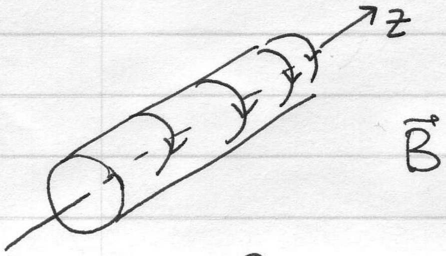
Lecture 10 14.02.05

NB: Chandrasekhar book

NOTES DISTRIBUTED

§12 Instabilities

1) Stability of the z pinch.



Equilibrium:

$$\vec{B} = B_\theta(r)\hat{\theta}, \quad \vec{J} = j_z(r)\hat{z}, \quad J_z(r) = \frac{c}{4\pi}\frac{1}{r}\frac{\partial}{\partial r} rB_\theta$$

$$\frac{\partial p}{\partial r} = -\frac{1}{c} j_z B_\theta = -\frac{B_\theta}{4\pi r}\frac{\partial}{\partial r} rB_\theta$$

System is symmetric in z and θ, so can Fourier transform:

$$\vec{\xi} = \sum_{m,k} \vec{\xi}_{mk}(r) e^{i(m\theta + kz)}$$

NB $\int_{-L_z}^{L_z} = 2L_z$

$$\boxed{\begin{aligned}\delta W_2[\vec{\xi},\vec{\xi}] &= - \sum_{mk} 2\pi L_z \int_0^\infty dr\, r\, \vec{\xi}^*_{mk}\cdot \vec{F}[\vec{\xi}_{mk}] = \\ &= +\frac{1}{2}\sum_{mk}\int d^3x \left[\gamma p |\nabla\cdot\vec{\xi}_{mk}|^2 + \underbrace{\frac{|\vec{Q}|^2}{4\pi} + (\nabla\cdot\vec{\xi}^*_{mk})\vec{\xi}_{mk}\cdot\nabla p + \frac{\vec{J}\cdot(\vec{\xi}^*_{mk}\times\vec{Q})}{c}}_{(*)}\right]\end{aligned}}$$

- $\nabla\cdot\vec{\xi} = \frac{1}{r}\frac{\partial}{\partial r} r\xi_r + \frac{im}{r}\xi_\theta + ik\xi_z$

- $\vec{Q} = \nabla\times(\vec{\xi}\times\vec{B}) = \nabla\times \begin{bmatrix}\xi_r\\ \xi_\theta\\ \xi_z\end{bmatrix}\times\begin{bmatrix}0\\ B_\theta\\ 0\end{bmatrix} = \nabla\times\begin{bmatrix}-\xi_z B_\theta\\ 0\\ \xi_r B_\theta\end{bmatrix} =$

$$= \hat{r}\left(\frac{1}{r} im\,\xi_r B_\theta\right) + \hat{\theta}\left(-ik\xi_z B_\theta - \frac{\partial}{\partial r}\xi_r B_\theta\right) + \hat{z}\left(-\frac{1}{r} im\,\xi_z B_\theta\right)$$

- $\frac{1}{c}\vec{J}\cdot(\vec{\xi}^*\times\vec{Q}) = \frac{1}{4\pi}\left(\frac{1}{r}\frac{\partial}{\partial r} rB_\theta\right)\left(\xi_r^* Q_\theta - \xi_\theta^* Q_r\right) =$

$$= \frac{1}{4\pi}\left(\frac{1}{r}\frac{\partial}{\partial r} rB_\theta\right)\left(-ik\xi_r^*\xi_z B_\theta - |\xi_r|^2\frac{\partial B_\theta}{\partial r} - B_\theta\xi_r^*\frac{\partial\xi_r}{\partial r} - \frac{im}{r}\xi_r\xi_\theta^* B_\theta\right)$$

