = B_{0y} + \delta B_y = \frac{\partial\psi}{\partial x} = \psi_0'(x) + \psi_1'\cos ky$$

$$B_x = -\frac{\partial\psi}{\partial y} = k\psi_1\sin ky$$

$\left(\text{NB: } \frac{c}{4\pi} = 1\right)$

$$j_z = \nabla_\perp^2\psi = \psi_0'' - k^2\psi_1\cos ky + \psi_1''\cos ky = $$
$$= \psi_0'' + (\psi_1'' - k^2\psi_1)\cos ky$$

Eq. (2): linearized

$$\frac{\partial}{\partial t}\psi_1\cos ky = \vec{B}_0\cdot\nabla_\perp\phi + \eta j_z = \psi_0'(x)\frac{\partial\phi}{\partial y} + \eta(\psi_1'' - k^2\psi_1)\cos ky$$

$$\Downarrow$$

$$\boxed{\phi = \phi_1(t,x)\sin ky}$$

and Eq. (2) becomes

$$\longrightarrow \boxed{\frac{\partial \psi_1}{\partial t} = k\underbrace{\psi_0'(x)}_{B_{y0}}\phi_1 + \eta\left(\underbrace{\psi_1''}_{\text{large}} - k^2\psi_1\right)} \qquad (3)$$

Eq. (1) linearised:

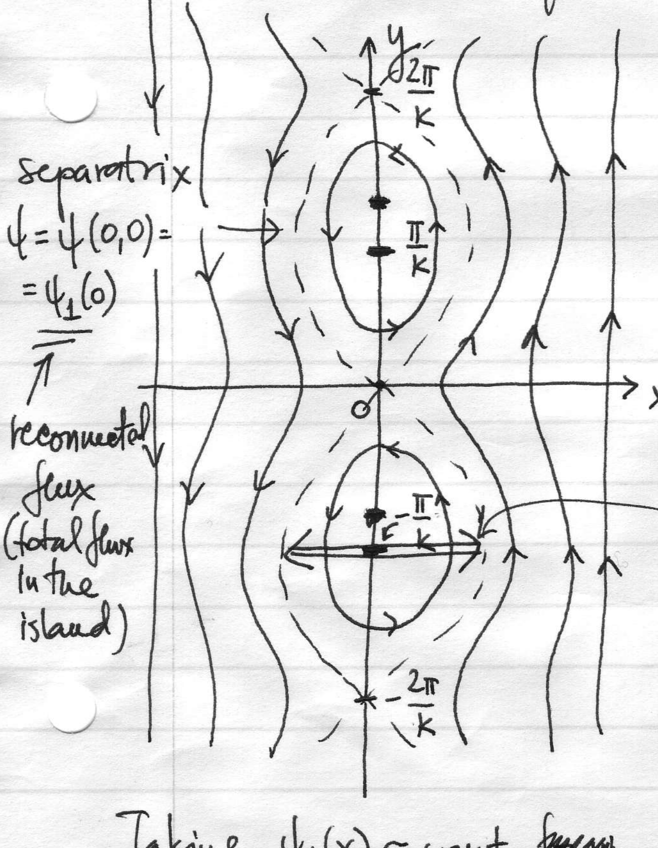
$$\frac{\partial}{\partial t}\left(\phi_1'' - k^2\phi_1\right)\sin ky = \psi_0'(x)\underbrace{\frac{\partial j_z}{\partial y}}_{= -k\left(\psi_1''-k^2\psi_1\right)\sin ky} + \overbrace{\delta B_x}^{= k\psi_1 \sin ky}\underbrace{\frac{\partial j_{z0}}{\partial x}}_{= \psi_0'''(x)} + \cancel{\delta B_y \frac{\partial j_{z0}}{\partial y}}^{\,0} =$$

$$\longrightarrow \boxed{\frac{\partial}{\partial t}\left(\underbrace{\phi_1''}_{\text{large}} - k^2\phi_1\right) = -k\psi_0'(x)\left(\psi_1''-k^2\psi_1\right) + k\psi_0'''(x)\psi_1 =}$$

$$\boxed{= -k\underbrace{\psi_0'(x)}_{B_{y0}}\left[\underbrace{\psi_1''}_{\text{large}} - k^2\psi_1 - \frac{\psi_0'''(x)}{\psi_0'(x)}\psi_1\right]} \qquad (4)$$

