NOTES DISTRIBUTED

§14. Alfvénic States / Finite-Amp. Alfvén Waves.

1) Let us go back to the exact Lagrangian MHD:

$$\rho_0 \frac{\partial^2 \vec{\xi}}{\partial t^2} = -J\,(\nabla_0 \vec{x})^{-1}\cdot\nabla_0\left(\frac{p_0}{J^\gamma} + \frac{|\vec{B}_0\cdot\nabla_0\vec{x}|^2}{8\pi J^2}\right) + \frac{1}{4\pi}\vec{B}_0\cdot\nabla_0\left(\frac{\vec{B}_0}{J}\cdot\nabla_0\vec{x}\right)$$

Consider the straight-field equilibrium, $\vec{B}_0 = B_0\hat{z}$ and assume incompressibility: $J = 1 = \text{const}$.

Then

$$\frac{\partial^2\vec{\xi}}{\partial t^2} = -(\nabla_0\vec{x})^{-1}\cdot\nabla_0\tilde{p} + \underbrace{\frac{B_0^2}{4\pi\rho_0}}_{v_A^2}\underbrace{\frac{\partial^2}{\partial z_0^2}\vec{x}}_{=\,\frac{\partial^2\vec{\xi}}{\partial z_0^2}}$$

$\tilde{p} \to \frac{1}{\rho_0}\left(p + \frac{B^2}{8\pi}\right)$ total pressure.

$$\frac{\partial^2\vec{\xi}}{\partial t^2} - v_A^2\frac{\partial^2\vec{\xi}}{\partial z_0^2} = -(\nabla_0\vec{x})^{-1}\cdot\nabla_0\tilde{p}$$

$$\boxed{(\mathbb{1} + \nabla_0\vec{\xi})\cdot\left(\frac{\partial}{\partial t} - v_A\frac{\partial}{\partial z_0}\right)\left(\frac{\partial}{\partial t} + v_A\frac{\partial}{\partial z_0}\right)\vec{\xi} = -\nabla_0\tilde{p}}$$

$\tilde{p}$ is determined by the incompressibility constraint:

$$\boxed{J = 1}$$

Exact solutions: $\vec{\xi} = \vec{\xi}_+(x_0, y_0, z_0 - v_A t)$

$\vec{\xi} = \vec{\xi}_-(x_0, y_0, z_0 + v_A t)$

with the constraint that $\vec{\xi}_\pm(x_0, y_0, z_0)$ is incompressible.

It will stay incompressible because ~~[illegible]~~ time evolution simply amounts to transformation to a frame moving with $\pm v_A$ along z.

Thus, $\nabla_0\tilde{p} = 0$ is OK.

There are finite-amplitude Alfvén waves (w. packets)

NB: $\vec{\xi}_+$ or $\vec{\xi}_-$ alone are solutions.

but $\vec{\xi}_+ + \vec{\xi}_-$ is <u>not</u> a solution because

J will contain interaction between them!

2) It is useful to look at these results in the Eulerian framework. Consider incompressible MHD:

$$\begin{cases} \dfrac{d\vec{u}}{dt} = -\nabla\tilde{p} + \vec{B}\cdot\nabla\vec{B} + \nu\nabla^2\vec{B}, & \text{where } \dfrac{\vec{B}}{\sqrt{4\pi\rho}} \mapsto \vec{B} \\ \dfrac{d\vec{B}}{dt} = \vec{B}\cdot\nabla\vec{u} + \eta\nabla^2\vec{B} & \nabla\cdot\vec{u} = 0. \end{cases}$$

Let $\vec{B} = B_0\hat{z} + \delta\vec{B}$.

Introduce <u>Elsässer variables</u>: $\vec{z}_+ = \vec{u} + \delta\vec{B}$, $\vec{z}_- = \vec{u} - \delta\vec{B}$.