- $(\nabla\cdot\vec{\xi}^*)\vec{\xi}\cdot\nabla p = \left(\frac{1}{r}\frac{\partial}{\partial r} r\xi_r^* - \frac{im}{r}\xi_\theta^* - ik\xi_z^*\right)\xi_r\frac{\partial p}{\partial r} =$

$$= \frac{1}{4\pi}\left(\frac{1}{r}\frac{\partial}{\partial r} rB_\theta\right)\left(-\xi_r\frac{\partial\xi_r^*}{\partial r}B_\theta + \frac{im}{r}\xi_\theta^*\xi_r B_\theta + ik\xi_z^*\xi_r B_\theta\right) + \frac{|\xi_r|^2}{r}\frac{\partial p}{\partial r}$$

$$\bullet\ \frac{|\vec{Q}|^2}{4\pi} = \frac{1}{4\pi}\Bigg[\frac{m^2}{r^2}|\xi_r|^2 B_\theta^2 + \frac{m^2}{r^2}|\xi_z|^2 B_\theta^2 + \underbrace{\left|\xi_r\frac{\partial B_\theta}{\partial r} + B_\theta\frac{\partial \xi_r}{\partial r} + ik\xi_z B_\theta\right|^2}\Bigg]$$

$$B_\theta^2\left|\frac{\partial \xi_r}{\partial r} + ik\xi_z\right|^2 + |\xi_r|^2\left(\frac{\partial B_\theta}{\partial r}\right)^2 + \xi_r^*\frac{\partial \xi_r}{\partial r}B_\theta\frac{\partial B_\theta}{\partial r} + ik\xi_z\xi_r^* B_\theta\frac{\partial B_\theta}{\partial r} + \xi_r\frac{\partial \xi_r^*}{\partial r}B_\theta\frac{\partial B_\theta}{\partial r} - ik\xi_z^*\xi_r B_\theta\frac{\partial B_\theta}{\partial r}$$

$$(*) = \frac{|\xi_r|^2}{r}\frac{\partial p}{\partial r} + \frac{1}{4\pi}\left(\frac{1}{r}\frac{\partial}{\partial r}rB_\theta\right)\left(-\xi_r\frac{\partial \xi_r^*}{\partial r}B_\theta + \cancel{\frac{im}{r}\xi_\theta^*\xi_r B_\theta} + ik\xi_z^*\xi_r B_\theta - ik\xi_r^*\xi_z B_\theta - |\xi_r|^2\frac{\partial B_\theta}{\partial r} - B_\theta\xi_r^*\frac{\partial \xi_r}{\partial r} - \cancel{\frac{im}{r}\xi_r\xi_\theta^* B_\theta}\right) +$$

$$+ \frac{1}{4\pi}\Bigg[B_\theta^2\frac{m^2}{r^2}\left(|\xi_r|^2 + |\xi_z|^2\right) + B_\theta^2\left|\frac{\partial \xi_r}{\partial r} + ik\xi_z\right|^2 + \cancel{|\xi_r|^2\left(\frac{\partial B_\theta}{\partial r}\right)^2} + \cancel{\left(\xi_r^*\frac{\partial \xi_r}{\partial r} + \xi_r\frac{\partial \xi_r^*}{\partial r}\right)B_\theta\frac{\partial B_\theta}{\partial r}} + \cancel{ik\left(\xi_z\xi_r^* - \xi_z^*\xi_r\right)B_\theta\frac{\partial B_\theta}{\partial r}}\Bigg] =$$

$$\frac{1}{4\pi}\Bigg[-\frac{B_\theta^2}{r}\left(\xi_r\frac{\partial \xi_r^*}{\partial r} + \xi_r^*\frac{\partial \xi_r}{\partial r}\right) - \cancel{B_\theta\frac{\partial B_\theta}{\partial r}\left(\xi_r\frac{\partial \xi_r^*}{\partial r} + \xi_r^*\frac{\partial \xi_r}{\partial r}\right)} - \underbrace{\frac{B_\theta}{r}\frac{\partial B_\theta}{\partial r}|\xi_r|^2} - \cancel{\left(\frac{\partial B_\theta}{\partial r}\right)^2|\xi_r|^2} + ik\left(\xi_z^*\xi_r - \xi_r^*\xi_z\right)\left(\frac{B_\theta^2}{r} + \cancel{B_\theta\frac{\partial B_\theta}{\partial r}}\right)\Bigg]$$