~~[illegible]~~

What does the perturbation look like?



$$\psi(x,y) = \psi_0(x) + \psi_1(t,x)\cos ky$$

[NB: lines of const ψ are the field lines of $\vec{B}_\perp = \hat{z}\times\nabla\psi \perp \nabla\psi$]

$B_{y0}(x) = \psi_0'(x)$ is odd, so ψ_0 even

$$\psi_0(x) \approx \frac{1}{2}\psi_0''(0)x^2 + \ldots$$

island width w.

$$\psi_1(0) = \psi\left(\frac{w}{2},\frac{\pi}{k}\right) = \frac{1}{2}\psi_0''(0)\frac{w^2}{4} - \psi_1\left(\frac{w}{2}\right)$$

$\Downarrow$

Taking $\psi_1(x) \approx \text{const}$, ~~[illegible]~~

$$w = 4\sqrt{\frac{\psi_1(0)}{\psi_0''(0)}}$$ small for small ψ_1

Consider exponentially growing solutions: $\psi_1, \phi_1 \propto e^{\gamma t}$.

Time scales in the problem:

Alfvén: $\tau_A = (k B_{0y})^{-1}$ NB $k \sim \frac{1}{L}$ system size.

resistive: $\tau_R = (\eta k^2)^{-1}$

mode growth: γ

Assume/guess: $\tau_A \ll \frac{1}{\gamma} \ll \tau_R$

Spatial scales: system size $L \sim \frac{1}{k}$

resistive

$$\delta \sim \sqrt{\frac{\eta}{\gamma}} \sim \frac{1}{k}\frac{1}{\sqrt{\gamma \tau_R}} \ll \frac{1}{k}$$

Outer solution: $\frac{\partial}{\partial x} \sim k \sim \frac{1}{L}$

Eq. (3) $\Rightarrow$ $\boxed{\gamma \psi_1 \approx \underbrace{k \psi_0'(x)}_{k B_0(x)} \phi_1}$ (5) (resistivity unimportant)

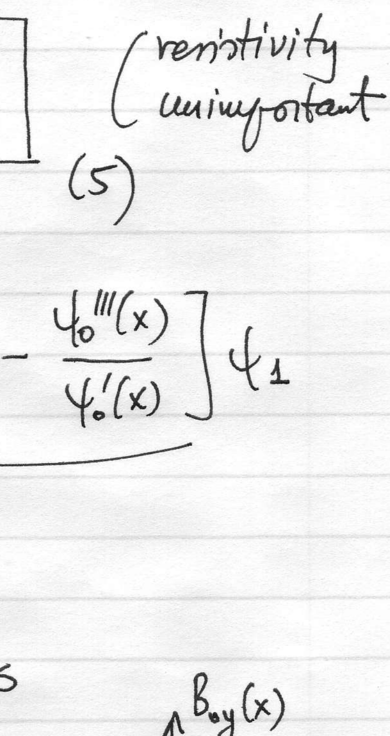
Substitute this into eq. (4):

$$\underbrace{\gamma^2 \left(\frac{\partial^2}{\partial x^2} - k^2\right)\frac{\psi_1}{k\psi_0'(x)}}_{\sim\ \gamma^2 \tau_A k^2 \psi_1} = -\underbrace{k\psi_0'(x)\left[\frac{\partial^2}{\partial x^2} - k^2 - \frac{\psi_0'''(x)}{\psi_0'(x)}\right]\psi_1}_{\sim\ \frac{1}{\tau_A} k^2 \psi_1}$$

