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§11. The Energy Principle.

Now suppose that ρ_0, p_0, B_0 represent an equilibrium. We'd like to know if this equilibrium is stable. For this, we consider small perturbations of the eq. and see if they grow. In the Lagr. formalism, this means that we consider small displacements $\vec{\xi}$.

Energy:

$$\mathcal{E} = \underbrace{\int d^3x_0 \frac{1}{2}\rho_0\left(\frac{\partial\vec{\xi}}{\partial t}\right)^2}_{\text{kinetic}} + \underbrace{\int d^3x_0 \left[\frac{p_0 J^{1-\gamma}}{\gamma-1} + \frac{|\vec{B}_0\cdot\nabla\vec{x}|^2}{8\pi J}\right]}_{\substack{\text{potential} \\ W}}$$

Keep terms up to 2 order in $\vec{\xi}$:

$$\mathcal{E} = \int d^3x_0 \frac{1}{2}\rho_0\left(\frac{\partial\vec{\xi}}{\partial t}\right)^2 + W_0 + \delta W_1[\vec{\xi}] + \delta W_2[\vec{\xi},\vec{\xi}] + \ldots$$

Energy is conserved (to all orders):

$$\frac{d\mathcal{E}}{dt} = \int d^3x_0\, \rho_0 \frac{\partial\vec{\xi}}{\partial t}\cdot\underbrace{\frac{\partial^2\vec{\xi}}{\partial t^2}}_{=\vec{F}[\vec{\xi}]} + \delta W_1\left[\frac{\partial\vec{\xi}}{\partial t}\right] + \delta W_2\left[\frac{\partial\vec{\xi}}{\partial t},\vec{\xi}\right] + \delta W_2\left[\vec{\xi},\frac{\partial\vec{\xi}}{\partial t}\right] + \ldots$$

$$= 0$$

~~This should be zero~~ This must be true at all times, including $t=0$, when $\vec{\xi}$ and $\frac{\partial\vec{\xi}}{\partial t}$ can be chosen independently. Thus, for arbitrary functions $\vec{\xi}$ and $\vec{\eta}$,

$$\int d^3x_0\, \vec{\eta}\cdot\vec{F}[\vec{\xi}] + \delta W_1[\vec{\eta}] + \delta W_2[\vec{\eta},\vec{\xi}] + \delta W_2[\vec{\xi},\vec{\eta}] + \text{higher order} = 0$$

1st order: $\delta W_1[\vec{\eta}] = 0$

2nd order: $\int d^3x_0\, \vec{\eta}\cdot\vec{F}[\vec{\xi}] = -\delta W_2[\vec{\eta},\vec{\xi}] - \delta W_2[\vec{\xi},\vec{\eta}]$ (*)

- Set $\vec{\eta} = \vec{\xi}$. This gives

$$\delta W_2[\vec{\xi},\vec{\xi}] = -\frac{1}{2}\int d^3x_0\, \vec{\xi}\cdot\vec{F}[\vec{\xi}]$$

work done by plasma to set up displ. $\vec{\xi}$.

- In (*), rhs symmetric wrt $\vec{\eta} \leftrightarrow \vec{\xi}$ $\Rightarrow$ lhs is too:

$$\int d^3x_0\, \vec{\eta}\cdot\vec{F}[\vec{\xi}] = \int d^3x_0\, \vec{\xi}\cdot\vec{F}[\vec{\eta}]$$

$$\Downarrow$$

Operator $\vec{F}[\vec{\xi}]$ is self-adjoint

11.02.04

Lecture 9

Consider normal modes of the operator $\vec{F}$:

$$\rho_0 \frac{\partial^2\vec{\xi}}{\partial t^2} = \vec{F}[\vec{\xi}],$$

Normal modes: $\vec{\xi}_{(t,\vec{x}_0)} = \vec{\xi}_n^{(\vec{x}_0)} e^{-i\omega_n t}$

$$-\rho_0\omega_n^2\vec{\xi}_n = \vec{F}[\vec{\xi}_n]$$

1) • Eigenvalues are real.