$$-\underbrace{\left(\frac{B_\theta}{r}\frac{\partial}{\partial r}rB_\theta\right)}_{=\,4\pi\frac{\partial p}{\partial r}}\frac{|\xi_r|^2}{r} + \frac{B_\theta^2}{r^2}|\xi_r|^2$$

$$= 2\frac{\partial p}{\partial r}\frac{|\xi_r|^2}{r} + \frac{B_\theta^2}{4\pi}\frac{m^2}{r^2}\left(|\xi_r|^2+|\xi_z|^2\right) +$$

$$+\frac{B_\theta^2}{4\pi}\Big[\underbrace{\left|\frac{\partial \xi_r}{\partial r}+ik\xi_z\right|^2 - \frac{1}{r}\left(\xi_r\frac{\partial\xi_r^*}{\partial r}+\xi_r^*\frac{\partial\xi_r}{\partial r}\right)+\frac{1}{r^2}|\xi_r|^2+\frac{ik}{r}\left(\xi_z^*\xi_r-\xi_r^*\xi_z\right)}_{=}\Big]$$

$$\left|-\frac{\xi_r}{r}+\frac{\partial\xi_r}{\partial r}+ik\xi_z\right|^2 = \left|r\frac{\partial}{\partial r}\frac{\xi_r}{r}+ik\xi_z\right|^2$$

Thus,

$$\boxed{\delta W_2 = \sum_{mk} 2\pi L_z\int_0^\infty dr\,r\Big[\gamma p\underbrace{\left|\frac{1}{r}\frac{\partial}{\partial r}r\xi_r+\frac{im}{r}\xi_\theta+ik\xi_z\right|^2}_{(1)} + 2\frac{\partial p}{\partial r}\frac{|\xi_r|^2}{r}+\frac{B_\theta^2}{4\pi}\frac{m^2}{r^2}\left(|\xi_r|^2+|\xi_z|^2\right)+\frac{B_\theta^2}{4\pi}\left|r\frac{\partial}{\partial r}\frac{\xi_r}{r}+ik\xi_z\right|^2\Big]}$$

$\boxed{m=0}$

$$\delta W_{2(0,k)} = 2\pi L_z\int_0^\infty dr\,r\Big[\gamma p\left|\frac{1}{r}\frac{\partial}{\partial r}r\xi_r+ik\xi_z\right|^2+2\frac{\partial p}{\partial r}\frac{|\xi_r|^2}{r}+$$
$$+\frac{B_\theta^2}{4\pi}\left|r\frac{\partial}{\partial r}\frac{\xi_r}{r}+ik\xi_z\right|^2\Big]$$

ξ_z only appears algebraically: can minimise the integrand by setting $\frac{\partial}{\partial\xi_z}[\ldots]=0$ and $\frac{\partial}{\partial\xi_z^*}[\ldots]=0$

$$\gamma p\left[-i\cancel{k}\left(\frac{1}{r}\frac{\partial}{\partial r}r\xi_r+ik\xi_z\right)\right]+$$
$$+\frac{B_\theta^2}{4\pi}\left[-i\cancel{k}\left(r\frac{\partial}{\partial r}\frac{\xi_r}{r}+ik\xi_z\right)\right]=0$$

$$\left(\gamma p+\frac{B_\theta^2}{4\pi}\right)ik\xi_z+\gamma p\underbrace{\frac{1}{r}\frac{\partial}{\partial r}r\xi_r}_{=\frac{\xi_r}{r}+\frac{\partial\xi_r}{\partial r}}+\frac{B_\theta^2}{4\pi}\underbrace{r\frac{\partial}{\partial r}\frac{\xi_r}{r}}_{=\frac{\partial\xi_r}{\partial r}-\frac{\xi_r}{r}}=0$$

$$ik\xi_z = -\frac{\gamma p - B_\theta^2/4\pi}{\gamma p + B_\theta^2/4\pi}\frac{\xi_r}{r} - \frac{\partial \xi_r}{\partial r}$$