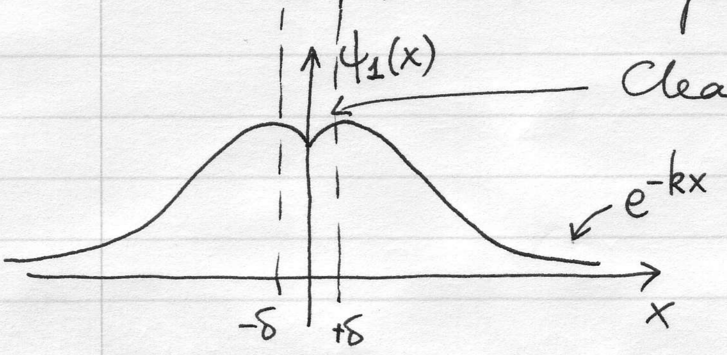
small because $\gamma \tau_A \ll 1$

Then the eqn is

$$\boxed{\psi_1'' = \left[k^2 + \underbrace{\frac{\psi_0'''(x)}{\psi_0'(x)}}_{\frac{B_{0y}''(x)}{B_{0y}(x)}}\right]\psi_1} \quad (6)$$



Solution depends on the particular profile of $B_{0y}(x)$. However, it will always look roughly like this:



Clearly, we must have

$$\psi_1'(+\delta) > 0$$

and $\psi_1'(-\delta) < 0$

because $\psi_1' = \delta B_y$ and that reverses direction

The derivative of the outer solution will have a jump:

$$\Delta' = \frac{\psi_{1_{out}}'(+0) - \psi_{1_{out}}'(-0)}{\psi_{1_{out}}(0)} = \left[\frac{d}{dx} \ln \psi_1\right]_{-0}^{+0} \qquad (7)$$

This quantity is a parameter of the problem (depends on k and on the $B_{0y}(x)$ profile).

Now we must solve in the inner region and match the inner solution so that

$$\frac{1}{\psi_1(0)} \int_{-\delta\cdot\infty}^{+\delta\cdot\infty} dx \, \frac{d^2}{dx^2} \psi_{1in}(x) = \Delta' \qquad (8)$$

<u>Inner Solution</u>: $\frac{\partial}{\partial x} \sim \frac{1}{\delta} \gg k, \frac{1}{L}$

Eq. (4): ϕ_1'' and ψ_1'' terms dominate.

Also, $\psi_0'(x) = B_{0y}(x) \approx +\overbrace{\psi_0''(0)}^{= B_{0y}'(0)}\, x$

Then

$$\boxed{\gamma \phi_1'' = -k \psi_0''(0)\, x\, \psi_1''} \qquad (9)$$

Use this in eq. (3): substituting $\psi_1'' = -\dfrac{\gamma\phi_1''}{kx\psi_0''(0)}$,

we get

$$\boxed{\gamma\psi_1 = kx\psi_0''(0)\,\phi_1 - \eta\frac{\gamma\phi_1''}{kx\psi_0''(0)}} \qquad (10)$$

So now we have to solve the eigenvalue problem (9-10) subject to ~~boundary~~ matching condition (8).

This is ~~the~~ easiest if we can expand $\psi_1(x)$ in the domain where we are solving:

$$\psi_1(x) \approx \psi_1(0) + \ldots$$

This is called <u>constant-ψ approximation</u>.

(it is not valid when Δ' is very big - solution will need to (and can) be generalized then)

We can do everything exactly now, but first let's do ~~one~~ a qualitative estimate:

Rewrite eq. (10) so:

$$\underbrace{\phi_1''}_{\displaystyle \frac{\phi_1}{\delta^2}} = \underbrace{\frac{[k\psi_0''(0)]^2x^2}{\eta\gamma}}_{\displaystyle \frac{[kB_0']^2\delta^2}{\eta\gamma}}\phi_1 - \overbrace{\frac{1}{\eta}k\psi_0''(0)\psi_1(0)\,x}^{\sim \frac{\delta}{\eta}[kB_0']\psi_1} \qquad (11)$$

$$\frac{\phi_1}{\delta^2} \sim \frac{[kB_0']^2\delta^2}{\eta\gamma} \Rightarrow \boxed{\frac{\delta^4}{\gamma} \sim \eta\,[kB_0']^{-2}} \qquad (12)$$