Pf. $-\rho_0\omega_n^2\vec{\xi}_n = \vec{F}[\vec{\xi}_n] \quad \Big|\cdot\vec{\xi}_n^*$

$-\rho_0\omega_n^{2*}\vec{\xi}_n^* = \vec{F}[\vec{\xi}_n^*] \quad \Big|\cdot\vec{\xi}_n$

substract

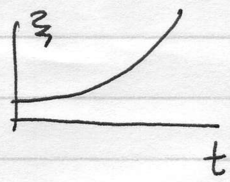
s-a.

$$-(\omega_n^2 - \omega_n^{2*})\underbrace{\int d^3x_0 |\vec{\xi}|^2}_{>0} = \int d^3x_0\, \vec{\xi}_n^*\vec{F}[\vec{\xi}_n] - \int d^3x_0\, \vec{\xi}_n\vec{F}[\vec{\xi}_n^*] = 0$$

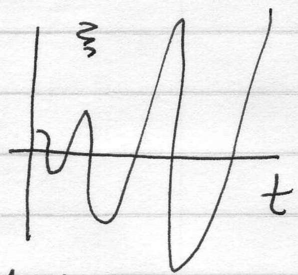
So $\omega_n^2 = \omega_n^{2*}$ q.e.d.

growth (instability) : $\omega_n^2 < 0$

ω_n^2 real means instabilities look like this:



, not like this:



(no oscillations!)

2) • Eigenvectors are orthogonal

Pf. $-\rho_0 \omega_n^2 \vec{\xi}_n = \vec{F}[\vec{\xi}_n] \quad \Big| \cdot \vec{\xi}_m$ subtract

$-\rho_0 \omega_m^2 \vec{\xi}_m = \vec{F}[\vec{\xi}_m] \quad \Big| \cdot \vec{\xi}_n$

$$-\underbrace{(\omega_n^2 - \omega_m^2)}_{\neq 0 \; n \neq m} \underbrace{\int d^3x_0 \, \rho_0 \vec{\xi}_n \cdot \vec{\xi}_m}_{= 0} = \int d^3x_0 \, \vec{\xi}_m \vec{F}[\vec{\xi}_n] - \int d^3x_0 \, \vec{\xi}_n \vec{F}[\vec{\xi}_m] = 0$$

q.e.d.

3) • Energy principle.

Assume completeness of $\{\vec{\xi}_n\}$. Then $\forall \vec{\xi} = \sum_n a_n(t) \vec{\xi}_n(\vec{x}_0)$ (at fixed t).

$$\delta W_2[\vec{\xi}, \vec{\xi}] = -\frac{1}{2} \int d^3x_0 \left(\sum_n a_n \vec{\xi}_n \right) \cdot \vec{F}\left[\sum_m a_m \vec{\xi}_m \right] =$$

$$= -\frac{1}{2} \sum_{n,m} \int d^3x_0 \, a_n a_m \vec{\xi}_n \cdot \vec{F}[\vec{\xi}_m] =$$

$$= +\frac{1}{2} \sum_{n,m} \underbrace{\int d^3x_0 \, \rho_0 \vec{\xi}_n \cdot \vec{\xi}_m}_{= \delta_{nm} \int d^3x_0 \, \rho_0 |\vec{\xi}_n|^2} \omega_m^2 a_n a_m =$$

$$= \frac{1}{2} \sum_n \omega_n^2 a_n^2 \int d^3x_0 \, \rho_0 |\vec{\xi}_n|^2$$

Introduce now

$$K[\vec{\xi},\vec{\xi}] = \frac{1}{2}\int d^3x_0\, \rho_0 |\vec{\xi}|^2 = \frac{1}{2}\sum_n a_n^2 \int d^3x_0\, \rho_0 |\vec{\xi}_n|^2$$

Then, arranging $\omega_1^2 < \omega_2^2 < \dots$, we have

$$\omega_1^2 = \min_{\vec{\xi}} \frac{\delta W_2[\vec{\xi},\vec{\xi}]}{K[\vec{\xi},\vec{\xi}]}$$

$$\boxed{\delta W_2[\vec{\xi},\vec{\xi}] > 0 \ \forall \vec{\xi} \iff \text{stability}}$$

(i.e. stable eq. corresponds to local min. of energy)

Pf.