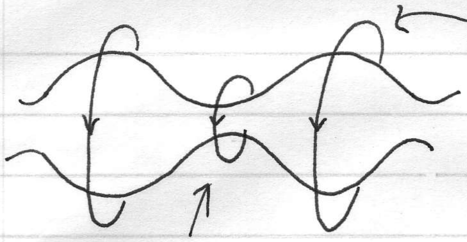
$$\delta W_{2(0,k)} = 2\pi L_z \int_0^\infty dr\, r \left[\gamma p \left(\frac{B_\theta^2/2\pi}{\gamma p + B_\theta^2/4\pi}\right)^2 \frac{|\xi_r|^2}{r^2} + 2\frac{\partial p}{\partial r}\frac{|\xi_r|^2}{r} + \right.$$

$$\left. + \frac{B_\theta^2}{4\pi}\left(\frac{2\gamma p}{\gamma p + B_\theta^2/4\pi}\right)^2 \frac{|\xi_r|^2}{r^2}\right] =$$

$$= 2\pi L_z \int_0^\infty dr \frac{|\xi_r|^2}{r}\left[2r\frac{\partial p}{\partial r} + 4\gamma p \frac{B_\theta^2}{4\pi}\frac{1}{\gamma p + \frac{B_\theta^2}{4\pi}}\right] < 0$$

Instability if $\boxed{-r\frac{\partial \ln p}{\partial r} > 2\gamma \frac{B_\theta^2/4\pi}{\gamma p + B_\theta^2/4\pi}}$

This is called the "<u>sausage mode</u>"



$\boxed{m \neq 0}$ First minimize wrt ξ_θ : it only appears in the term ①.

$$\frac{1}{r}\frac{\partial}{\partial r} r\xi_r + \frac{im}{r}\xi_\theta + ik\xi_z = 0 \qquad (\nabla\cdot\vec{\xi} = 0 \text{ incompr.})$$

Now minimize the remainder wrt ξ_z : just

$$\frac{\partial}{\partial \xi_z}[\ldots] = 0 \quad \text{and} \quad \frac{\partial}{\partial \xi_z^*}[\ldots] = 0$$

$$\cancel{\frac{B_\theta^2}{4\pi}}\frac{m^2}{r^2}\xi_z + \cancel{\frac{B_\theta^2}{4\pi}}(-ik)\left(r\frac{\partial}{\partial r}\frac{\xi_r}{r} + ik\xi_z\right) = 0$$

$$\left(\frac{m^2}{r^2} + k^2\right)\xi_z = ikr\frac{\partial}{\partial r}\frac{\xi_r}{r}$$

$$\xi_z = \frac{ikr^3}{m^2+k^2r^2}\frac{\partial}{\partial r}\frac{\xi_r}{r} =$$

Substitute this:

$$\boxed{\delta W_2} = \sum_{mk} 2\pi L_z \int_0^\infty dr\, r\left[2\frac{\partial p}{\partial r}\frac{|\xi_r|^2}{r} + \frac{B_\theta^2}{4\pi}\frac{m^2}{r^2}\left(|\xi_r|^2 + \frac{k^2r^6}{(m^2+k^2r^2)^2}\left|\frac{\partial}{\partial r}\frac{\xi_r}{r}\right|^2\right)\right.$$
$$\left. + \frac{B_\theta^2}{4\pi}\left(1 - \frac{k^2r^2}{m^2+k^2r^2}\right)^2\left|r\frac{\partial}{\partial r}\frac{\xi_r}{r}\right|^2\right] =$$

