NOTES DISTRIBUTED

16.02.05

Lecture 11

§13. MHD Waves

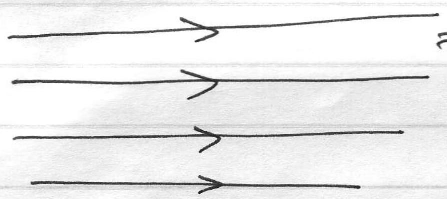
We had the following equation for the linearised momentum equation:

$$\rho_0 \frac{\partial^2 \vec{\xi}}{\partial t^2} = \vec{F}[\vec{\xi}] = \underbrace{\nabla(\gamma p_0 \nabla\cdot\vec{\xi} + \cancel{\vec{\xi}\cdot\nabla p_0})}_{= -\delta p} + \underbrace{\frac{-\nabla \vec{B}_0\cdot\vec{Q} + \vec{B}_0\cdot\nabla\vec{Q} + \cancel{\vec{Q}\cdot\nabla\vec{B}_0}}{4\pi}}_{= \delta\frac{(\nabla\times\vec{B})\times\vec{B}}{4\pi}}$$

where $\vec{Q} = \nabla\times(\vec{\xi}\times\vec{B}_0) = -\cancel{\vec{\xi}\cdot\nabla\vec{B}_0} + \vec{B}_0\cdot\nabla\vec{\xi} - \vec{B}_0\nabla\cdot\vec{\xi} = \delta\vec{B}$

(Note that ∇ and ∇_0 are the same in this order)

Now consider the following homogeneous equilibrium:



$\vec{B}_0 = B_0\hat{z} = \text{const}$, $\rho_0 = \text{const}$, $p_0 = \text{const}$

$$\frac{\partial^2\vec{\xi}}{\partial t^2} = \underbrace{\frac{\gamma p_0}{\rho_0}}_{\substack{= c_s^2 \\ \text{sound speed}}} \nabla\nabla\cdot\vec{\xi} + \underbrace{\frac{B_0^2}{4\pi\rho_0}}_{\substack{= v_A^2 \\ \text{Alfvén speed}}} \underbrace{\left[-\nabla\Big(\underbrace{\nabla_\parallel\xi_\parallel - \nabla\cdot\vec{\xi}}_{= -\nabla_\perp\cdot\vec{\xi}_\perp}\Big) + \nabla_\parallel\Big(\underbrace{\nabla_\parallel\vec{\xi} - \hat{z}\nabla\cdot\vec{\xi}}_{= \nabla_\parallel\vec{\xi}_\perp - \hat{z}\nabla_\perp\cdot\vec{\xi}_\perp}\Big)\right]}_{= \nabla\nabla_\perp\cdot\vec{\xi}_\perp + \nabla_\parallel^2\vec{\xi}_\perp - \hat{z}\nabla_\parallel\nabla_\perp\cdot\vec{\xi}_\perp = \nabla_\perp\nabla_\perp\cdot\vec{\xi}_\perp + \nabla_\parallel^2\vec{\xi}_\perp}$$

$$= c_s^2\nabla\nabla\cdot\vec{\xi} + v_A^2\left(\nabla_\perp\nabla_\perp\cdot\vec{\xi}_\perp + \nabla_\parallel^2\vec{\xi}_\perp\right)$$

NB: $\delta\vec{B} = \vec{Q} = B_0\left(\nabla_\parallel\vec{\xi} - \hat{z}\nabla\cdot\vec{\xi}\right) = B_0\left(\nabla_\parallel\vec{\xi}_\perp - \hat{z}\nabla_\perp\cdot\vec{\xi}_\perp\right)$