Suff.: $\delta W_2 > 0\ \forall \vec{\xi} \mapsto \omega_1^2 > 0 \mapsto$ all $\omega_n^2 > 0$

Nec.: all $\omega_n^2 > 0 \mapsto \delta W_2 > 0$ q.e.d.

Note: This is actually true w/o assuming completeness of eigenmodes (see refs.)

Now that we know we need δW_2 to check for stability, let us calculate δW_2 explicitly. For that, all we need to do is to calculate $\vec{F}[\vec{\xi}]$ up to first order in $\vec{\xi}$.

$$\vec{F}[\vec{\xi}] = -J\,(\nabla_0\vec{x})^{-1}\cdot\nabla_0\left(\frac{p_0}{J^\gamma} + \frac{|\vec{B}_0\cdot\nabla_0\vec{x}|^2}{8\pi J^2}\right) + $$
$$+\frac{1}{4\pi}\vec{B}_0\cdot\nabla_0\left(\frac{\vec{B}_0}{J}\cdot\nabla_0\vec{x}\right)$$

$$\boxed{J} = \frac{1}{6}\epsilon_{ijk}\epsilon_{mnl}\left(\delta_{im} + \frac{\partial \xi_i}{\partial x_{om}} + \dots\right)\left(\delta_{jn} + \frac{\partial \xi_j}{\partial x_{on}} + \dots\right)\left(\delta_{kl} + \frac{\partial \xi_k}{\partial x_{ol}} + \dots\right)$$

$$= 1 + \frac{1}{6}\left(\epsilon_{ijk}\epsilon_{mjk}\frac{\partial \xi_i}{\partial x_{om}} + \epsilon_{ijk}\epsilon_{ink}\frac{\partial \xi_j}{\partial x_{on}} + \epsilon_{ijk}\epsilon_{ijl}\frac{\partial \xi_k}{\partial x_{ol}} + \dots\right)$$

$$= 1 + \frac{1}{2}\epsilon_{ijk}\epsilon_{ijl}\frac{\partial \xi_k}{\partial x_{ol}} + \dots = \boxed{1 + \nabla_o \cdot \vec{\xi} + \dots}$$

Use $\epsilon_{ijk}\epsilon_{inl} = \delta_{jn}\delta_{kl} - \delta_{jl}\delta_{kn}$

$$\epsilon_{ijk}\epsilon_{ijl} = 3\delta_{kl} - \delta_{kl} = 2\delta_{kl}$$

[Alternatively, this can be calculated so:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0$$

$$\frac{\partial \rho_L}{\partial t} = -\rho_L \nabla\cdot\vec{u} = -\rho_L \nabla\cdot\frac{\partial \vec{\xi}}{\partial t}$$

$$\rho_L = \frac{\rho_o}{J} = \rho_o + \delta\rho_L \Rightarrow \frac{1}{J} = 1 + \frac{\delta\rho_L}{\rho_o} \Rightarrow J = 1 - \frac{\delta\rho_L}{\rho_o} + \dots$$

To linear order $\frac{\partial \rho_L}{\partial t} = \frac{\partial \delta\rho_L}{\partial t} = -\rho_o \nabla_o\cdot\frac{\partial \vec{\xi}}{\partial t}$

$$\delta\rho_L = -\rho_o \nabla_o\cdot\vec{\xi} \Rightarrow J = 1 + \nabla_o\cdot\vec{\xi} + \dots \;]$$