$$= \sum_{mk} 2\pi L_z \int_0^\infty dr\, r\left\{2\frac{\partial p}{\partial r}\frac{|\xi_r|^2}{r} + \frac{B_\theta^2}{4\pi}\frac{m^2}{r^2}|\xi_r|^2 + \right.$$
$$\left. + \frac{B_\theta^2}{4\pi}\left[\frac{m^2k^2r^2}{(m^2+k^2r^2)^2} + \frac{m^4}{(m^2+k^2r^2)^2}\right]\left|r\frac{\partial}{\partial r}\frac{\xi_r}{r}\right|^2\right\} =$$

$$\boxed{= \sum_{mk} 2\pi L_z \int_0^\infty dr\, r\left[\frac{1}{r^2}\left(2r\frac{\partial p}{\partial r} + m^2\frac{B_\theta^2}{4\pi}\right)|\xi_r|^2 + \underbrace{\frac{m^2r^2}{m^2+k^2r^2}\frac{B_\theta^2}{4\pi}\left|\frac{\partial}{\partial r}\frac{\xi_r}{r}\right|^2}\right]}$$

This term is positive def. ⇒ stabilising

Most unstable modes have $k \to \infty$.

So instability when

$$\boxed{2r\frac{\partial p}{\partial r} < -m^2\frac{B_\theta^2}{4\pi}}$$

$$\frac{\partial p}{\partial r} = -\frac{B_\theta}{4\pi r}\frac{\partial}{\partial r}rB_\theta \qquad -2\frac{\cancel{B_\theta}}{\cancel{4\pi}}\frac{\partial}{\partial r}rB_\theta < -m^2\frac{B_\theta^{\cancel{2}}}{\cancel{4\pi}}$$

$$m^2B_\theta - 2B_\theta - \underbrace{2r\frac{\partial B_\theta}{\partial r}}_{=\,2B_\theta - 2r^2\frac{\partial}{\partial r}\frac{B_\theta}{r}} < 0$$

$$\boxed{\frac{r^2}{B_\theta}\frac{\partial}{\partial r}\frac{B_\theta}{r} > \frac{1}{2}(m^2-4)}$$

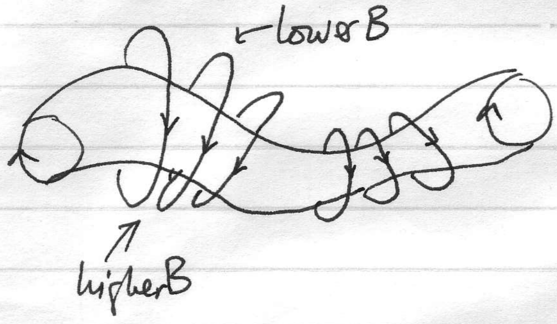
$m=1$ most unstable

What does this look like? $\xi_r = \xi_r(r)e^{i(\theta+kz)}$

ξ_z is small as $k\to\infty$

$$\left(i\xi_\theta = -\frac{\partial}{\partial r}r\xi_r\right\} - ikr\xi_z = -\xi_r - r\xi_r' + \frac{k^2r^4}{m^2+k^2r^2}\frac{\partial}{\partial r}\frac{\xi_r}{r} \approx$$

$$\left.\approx -\xi_r - r\xi_r' + r^2\frac{\partial}{\partial r}\frac{\xi_r}{r} = -\xi_r - \cancel{r\xi_r'} + \cancel{r\xi_r'} - \xi_r = -2\xi_r\right)$$

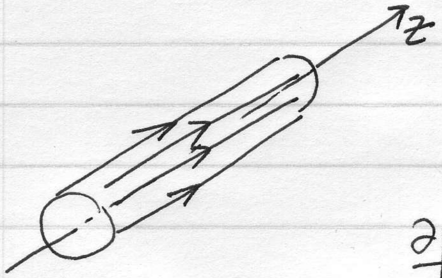


magnetic pressure in areas of higher B pushes flux outward.