Fourier transform: $\vec{\xi} \sim \vec{\xi}_{k\omega}\, e^{-i\omega t + i\vec{k}\cdot\vec{x}}$

$$\omega^2 \vec{\xi} = c_s^2 \vec{k}\,\vec{k}\cdot\vec{\xi} + v_A^2 \left(\vec{k}_\perp \vec{k}_\perp\cdot\vec{\xi}_\perp + k_\parallel^2 \vec{\xi}_\perp\right)$$

or
$$\begin{cases} \omega^2 \xi_\parallel = c_s^2 \left[k_\parallel (\vec{k}_\perp\cdot\vec{\xi}_\perp) + k_\parallel^2 \xi_\parallel\right] & (1) \\ \omega^2 \vec{\xi}_\perp = (c_s^2 + v_A^2)\,\vec{k}_\perp (\vec{k}_\perp\cdot\vec{\xi}_\perp) + c_s^2 \vec{k}_\perp k_\parallel \xi_\parallel + v_A^2 k_\parallel^2 \vec{\xi}_\perp & (2) \end{cases}$$

$$\frac{\delta\vec{B}}{B_0} = i k_\parallel \vec{\xi}_\perp - \hat{z}\, i (\vec{k}_\perp\cdot\vec{\xi}_\perp)$$

Note that

$$\frac{\delta\vec{B}}{B_0} = \frac{\delta(B\hat{b})}{B_0} = \frac{B_0\delta\hat{b}}{B_0} + \underset{\parallel\, \hat{z}}{\hat{b}}\frac{\delta B}{B_0} = \delta\hat{b} + \hat{z}\frac{\delta B}{B_0}$$

and $\hat{b}^2 = 1 = (\hat{z}+\delta\hat{b})\cdot(\hat{z}+\delta\hat{b}) \simeq 1 + 2\hat{z}\cdot\delta\hat{b} = 1 \Rightarrow \delta\hat{b} \perp \hat{z}$

So: $\underset{\text{field direction}}{\delta\hat{b} = i k_\parallel \vec{\xi}_\perp}$ and $\underset{\text{field strength.}}{\frac{\delta B}{B_0} = -i(\vec{k}_\perp\cdot\vec{\xi}_\perp)}$

Also $\frac{\delta p}{p_0} = -i\gamma\,\vec{k}\cdot\vec{\xi}$ and $\frac{\delta\rho}{\rho_0} = -i\vec{k}\cdot\vec{\xi}$

First consider simple special cases.

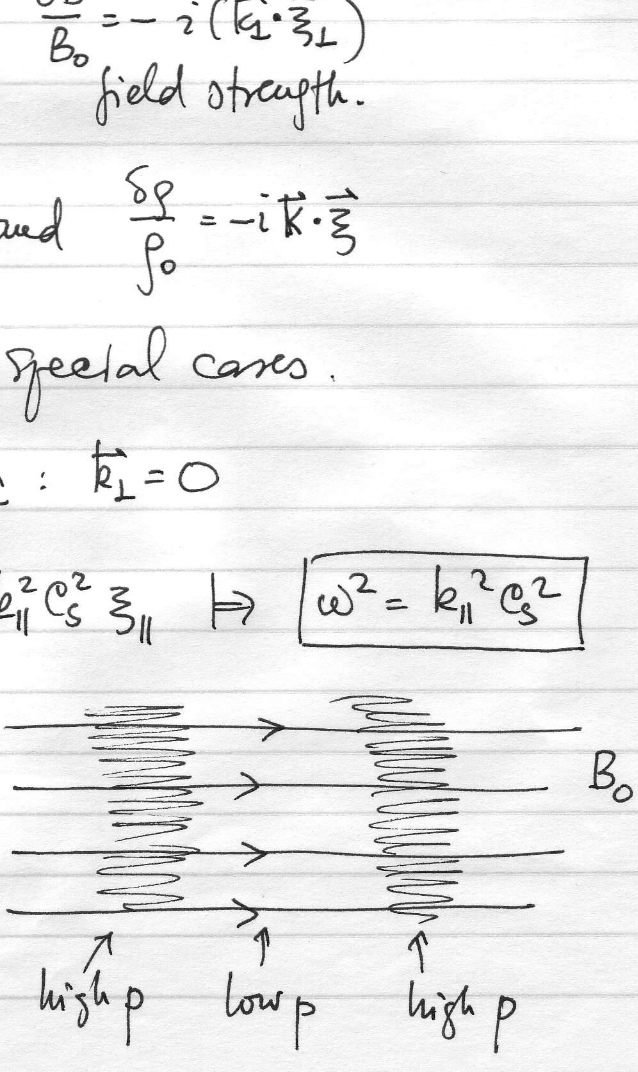
1) <u>Parallel propagation</u>: $\vec{k}_\perp = 0$