~~Also~~ $\cancel{\frac{\partial x_i}{\partial x_{oj}} = \delta_{ij} + \frac{\partial \xi_i}{\partial x_{oj}}}$

$$\cancel{\nabla = (\nabla_o \vec{x})^{-1}\cdot\nabla_o = (\nabla\vec{x}_o)\cdot\nabla_o \Rightarrow \frac{\partial}{\partial x_i} = \frac{\partial x_{oj}}{\partial x_i}\frac{\partial}{\partial x_{oj}}} =$$

$$\cancel{= \left(\frac{\partial x_i}{\partial x_{oj}}\right)^{-1}\frac{\partial}{\partial x_{oj}} = \left(\delta_{ij} + \frac{\partial \xi_i}{\partial x_{oj}}\right)^{-1}\frac{\partial}{\partial x_{oj}} \approx \left(\delta_{ij} - \frac{\partial \xi_i}{\partial x_{oj}}\right)\frac{\partial}{\partial x_{oj}}}$$

Also $\nabla_o \vec{x}_o = \mathbb{1} + \nabla_o \vec{\xi}$

$$\boxed{\nabla \vec{x}_o = (\nabla_o \vec{x}_o)^{-1} = \mathbb{1} - \nabla_o \vec{\xi}}$$

$$\frac{\partial x_{0j}}{\partial x_i}\overbrace{\frac{\partial x_i}{\partial x_{0k}}}^{=\,\delta_{ik}+\frac{\partial \xi_i}{\partial x_{0k}}} = \delta_{jk}$$

$$\frac{\partial x_{0j}}{\partial x_k} + \underbrace{\frac{\partial x_{0j}}{\partial x_i}}_{=\,\delta_{ji}\text{ to lowest order.}}\frac{\partial \xi_i}{\partial x_{0k}} = \delta_{jk}$$

$$\frac{\partial x_{0j}}{\partial x_k} = \delta_{jk} - \frac{\partial \xi_j}{\partial x_{0k}}$$

$$\frac{\partial}{\partial x_i} = \frac{\partial x_{0j}}{\partial x_i}\frac{\partial}{\partial x_{0j}} = \left(\delta_{ji} - \frac{\partial \xi_j}{\partial x_{0i}}\right)\frac{\partial}{\partial x_{0j}}$$

$$\rho_0\vec g(\vec x) = \rho_0\vec g(\vec x_0+\vec\xi) = \rho_0\underbrace{\vec g(\vec x_0)}_{=\vec g_0} + \rho_0\vec\xi\cdot\nabla_0\vec g_0$$

So, the force operator is

$$\boxed{\vec F[\vec\xi] = -\underbrace{(1+\nabla_0\cdot\vec\xi)(\mathbb{1}-\nabla_0\vec\xi)}_{=\mathbb{1}+\mathbb{1}\nabla_0\cdot\vec\xi-\nabla_0\vec\xi}\cdot\nabla_0\Big[p_0(1-\gamma\nabla_0\cdot\vec\xi) + \frac{1}{8\pi}\underbrace{(1-2\nabla_0\cdot\vec\xi)|\vec B_0+\vec B_0\cdot\nabla_0\vec\xi|^2}\Big] + \frac{1}{4\pi}\vec B_0\cdot\nabla_0\Big[\underbrace{(1-\nabla_0\cdot\vec\xi)(\vec B_0+\vec B_0\cdot\nabla_0\vec\xi)}_{=\vec B_0-\vec B_0\nabla_0\cdot\vec\xi+\vec B_0\cdot\nabla_0\vec\xi}\Big] + \rho_0\vec g}$$

$$= (1-2\nabla_0\cdot\vec\xi)(B_0^2+2\vec B_0\vec B_0:\nabla\vec\xi) = B_0^2+2\vec B_0\vec B_0:\nabla_0\vec\xi-2B_0^2\nabla_0\cdot\vec\xi$$

$$= -\nabla_0\Big[\cancel{p_0}-\gamma p_0\nabla_0\cdot\vec\xi+\cancel{\frac{B_0^2}{8\pi}}+\frac{\vec B_0\vec B_0}{4\pi}:\nabla_0\vec\xi-\frac{B_0^2}{4\pi}\nabla\cdot\vec\xi\Big]$$