Then

$$\frac{\partial\vec{z}_\pm}{\partial t} = \frac{\partial\vec{u}}{\partial t} \pm \frac{\partial\delta\vec{B}}{\partial t} = -\vec{u}\cdot\nabla\vec{u} - \nabla\tilde{p} + \vec{B}_0\cdot\nabla\delta\vec{B} + \delta\vec{B}\cdot\nabla\delta\vec{B} + \nu\nabla^2\vec{u}$$

$$\mp\vec{u}\cdot\nabla\delta\vec{B} \pm \vec{B}_0\cdot\nabla\vec{u} \pm \delta\vec{B}\cdot\nabla\vec{u} \pm \eta\nabla^2\delta\vec{B} =$$

$$= \pm\vec{B}_0\cdot\nabla\vec{z}_\pm - \nabla p - \vec{z}_\mp\cdot\nabla\vec{u} \mp \vec{z}_\mp\cdot\nabla\delta\vec{B} +$$

$$+ \nu\nabla^2\frac{\vec{z}_+ + \vec{z}_-}{2} \pm \eta\nabla^2\frac{\vec{z}_+ - \vec{z}_-}{2} =$$

$$= \pm\underbrace{\vec{B}_0\cdot\nabla}_{= v_A\frac{\partial}{\partial z}}\vec{z}_\pm - \nabla p - \vec{z}_\mp\cdot\nabla\vec{z}_\pm + \underbrace{\frac{\nu\pm\eta}{2}\nabla^2\vec{z}_+ + \frac{\nu\mp\eta}{2}\nabla^2\vec{z}_-}_{= \frac{\nu+\eta}{2}\nabla^2\vec{z}_\pm + \frac{\nu-\eta}{2}\nabla^2\vec{z}_\mp}$$

Thus,

$$\nabla\cdot\vec{z}_\pm = 0 \qquad \boxed{\frac{\partial\vec{z}_\pm}{\partial t} \mp v_A\frac{\partial\vec{z}_\pm}{\partial z} + \vec{z}_\mp\cdot\nabla\vec{z}_\pm = -\nabla p + \frac{\nu+\eta}{2}\nabla^2\vec{z}_\pm + \frac{\nu-\eta}{2}\nabla^2\vec{z}_\mp}$$

NB:
$$\vec{u} \pm \delta\vec{B} = \frac{\partial \vec{\xi}}{\partial t} \pm \left[\vec{B}_0 \cdot \nabla_0 \vec{x} - \vec{B}_0\right] = $$
$$= \frac{\partial \vec{\xi}}{\partial t} \pm V_A \frac{\partial \vec{\xi}}{\partial z}$$

Exact solutions: $\vec{z}_{\mp} = \vec{u} \mp \delta\vec{B} = 0$

then
$$\frac{\partial \vec{z}_\pm}{\partial t} \mp V_A \frac{\partial \vec{z}_\pm}{\partial z} = -\cancel{\nabla p} + \frac{\nu+\eta}{2}\nabla^2 \vec{z}_\pm$$

$\cancel{\nabla p} \to 0$

$\nabla \cdot \vec{z}_\pm = 0$ satisfied always if satisfied initially.

This is simply Alfvén waves + diffusion.

$\vec{u} = \pm\delta\vec{B}$ are called Alfvénic (or Elsasser) states.

<u>Energetics of Elsässer variables:</u>

$$|\vec{z}_\pm|^2 = u^2 \pm 2\vec{u}\cdot\delta\vec{B} + \delta B^2$$

$$|\vec{z}_+|^2 + |\vec{z}_-|^2 = u^2 + \delta B^2 \quad \text{total energy} \quad \text{(when integrated)}$$

$$|\vec{z}_+|^2 - |\vec{z}_-|^2 = 4\vec{u}\cdot\delta\vec{B} \quad \text{cross-helicity}$$

Thus, the "energy" of each of the E. variables is conserved. Cross-helicity measures the imbalance between + and − fields.

[With these subjects we come very close to the MHD turbulence, but we have to postpone that and do Resistive MHD and HD Turbulence first.]