This is called the "<u>kink mode</u>".

Exam

2) Stability of the θ pinch.



$$\vec{B} = B_z(r)\hat{z} \quad , \quad \vec{J} = j_\theta(r)\hat{\theta} \quad , \quad j_\theta = -\frac{c}{4\pi}\frac{\partial B_z}{\partial r}$$

$$\frac{\partial p}{\partial r} = -\frac{\partial}{\partial r}\frac{B_z^2}{8\pi}$$

Again

$$\delta W_{2_{mk}} = 2\pi L_z \int_0^\infty dr\, r \Big[\gamma p |\nabla\cdot\vec{\xi}|^2 + \underbrace{\frac{|\vec{Q}|^2}{4\pi} + (\nabla\cdot\vec{\xi}^*_{mk})\,\vec{\xi}_{mk}\cdot\nabla p + \frac{\vec{J}\cdot(\vec{\xi}_m^*\times\vec{Q})}{c}}_{= (*)}\Big]_{mk}$$

- $$\vec{Q} = \nabla\times(\vec{\xi}\times\vec{B}) = \nabla\times\begin{bmatrix}\xi_r\\ \xi_\theta\\ \xi_z\end{bmatrix}\times\begin{bmatrix}0\\0\\B_z\end{bmatrix} = \nabla\times\begin{bmatrix}\xi_\theta B_z\\ -\xi_r B_z\\ 0\end{bmatrix} =$$

$$= \hat{r}\left(+ik\xi_r B_z\right) + \hat{\theta}\left(ik\xi_\theta B_z\right) + \hat{z}\left(-\frac{1}{r}\frac{\partial}{\partial r}(r\xi_r B_z) - \frac{im}{r}\xi_\theta B_z\right)$$

- $$\frac{1}{c}\vec{J}\cdot(\vec{\xi}^*\times\vec{Q}) = -\frac{1}{4\pi}\frac{\partial B_z}{\partial r}\left(-\xi_r^* Q_z + \xi_z^* Q_r\right) =$$

$$= \frac{1}{4\pi}\frac{\partial B_z}{\partial r}\Big(\underbrace{-\frac{\xi_r^*}{r}\underbrace{\frac{\partial}{\partial r}r\xi_r B_z}_{=\ \xi_r B_z + B_z r\frac{\partial \xi_r}{\partial r} + r\xi_r\frac{\partial B_z}{\partial r}}}_{=\ -\frac{|\xi_r|^2}{r}B_z - B_z\xi_r^*\frac{\partial\xi_r}{\partial r} - |\xi_r|^2\frac{\partial B_z}{\partial r}} - \frac{im}{r}\xi_\theta\xi_r^* B_z - ik\xi_r\xi_z^* B_z\Big)$$

$$= -\frac{1}{4\pi}|\xi_r|^2\left(\frac{\partial B_z}{\partial r}\right)^2 + \frac{\partial p}{\partial r}\left(\frac{|\xi_r|^2}{r} + \xi_r^*\frac{\partial \xi_r}{\partial r} + \frac{im}{r}\xi_\theta\xi_r^* + ik\xi_r\xi_z^*\right)$$

- $$(\nabla\cdot\vec{\xi}^*)\,\vec{\xi}\cdot\nabla p = \left(\frac{1}{r}\frac{\partial}{\partial r}r\xi_r^* - \frac{im}{r}\xi_\theta^* - ik\xi_z^*\right)\xi_r\frac{\partial p}{\partial r} =$$

$$= \frac{\partial p}{\partial r}\left(\frac{|\xi_r|^2}{r} + \xi_r\frac{\partial \xi_r^*}{\partial r} - \frac{im}{r}\xi_\theta^*\xi_r - ik\xi_z^*\xi_r\right)$$