$$-(\mathbb{1}\nabla_0\cdot\vec\xi-\nabla_0\vec\xi)\cdot\nabla_0\left(p_0+\frac{B_0^2}{8\pi}\right) + \quad \longrightarrow \quad \frac{\vec B_0\cdot\nabla_0\vec B_0}{4\pi}+\rho_0\vec g_0$$

only for the $\nabla\cdot\vec\xi$ term

$$+\cancel{\frac{\vec B_0\cdot\nabla_0\vec B_0}{4\pi}}+\frac{1}{4\pi}\vec B_0\cdot\nabla_0(\vec B_0\cdot\nabla_0\vec\xi-\vec B_0\nabla_0\cdot\vec\xi)+\cancel{\rho_0\vec g_0}$$
$$+\rho_0\vec\xi\cdot\nabla_0\vec g_0$$

$$= \nabla_0\gamma p_0\nabla_0\cdot\vec\xi-\nabla_0\Big[\frac{\vec B_0}{4\pi}\cdot(\vec B_0\cdot\nabla_0\vec\xi-\vec B_0\nabla_0\cdot\vec\xi)\Big]$$
$$+\frac{\vec B_0}{4\pi}\cdot\nabla_0(\vec B_0\cdot\nabla_0\vec\xi-\vec B_0\nabla_0\cdot\vec\xi)-\frac{\vec B_0\cdot\nabla_0\vec B_0}{4\pi}\nabla_0\cdot\vec\xi$$
$$-\rho_0\vec g_0\nabla_0\cdot\vec\xi+\rho_0\vec\xi\cdot\nabla_0\vec g_0+\nabla_0\Big[\vec\xi\cdot\nabla_0\left(p_0+\frac{B_0^2}{8\pi}\right)\Big]-\vec\xi\cdot\nabla_0\nabla_0\left(p_0+\frac{B_0^2}{8\pi}\right)$$

$$= \nabla_0(\gamma p_0\nabla_0\cdot\vec\xi+\vec\xi\cdot\nabla_0 p_0)-\nabla_0\Big[\frac{\vec B_0}{4\pi}\cdot\underbrace{(-\vec\xi\cdot\nabla_0\vec B_0+\vec B_0\cdot\nabla_0\vec\xi-\vec B_0\nabla_0\cdot\vec\xi)}_{=\vec Q}\Big]$$
$$+\frac{\vec B_0}{4\pi}\cdot\nabla_0(\vec B_0\cdot\nabla_0\vec\xi-\vec B_0\nabla_0\cdot\vec\xi)-\frac{\vec B_0\cdot\nabla_0\vec B_0}{4\pi}\nabla_0\cdot\vec\xi$$
$$-\rho_0\vec g_0\nabla_0\cdot\vec\xi-\vec\xi\cdot\nabla_0\frac{\vec B_0\cdot\nabla_0\vec B_0}{4\pi}\underbrace{-\vec\xi\cdot\nabla_0(\rho_0\vec g_0)+\rho_0\vec\xi\cdot\nabla_0\vec g_0}_{=-(\vec\xi\cdot\nabla_0\rho_0)\vec g_0} =$$

Gravity term:

$$\boxed{\rho_0 \vec{g}} = -\rho_0 \nabla \Phi = -\rho_0 \left(\nabla_0 \vec{x}\right)^{-1} \cdot \nabla_0 \Phi = -\rho_0 \left(\mathbb{1} - \nabla_0 \vec{\xi}\right) \cdot \nabla_0 \Phi$$

$$= -\rho_0 \nabla_0 \Phi + \rho_0 \underbrace{\left(\nabla_0 \vec{\xi}\right) \cdot \nabla_0 \Phi_0}_{=\ \nabla_0 \left(\vec{\xi} \cdot \nabla_0 \Phi_0\right) - \vec{\xi} \cdot \nabla_0 \nabla_0 \Phi_0} =$$