Eq. (9): $\underbrace{\gamma\phi_1''}_{\displaystyle \frac{\gamma}{\delta^2}\phi_1} = \underbrace{-k\psi_0''(0)\,x\,\psi_1''}_{\displaystyle kB_{0y}'\delta\,\frac{\Delta'\psi_1}{\delta}}$

from second term on rhs of (11)
$\psi_1 \sim \frac{\eta}{\delta^3}[kB_{0y}']^{-1}\phi_1$

$$\frac{\gamma}{\delta^2}\phi_1 \sim kB_{0y}'\cancel{\delta}\,\frac{\Delta'\psi_1}{\cancel{\delta}} \sim \frac{\eta\Delta'}{\delta^3}\phi_1 \Rightarrow \boxed{\gamma\delta \sim \eta\Delta'} \qquad (13)$$

NB: The estimate $\psi_1'' \sim \frac{\Delta' \psi_1}{\delta}$ came from the matching condition (8):

$$\underbrace{\frac{1}{\psi_1(0)} \int_{-\delta}^{\delta} dx \, \frac{d^2}{dx^2}\psi_1}_{\sim \delta \psi_1''} = \Delta' \quad \Rightarrow \psi_1'' \sim \frac{\Delta'}{\delta}\psi_1(0)$$

So, now, from (12) and (13),

$$\delta^5 \sim \eta^2 \Delta' \left[k B_{oy}'\right]^{-2} \Rightarrow \boxed{\delta \sim \eta^{2/5} (\Delta')^{1/5} \left[k B_{oy}'\right]^{-2/5}} \quad (14)$$

and

$$\gamma \sim \delta^{-1} \eta \Delta' \Rightarrow \boxed{\gamma \sim \eta^{3/5} (\Delta')^{4/5} \left[k B_{oy}'\right]^{2/5}} \quad (15)$$

Now we do the exact solution.

Introduce the inner variable X, s.t. $x = \delta X$, where δ is small (boundary layer width)

$$\text{Eq. (11)}: \quad \frac{1}{\delta^2}\frac{\partial^2 \phi_1}{\partial X^2} = \left[k B_{oy}'\right]^2 \frac{\delta^2}{\eta\gamma} X^2 \phi_1 - \underbrace{\frac{\delta}{\eta}\left[k B_{oy}'\right]\psi_1(0) X}_{\text{divide through by this}}$$

$$-\frac{\eta}{\delta^3}\left[k B_{oy}'\right]^{-1} \frac{\partial^2}{\partial X^2}\frac{\phi_1}{\psi_1(0)} = X\left(1 - \underbrace{X \frac{\delta}{\gamma}\left[k B_{oy}'\right]\frac{\phi_1}{\psi_1(0)}}_{= -\chi(X) \text{ by definition}}\right)$$

So, for $\chi(X) = -\frac{\delta}{\gamma}\left[k B_{oy}'\right]\frac{\phi_1}{\psi_1(0)}$, (16)

we have

$$\underbrace{\frac{\eta}{\delta^3}\left[k B_{oy}'\right]^{-1}\frac{\gamma}{\delta}\left[k B_{oy}'\right]^{-1}}\frac{\partial^2 \chi}{\partial X^2} = X(1 + \chi X)$$

Define δ so that this is 1: $\boxed{\delta^4 = \eta\gamma\left[k B_{oy}'\right]^{-2}}$ (17) [same as (12)]

So, we get $\boxed{\dfrac{\partial^2 \chi}{\partial X^2} = X(1+\chi X)}$ (18)

This equation contains no ~~more~~ parameters.

Matching condition (8) in these variables becomes:

$$\Delta' = \frac{1}{\psi_1(0)} \int_{-\delta\cdot\infty}^{\delta\cdot\infty} dx\, \underbrace{\psi_1''(x)}_{\text{substitute from (9)}} = -\frac{1}{\psi_1(0)} \int_{-\delta\cdot\infty}^{\delta\cdot\infty} dx \frac{\gamma \phi_1''}{k x \underbrace{\psi_0''(0)}_{B'_{0y}}} =$$

$$= -\frac{1}{\psi_1(0)} \int_{-\infty}^{+\infty} dX\, \gamma \left[k B'_{0y}\right]^{-1} \frac{1}{\delta^2} \frac{1}{X} \frac{\partial^2 \phi_1}{\partial X^2} \overset{\text{use (16)}}{=}$$