- <u>Sound wave</u>: $\omega^2 \xi_\parallel = k_\parallel^2 c_s^2 \xi_\parallel \Rightarrow \boxed{\omega^2 = k_\parallel^2 c_s^2}$ (1)

$\vec{\xi}_\perp = 0$

$\delta\vec{B} = 0$

$\frac{\delta p}{p_0} = -i\gamma k_\parallel \xi_\parallel$



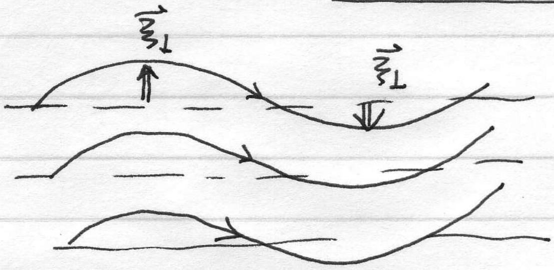
• Alfvén wave : $\omega^2 \vec{\xi}_\perp = k_\parallel^2 V_A^2 \vec{\xi}_\perp \Rightarrow \boxed{\omega^2 = k_\parallel^2 V_A^2}$

$\uparrow$ ②

$\xi_\parallel = 0$

$\delta\hat{b} = i k_\parallel \vec{\xi}_\perp$, $\delta B = 0$

$\delta p = 0$ (incompressible)



NB: curvature force $\delta(\hat{b}\cdot\nabla\hat{b}) = i k_\parallel \delta\hat{b} =$
$= -k_\parallel^2 \vec{\xi}_\perp$

2) Perpendicular propagation : $k_\parallel = 0$

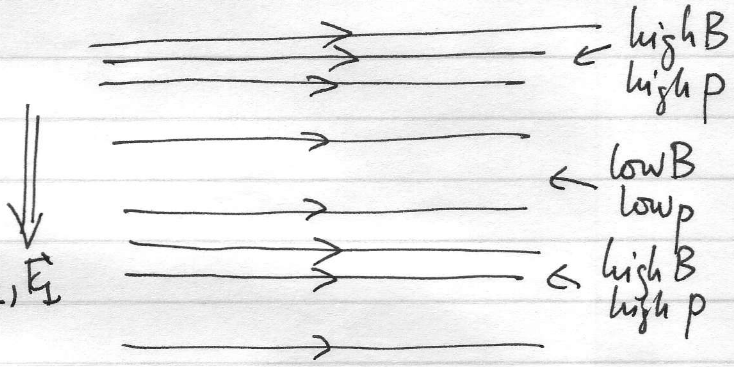
• Magnetosonic wave ② $\omega^2 \vec{\xi}_\perp = (c_s^2 + V_A^2)\, \vec{k}_\perp (\vec{k}_\perp \cdot \vec{\xi}_\perp) \Rightarrow \boxed{\omega^2 = (c_s^2 + V_A^2) k_\perp^2}$

① $\xi_\parallel = 0$

$\vec{\xi}_\perp \parallel \vec{k}_\perp$

$\delta\hat{b} = 0$, $\dfrac{\delta B}{B_0} = -i(\vec{k}_\perp \cdot \vec{\xi}_\perp)$

$\dfrac{\delta p}{p_0} = -i\gamma (\vec{k}_\perp \cdot \vec{\xi}_\perp)$



This is like a sound wave but with p and $\dfrac{B^2}{8\pi}$ combined.

3) Oblique propagation : general case.

$\vec{k} = (k_\perp, 0, k_\parallel)$

• Shear Alfvén wave. $②_y$: $\omega^2 \xi_y = k_\parallel^2 V_A^2 \xi_y \Rightarrow \boxed{\omega^2 = k_\parallel^2 V_A^2}$