$$= -\rho_0 \underbrace{\nabla_0 \Phi}_{} + \cancel{\rho_0 \nabla_0 \vec{\xi} \cdot \nabla_0 \Phi_0} + \rho_0 \vec{\xi} \cdot \nabla_0 \vec{g}_0 =$$

$$\Phi\left(\vec{x}(t, \vec{x}_0)\right) = \Phi\left(\vec{x}_0 + \vec{\xi}\right) = \Phi_0 + \cancel{\vec{\xi} \cdot \nabla_0 \Phi_0}$$

$$= \boxed{\rho_0 \vec{g}_0 + \rho_0 \vec{\xi} \cdot \nabla_0 \vec{g}_0}$$

$$= \nabla_0\left(\gamma p_0 \nabla_0\cdot\vec{\xi} + \vec{\xi}\cdot\nabla p_0\right) - \nabla_0 \frac{\vec{B}_0\cdot\vec{Q}}{4\pi} + \frac{\vec{B}_0}{4\pi}\cdot\nabla_0\Big(\vec{Q} + \underbrace{\vec{\xi}\cdot\nabla_0\vec{B}_0}\Big)$$

$$-\left(\nabla_0\cdot(\rho_0\vec{\xi})\right)\vec{g} \underbrace{- (\nabla\cdot\vec{\xi})\frac{\vec{B}_0\cdot\nabla_0\vec{B}_0}{4\pi} - \vec{\xi}\cdot\nabla_0\frac{\vec{B}_0\cdot\nabla_0\vec{B}_0}{4\pi}}$$

$$\frac{1}{4\pi}\Big[(\vec{B}_0\cdot\nabla_0\vec{\xi})\cdot\nabla_0\vec{B}_0 + \cancel{\vec{B}_0\vec{\xi}:\nabla_0\nabla_0\vec{B}_0} - (\nabla_0\cdot\vec{\xi})\vec{B}_0\cdot\nabla_0\vec{B}_0 - (\vec{\xi}\cdot\nabla_0\vec{B}_0)\cdot\nabla_0\vec{B} - \cancel{\vec{\xi}\vec{B}_0:\nabla_0\nabla_0\vec{B}_0}\Big] = \frac{\vec{Q}\cdot\nabla_0\vec{B}_0}{4\pi}$$

$$\boxed{= \nabla_0\left(\gamma p_0\nabla_0\cdot\vec{\xi} + \vec{\xi}\cdot\nabla p_0\right) - \nabla_0\frac{\vec{B}_0\cdot\vec{Q}}{4\pi} + \frac{\vec{B}_0\cdot\nabla_0\vec{Q} + \vec{Q}\cdot\nabla_0\vec{B}_0}{4\pi} - \left(\nabla_0\cdot(\rho_0\vec{\xi})\right)\vec{g}}$$

where $\vec{Q} = -\vec{\xi}\cdot\nabla_0\vec{B}_0 + \vec{B}_0\cdot\nabla_0\vec{\xi} - \vec{B}_0\nabla\cdot\vec{\xi} = \nabla_0\times(\vec{\xi}\times\vec{B}_0)$

[Note that another way to get this is

$$\frac{\partial^2\vec{\xi}}{\partial t^2} = -\nabla_0\delta p + \frac{(\nabla\times\delta\vec{B})\times\vec{B}_0 + (\nabla\times\vec{B}_0)\times\delta\vec{B}}{4\pi} + \delta(\rho\vec{g})$$

where δp, $\delta\vec{B}$ and $\delta(\rho\vec{g})$ are <u>Eulerian</u> perturbations

$$\delta p = -\vec{\xi}\cdot\nabla p_0 - \gamma p_0\nabla\cdot\vec{\xi}$$

$$\delta\vec{B} = \vec{Q}$$

$$\delta\rho = -\nabla\cdot(\rho_0\vec{\xi})$$

and $(\nabla\times\vec{Q})\times\vec{B}_0 + (\nabla\times\vec{B}_0)\times\vec{Q} = -(\nabla\vec{Q})\cdot\vec{B}_0 + \vec{B}_0\cdot\nabla\vec{Q} - (\nabla\vec{B}_0)\cdot\vec{Q} + \vec{Q}\cdot\nabla\vec{B}_0$

$$= -\nabla(\vec{Q}\cdot\vec{B}_0) + \vec{B}_0\cdot\nabla\vec{Q} + \vec{Q}\cdot\nabla\vec{B}_0$$