$$\bullet\ \frac{|\vec{Q}|^2}{4\pi} = \frac{1}{4\pi}\Big[B_z^2 k^2 |\xi_r|^2 + B_z^2 k^2 |\xi_\theta|^2 + \underbrace{\left|\frac{1}{r}\xi_r B_z + B_z \frac{\partial \xi_r}{\partial r} + \xi_r \frac{\partial B_z}{\partial r} + \frac{im}{r}\xi_\theta B_z\right|^2}_{\parallel}\Big]$$

$$\left|B_z\left(\frac{\xi_r}{r} + \frac{\partial \xi_r}{\partial r} + \frac{im\,\xi_\theta}{r}\right) + \xi_r \frac{\partial B_z}{\partial r}\right|^2 =$$

$$= B_z^2 \left|\frac{\xi_r}{r} + \frac{\partial \xi_r}{\partial r} + \frac{im\,\xi_\theta}{r}\right|^2 + |\xi_r|^2\left(\frac{\partial B_z}{\partial r}\right)^2 +$$

$$+ \left(2\frac{|\xi_r|^2}{r} + \xi_r \frac{\partial \xi_r^*}{\partial r} - \frac{im\,\xi_r \xi_\theta^*}{r} + \xi_r^* \frac{\partial \xi_r}{\partial r} + \frac{im\,\xi_r^* \xi_\theta}{r}\right)\frac{\partial}{\partial r}\frac{B_z^2}{2}$$

$$= \frac{B_z^2}{4\pi}\left[k^2\left(|\xi_r|^2 + |\xi_\theta|^2\right) + \left|\frac{\xi_r}{r} + \frac{\partial \xi_r}{\partial r} + \frac{im\,\xi_\theta}{r}\right|^2\right] + \cancel{\frac{1}{4\pi}|\xi_r|^2\left(\frac{\partial B_z}{\partial r}\right)^2}$$

$$- \frac{\partial p}{\partial r}\left[\cancel{2\frac{|\xi_r|^2}{r}} + \cancel{\xi_r \frac{\partial \xi_r^*}{\partial r}} + \frac{im}{r}\left(\cancel{\xi_r^* \xi_\theta} - \cancel{\xi_r \xi_\theta^*}\right) + \cancel{\xi_r^* \frac{\partial \xi_r}{\partial r}}\right]$$

$$(*) = (\ldots) + \frac{\partial p}{\partial r}\left[\cancel{\frac{|\xi_r|^2}{r}} + \cancel{\xi_r \frac{\partial \xi_r^*}{\partial r}} - \cancel{\frac{im}{r}\xi_\theta^*}\xi_r + \cancel{ik\,\xi_z^* \xi_r}\right]$$

$$- \cancel{\frac{1}{4\pi}|\xi_r|^2}\left(\frac{\partial B_z}{\partial r}\right)^2 + \frac{\partial p}{\partial r}\left[\cancel{\frac{|\xi_r|^2}{r}} + \cancel{\xi_r^* \frac{\partial \xi_r}{\partial r}} + \cancel{\frac{im}{r}}\xi_\theta \xi_r^* - \cancel{ik\,\xi_r \xi_z^*}\right]$$

$$= \frac{B_z^2}{4\pi}\left[k^2\left(|\xi_r|^2 + |\xi_\theta|^2\right) + \left|\frac{\xi_r}{r} + \frac{\partial \xi_r}{\partial r} + \frac{im\,\xi_\theta}{r}\right|^2\right]$$

$$\delta W_{2MK} = 2\pi L_z \int_0^\infty dr\,r \Big\{\gamma p \left|\frac{1}{r}\frac{\partial}{\partial r} r\xi_r + \frac{im}{r}\xi_\theta + ik\,\xi_z\right|^2 + \frac{B_z^2}{4\pi}\left[k^2\left(|\xi_r|^2 + |\xi_\theta|^2\right) + \left|\frac{\xi_r}{r} + \frac{\partial \xi_r}{\partial r} + \frac{im\,\xi_\theta}{r}\right|^2\right]\Big\} > 0$$

positive definite $\mapsto$ θ pinch always stable.