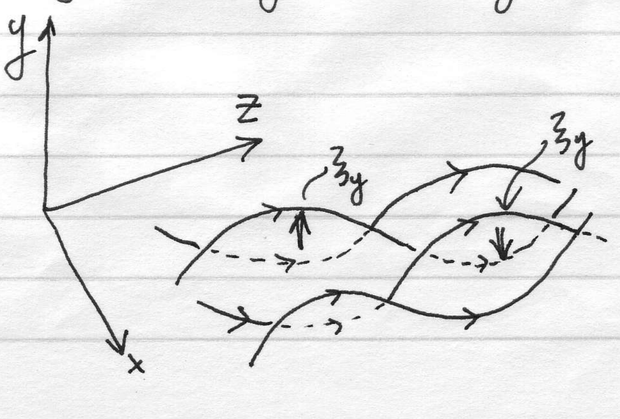
① $\Rightarrow \xi_\parallel = 0$

$②_x \Rightarrow \xi_x = 0$

$\vec{\xi}_\perp \perp \vec{k}_\perp$

$\delta\hat{b} = i k_\parallel \xi_y \hat{y}$, $\delta B = 0$

$\delta p = 0$ (incompressible)



Now ② $_x$: $\omega^2 \xi_x = (c_s^2 + V_A^2) k_\perp^2 \xi_x + c_s^2 k_\perp k_\parallel \xi_\parallel + V_A^2 k_\parallel^2 \xi_x$ ③

$\xi_y = 0$ ①: $\omega^2 \xi_\parallel = c_s^2 (k_\parallel k_\perp \xi_x + k_\parallel^2 \xi_\parallel)$ ④

$$\omega^2 \begin{pmatrix} \xi_x \\ \xi_\parallel \end{pmatrix} = \begin{pmatrix} c_s^2 k_\perp^2 + V_A^2 k^2 & c_s^2 k_\perp k_\parallel \\ c_s^2 k_\parallel k_\perp & c_s^2 k_\parallel^2 \end{pmatrix} \cdot \begin{pmatrix} \xi_x \\ \xi_\parallel \end{pmatrix}$$

$$\omega^4 - k^2 (c_s^2 + V_A^2)\omega^2 + c_s^2 V_A^2 k^2 k_\parallel^2 = 0$$

$$\boxed{\omega^2 = \frac{1}{2} k^2 \left[c_s^2 + V_A^2 \pm \sqrt{(c_s^2 + V_A^2)^2 - 4 c_s^2 V_A^2 \underbrace{\frac{k_\parallel^2}{k^2}}_{=\cos^2\theta}} \right]}$$

~~③ $\omega^2 \xi_x = c_s^2 k_\perp (k_\perp \xi_x + k_\parallel \xi_\parallel) + V_A^2 k^2 \xi_x$~~

~~④ $\omega^2 \xi_\parallel = c_s^2 k_\parallel (k_\perp \xi_x + k_\parallel \xi_\parallel)$~~

~~$\omega^2 \xi_x = \frac{k_\perp}{k_\parallel} \omega^2 \xi_\parallel + V_A^2 k^2 \xi_x$~~

⊕ fast wave

⊖ slow wave

NB: $\theta = \frac{\pi}{2}$ ($\perp$ propagation): ⊕ $\Rightarrow$ magnetosonic

⊖ $\Rightarrow$ $\omega = 0$ (entropy mode)

$\theta = 0$ ($\parallel$ propagation): $\omega^2 = \frac{1}{2} k^2 \left[c_s^2 + V_A^2 \pm \sqrt{(c_s^2 - V_A^2)^2} \right]$