]

Now we calculate δW_2.

$$\boxed{\delta W_2[\vec\xi,\vec\xi] = -\frac{1}{2}\int d^3x_0\,\vec\xi\cdot\vec F[\vec\xi]} =$$

because $\nabla_0\cdot\vec Q = 0$

$$= -\frac{1}{2}\int d^3x_0\,\vec\xi\cdot\left[\nabla_0(\gamma p_0\nabla_0\cdot\vec\xi + \vec\xi\cdot\nabla p_0) - \nabla_0\frac{\vec B_0\cdot\vec Q}{4\pi} + \nabla_0\cdot\frac{\vec B_0\vec Q + \vec Q\vec B_0}{4\pi} - (\nabla_0\cdot\rho_0\vec\xi)\vec g_0\right]$$

$$= \frac{1}{2}\int d^3x_0\left[\gamma p_0(\nabla_0\cdot\vec\xi)^2 + (\nabla_0\cdot\vec\xi)\vec\xi\cdot\nabla p_0 \underbrace{- \frac{\vec B_0\cdot\vec Q}{4\pi}\nabla_0\cdot\vec\xi + \frac{\vec Q\vec B}{4\pi}:\nabla_0\vec\xi + \frac{\vec B\vec Q}{4\pi}:\nabla_0\vec\xi} + (\vec\xi\cdot\vec g_0)\nabla_0\cdot(\rho_0\vec\xi)\right]$$

$$\frac{\vec Q}{4\pi}\cdot\left(-\vec B_0\nabla_0\cdot\vec\xi + \vec B_0\cdot\nabla_0\vec\xi - \vec\xi\cdot\nabla_0\vec B_0\right) + \frac{\vec\xi\cdot(\nabla_0\vec B_0)\cdot\vec Q + \vec Q\cdot(\nabla_0\vec\xi)\cdot\vec B_0}{4\pi}$$

$$= \frac{Q^2}{4\pi} + \frac{\vec\xi\cdot(\nabla_0\vec B_0)\cdot\vec Q + \cancel{\vec Q\cdot\nabla_0(\vec\xi\cdot\vec B_0)} - \vec Q\cdot(\nabla_0\vec B_0)\cdot\vec\xi}{4\pi} =$$

$$= \frac{Q^2}{4\pi} + \frac{\vec\xi\cdot[\vec Q\times(\nabla_0\times\vec B_0)]}{4\pi} = \frac{Q^2}{4\pi} + \frac{\vec j_0\cdot(\vec\xi\times\vec Q)}{c}$$

$$\boxed{= \frac{1}{2}\int d^3x_0\left[\gamma p_0(\nabla_0\cdot\vec\xi)^2 + (\nabla_0\cdot\vec\xi)\,\vec\xi\cdot\nabla_0 p_0 + \frac{Q^2}{4\pi} + \frac{\vec j_0\cdot(\vec\xi\times\vec Q)}{c} + (\vec\xi\cdot\vec g_0)\nabla_0\cdot(\rho_0\vec\xi)\right]}$$

(not unique!)

This is the standard form as given, e.g., in Kulsrud p. 161

$\Rightarrow$ Stability analysis is done in one of two ways:

1) substitute the eq. into $\delta W_2[\vec\xi,\vec\xi]$, check if $\delta W_2 < 0$ for any $\vec\xi$

2) do normal mode analysis (e.v. problem):

$$-\rho_0\omega^2\vec\xi = \vec F[\vec\xi]$$