$$= \gamma \left[k B'_{0y}\right]^{-1} \frac{1}{\delta^2} \frac{\gamma}{\delta} \left[k B'_{0y}\right]^{-1} \int_{-\infty}^{+\infty} dX \underbrace{\frac{1}{X} \frac{\partial^2 \chi}{\partial X^2}}_{\text{use (18)}} =$$

$$= \underbrace{\frac{\gamma^2}{\delta^3}}_{\text{use (17)}} \left[k B'_{0y}\right]^{-2} \underbrace{\int_{-\infty}^{+\infty} dX (1+\chi X)}_{\substack{\parallel \\ I \text{ some number (depends on nothing)}}}$$

$$\Delta' = I \left[k B'_{0y}\right]^{-2} \gamma^2 \eta^{-3/4} \gamma^{-3/4} \left[k B'_{0y}\right]^{3/2} =$$

$$= I \left[k B'_{0y}\right]^{-\frac{1}{2}} \gamma^{5/4} \eta^{-3/4}$$

$$\boxed{\gamma = \eta^{3/5} (\Delta')^{4/5} \left[k B'_{0y}\right]^{2/5} \cdot \underbrace{I^{-4/5}}_{\text{some number.}}} \quad (19) \quad \left[\text{same as (15)}\right]$$

Using (17) again

$$\boxed{\delta = \eta^{2/5} (\Delta')^{1/5} \left[k B'_{0y}\right]^{-2/5} \cdot I^{-1/5}} \quad (20) \quad \left[\text{same as (14)}\right]$$

Now we must check the assumption made at the beginning:

$$\tau_A \ll \frac{1}{\gamma} \ll \tau_R$$

$$\gamma \sim \underbrace{\eta^{3/5}}_{=(\tau_R k^2)^{-3/5}} (\Delta')^{4/5} \left[\underbrace{k B_{0y}}_{=\tau_A^{-1}} \frac{B'_{0y}}{B_{0y}} \right]^{2/5} = \tau_R^{-3/5} \tau_A^{-2/5} (\Delta')^{4/5} k^{-6/5} \left[\frac{B'_{0y}}{B_{0y}} \right]^{2/5}$$

$\Delta' \sim \frac{1}{L}$

scale length in y $k \sim \frac{1}{L}$

$\frac{1}{L}$ scale length in x

So

$$\gamma \sim \tau_R^{-3/5} \tau_A^{-2/5}$$

Suppose $\tau_A \ll \tau_R$ i.e. $S = \frac{\tau_R}{\tau_A} = \frac{k B_0}{\eta k^2} = \frac{v_A L}{\eta} \gg 1$

Lundquist number.

Then

$$\gamma \tau_A \sim S^{-3/5} \ll 1, \quad \gamma \tau_R \sim S^{2/5} \gg 1 \quad \text{OK}$$

A similar calculation gives

$$\delta \sim S^{-2/5} L \ll L \quad \text{OK}.$$

So our solution is consistent with our assumptions, which is nice.

In order to calculate I (if we insist on knowing it), we must solve eq. (18) and then do the integral.

The solution to this equation is, in fact, known. It is called the Rutherford-Furth solution:

$$\chi(X) = -\frac{X}{2}\int_0^1 d\mu\, e^{-\frac{1}{2}\mu X^2}\left(1-\mu^2\right)^{1/4} \qquad (21)$$

(you can check that it works!)

and

$$I = \int_{-\infty}^{+\infty} dX\,\left(1 + X\chi\right) = \frac{\pi\,\Gamma(3/4)}{\Gamma(1/4)} \qquad (22)$$

The solution for the linear tearing mode that I have described is called the Furth-Killeen-Rosenbluth or <u>FKR theory</u>.

It is valid only as long as the island width remains smaller than the resistive scale:

recall $w = 4\sqrt{\dfrac{\psi_1(0)}{\psi_0''(0)}} \equiv 4\sqrt{\dfrac{\psi_1(0)}{B_{0y}'}} \ll \delta$

or $\psi_1(0) \ll B_{0y}'\,\dfrac{\delta^2}{16}$

Since $\psi_1(0)$ is growing exponentially, this eventually breaks down and one must do nonlinear theory - see papers posted on the course blog.