high β $c_s^2 > V_A^2$: ⊕ $\Rightarrow$ sound

⊖ $\Rightarrow$ Alfvén

low β $c_s^2 < V_A^2$: ⊕ $\Rightarrow$ Alfvén

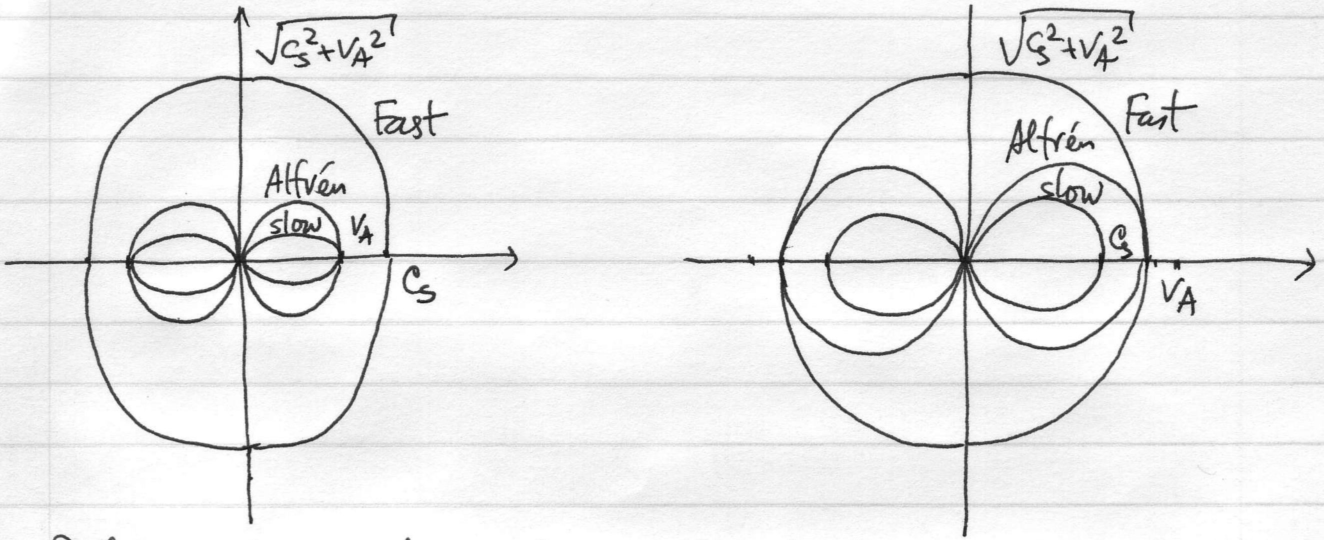
⊖ $\Rightarrow$ sound

NB:

$$\frac{c_s^2}{V_A^2} = \frac{\gamma p_0}{\rho_0} \frac{4\pi \rho_0}{B_0^2} = \frac{\gamma \beta}{2}$$

Friedricks diagrams:

$c_s^2 > V_A^2$ (high β) $\qquad\qquad$ $c_s^2 < V_A^2$ (low β)



Polar plot: radius $= \frac{\omega}{k}$ (phase velocity), angle $= \theta$

Incompressible limit: $c_s^2 \gg V_A^2$ $\qquad$ Lecture 12 $\qquad$ 18.02.05

$$\omega^2 = \frac{1}{2} k^2 (c_s^2 + V_A^2)\left[1 \pm \sqrt{1 - 4\frac{c_s^2 V_A^2}{(c_s^2+V_A^2)^2}\cos^2\theta}\,\right] \approx$$

$$\approx \frac{1}{2} k^2 c_s^2 \left[1 \pm \sqrt{1 - 4\frac{V_A^2}{c_s^2}\cos^2\theta}\,\right] \approx$$

$$\approx \frac{1}{2} k^2 c_s^2 \left[1 \pm 1 \mp 2\frac{V_A^2}{c_s^2}\cos^2\theta\right]$$

Fast: $\omega^2 = k^2 c_s^2$ sound wave

Slow: $\omega^2 = k^2 V_A^2 \cos^2\theta = k_\parallel^2 V_A^2$ pseudo-Alfvén wave

$$3 \Rightarrow \quad \omega^2 \xi_x = c_s^2 k_\perp \underbrace{(k_\perp \xi_x + k_\parallel \xi_\parallel)}_{= \vec{k}\cdot\vec{\xi}} + V_A^2 k^2 \xi_x \qquad 5$$

$$4 \Rightarrow \quad \omega^2 \xi_\parallel = c_s^2 k_\parallel \overbrace{(k_\perp \xi_x + k_\parallel \xi_\parallel)}^{=} \qquad 6$$

- Sound Wave : ⑤ $\Rightarrow \omega^2 \xi_x = c_s^2 k_\perp \vec{k}\cdot\vec{\xi}$
  ⑥ $\Rightarrow \omega^2 \xi_\parallel = c_s^2 k_\parallel \vec{k}\cdot\vec{\xi}$ $\Rightarrow \boxed{\omega^2 = k^2 c_s^2}$ OK

- Slow/pseudo-Alfvén wave : $\boxed{\omega^2 = k_\parallel^2 v_A^2}$

⑥ $\Rightarrow k_\parallel^2 v_A^2 \xi_\parallel = c_s^2 k_\parallel \vec{k}\cdot\vec{\xi} \Rightarrow \vec{k}\cdot\vec{\xi} = \frac{v_A^2}{c_s^2} k_\parallel \xi_\parallel \approx -\frac{v_A^2}{c_s^2} k_\perp \xi_x$

small (nearly incompressible)

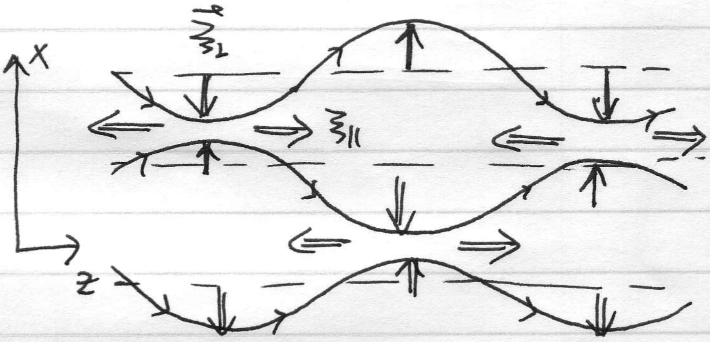
⑤ $\Rightarrow \omega^2 \xi_x = -k_\perp^2 v_A^2 \xi_x + k^2 v_A^2 \xi_x = k_\parallel^2 v_A^2 \xi_x$ OK

NB: While the disp. relation is the same as for shear Alfvén waves, the physics is different:

$\vec{\xi}_\perp = \xi_x \hat{x} \parallel \vec{k}_\perp$, $\xi_\parallel \approx -\frac{k_\perp}{k_\parallel}\xi_x$

$\delta\hat{b} = i k_\parallel \xi_x \hat{x}$, $\frac{\delta B}{B_0} = -i k_\perp \xi_x$

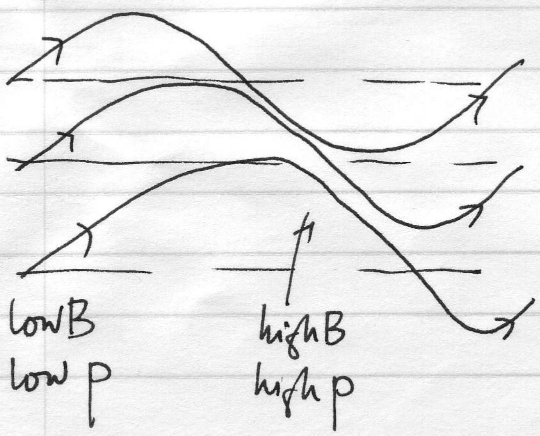
$\frac{\delta p}{p_0} = i\gamma \frac{v_A^2}{c_s^2} k_\perp \xi_x$ small



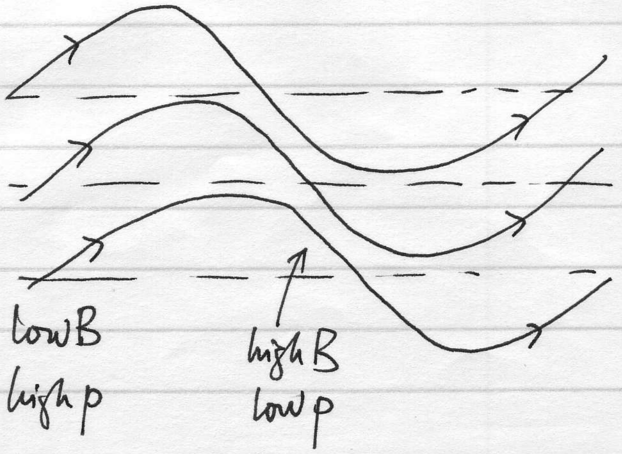
Exercise : Derive these waves directly from the incompressible form of the MHD equations.

More generally,

Fast waves:



Slow waves



NB: $\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{u} \times \vec{B})$ — ∥ velocities don't change field

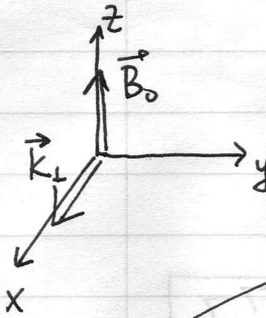
$\frac{\partial \vec{u}}{\partial t} = \dots \frac{1}{c} \vec{j} \times \vec{B}$ — no forces ∥ field.

$c_s^2 = \frac{\gamma p_0}{\rho_0}, \quad v_A^2 = \frac{B_0^2}{4\pi \rho_0}$

Review of last time:

$$\omega^2 \vec{\xi} = c_s^2 \vec{k}\, \vec{k} \cdot \vec{\xi} + v_A^2 \left(\vec{k}_\perp \vec{k}_\perp \cdot \vec{\xi}_\perp + k_\parallel^2 \vec{\xi}_\perp \right)$$

$$\Downarrow$$



$$\begin{cases} \omega^2 \xi_x = c_s^2 k_\perp \left(k_\perp \xi_x + k_\parallel \xi_\parallel \right) + v_A^2 k^2 \xi_x \\ \omega^2 \xi_y = v_A^2 k_\parallel^2 \xi_y \\ \omega^2 \xi_\parallel = c_s^2 k_\parallel \left(k_\perp \xi_x + k_\parallel \xi_\parallel \right) \end{cases}$$

$\boxed{\omega^2 = k_\parallel^2 v_A^2}$ and $\delta \hat{b} = i k_\parallel \vec{\xi}_\perp, \quad \frac{\delta B}{B_0} = -i k_\perp \xi_x$

shear Alfvén wave

$\delta B = 0, \delta p = 0, \delta \hat{b} \neq 0$ $\quad \frac{\delta p}{p_0} = \gamma \frac{\delta \rho}{\rho_0} = -i\gamma \left(k_\perp \xi_x + k_\parallel \xi_\parallel \right)$

$$\boxed{\omega^2 = \frac{1}{2} k^2 \left[c_s^2 + v_A^2 \pm \sqrt{\left(c_s^2 + v_A^2 \right)^2 - 4 c_s^2 v_A^2 \cos^2\theta} \right]} \qquad \cos^2\theta = \frac{k_\parallel^2}{k^2}$$

⊕ fast
⊖ slow

$\delta \hat{b} \neq 0, \ \delta B \neq 0, \ \delta p \neq 0$

		⊕ fast	⊖ slow	shear Alfvén
$\theta = \frac{\pi}{2}$ (⊥ prop)		magnetosonic $\omega^2 = k_\perp^2 (c_s^2 + v_A^2)$	$\omega = 0$	$\omega = 0$
$\theta = 0$ (∥ prop)	$c_s^2 > v_A^2$	sound $\omega^2 = k_\parallel^2 c_s^2$	Alfvén $\omega^2 = k_\parallel^2 v_A^2$	Alfvén $\omega^2 = k_\parallel^2 v_A^2$
	$c_s^2 < v_A^2$	Alfvén $\omega^2 = k_\parallel^2 v_A^2$	sound $\omega^2 = k_\parallel^2 c_s^2$	

$$\vec{u} = \frac{\partial \vec{\xi}}{\partial t}$$

- $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$

$$\frac{\partial \delta \rho}{\partial t} + \nabla \cdot \left(\rho_0 \frac{\partial \vec{\xi}}{\partial t}\right) = 0$$

$$\boxed{\delta \rho = - \nabla \cdot (\rho_0 \vec{\xi})}$$

- $$\frac{\partial p}{\partial t} + \vec{u} \cdot \nabla p + \gamma p \nabla \cdot \vec{u} = 0$$

$$\frac{\partial \delta p}{\partial t} + \frac{\partial \vec{\xi}}{\partial t} \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \frac{\partial \vec{\xi}}{\partial t} = 0$$

$$\boxed{\delta p = - \vec{\xi} \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \vec{\xi}}$$

- $$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{u} \times \vec{B})$$

$$\frac{\partial \delta \vec{B}}{\partial t} = \nabla \times \left(\frac{\delta \vec{\xi}}{\partial t} \times \vec{B}_0\right)$$

$$\boxed{\delta \vec{B} = \nabla \times (\vec{\xi} \times \vec{B}_0)}$$