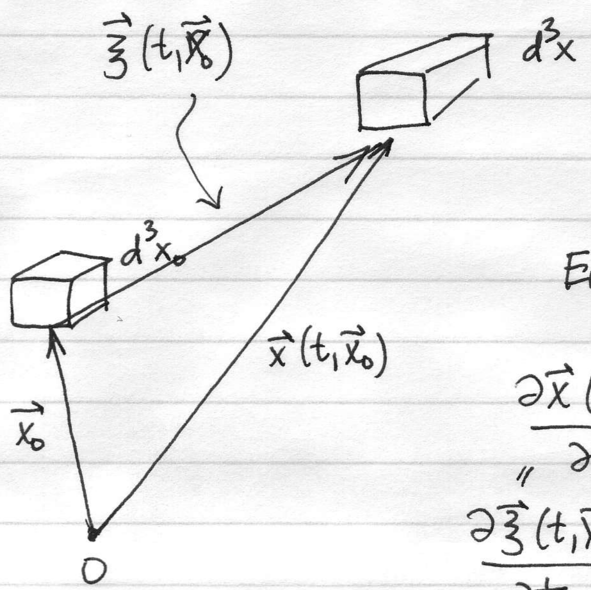
NOTES DISTRIBUTED

Lecture 8 9.02.04

§10 Lagrangian MHD



Lagrangian coords:

$$\underset{\text{Eulerian}}{\vec{x}} = \underset{\text{Lagr.}}{\vec{x}_0} + \underset{\text{displacement}}{\vec{\xi}(t,\vec{x}_0)}$$

$$\frac{\partial \vec{x}(t,\vec{x}_0)}{\partial t} = \vec{u}(t,\vec{x}(t,\vec{x}_0))$$

$$\frac{\partial \vec{x}(t,\vec{x}_0)}{\partial t} = \frac{\partial \vec{\xi}(t,\vec{x}_0)}{\partial t}$$

Transformation of derivatives: $\frac{\partial}{\partial x_{0i}} = \underbrace{\frac{\partial x_k}{\partial x_{0i}}}_{\text{strain matrix}} \overbrace{\frac{\partial}{\partial x_k}}^{=\nabla} \equiv \nabla_0$

$$\boxed{\nabla_0 = (\mathbb{1} + \nabla_0 \vec{\xi}) \cdot \nabla}$$

In these variables, we can integrate MHD equations.

① Conservation of mass:

$$\rho_0(\vec{x}_0)\, d^3x_0 = \rho_L(t,\vec{x}_0)\, d^3x$$

where $\rho_L(t,\vec{x}_0) = \rho(t,\vec{x}(t,\vec{x}_0))$

But $d^3x = J\, d^3x_0$, where $J = \left|\det \frac{\partial x_i}{\partial x_{0j}}\right| = \left|\det(\mathbb{1} + \nabla_0 \vec{\xi})\right|$

So
$$\boxed{\rho_L(t,\vec{x}_0) = \frac{\rho_0(\vec{x}_0)}{J(t,\vec{x}_0)}}$$
continuity eqn integrated.

② Take the adiabatic law for pressure:

$$\frac{d}{dt}\frac{p}{\rho^\gamma} = \frac{\partial}{\partial t}\frac{p_L(t,\vec{x}_0)}{\rho_L^\gamma(t,\vec{x}_0)} = 0$$

$$\frac{p_L}{\rho_L^\gamma} = \text{const} = \frac{p_0(\vec{x}_0)}{\rho_0^\gamma(\vec{x}_0)}$$

$$\frac{p_L}{\rho_L^\gamma} = \frac{p_L(t,\vec{x}_0)}{\rho_0^\gamma(\vec{x}_0)}\, J^\gamma(t,\vec{x}_0)$$

So $$\boxed{p_L(t,\vec{x}_0) = \frac{p_0(\vec{x}_0)}{J^\gamma(t,\vec{x}_0)}}$$ equ. of state integrated.

③ We have already proved the Lundquist theorem (Cauchy solution):

$$\frac{\vec{B}_L(t,\vec{x}_0)}{\underbrace{\rho_L(t,\vec{x}_0)}_{=\rho_0(\vec{x}_0)/J(t,\vec{x}_0)}} = \frac{\vec{B}_0(\vec{x}_0)}{\rho_0(\vec{x}_0)} \cdot \overbrace{\nabla_0 \vec{x}(t,\vec{x}_0)}^{=\mathbb{1}+\nabla_0\vec{\xi}}$$

$$\boxed{\vec{B}_L(t,\vec{x}_0) = \frac{\vec{B}_0(\vec{x}_0)\cdot(\mathbb{1}+\nabla_0\vec{\xi})}{J(t,\vec{x}_0)}}$$ induction equ. integrated.

④ Now deal with the momentum equation

$$\underbrace{\rho}_{=\frac{\rho_0}{J}} \underbrace{\frac{d}{dt}\vec{u}}_{=\frac{\partial^2\vec{\xi}}{\partial t^2}} = -\nabla\Big(\underbrace{p}_{=p_0 J^{-\gamma}} + \underbrace{\frac{B^2}{8\pi}}_{=\frac{|\vec{B}_0\cdot\nabla_0\vec{x}|^2}{8\pi J^2}}\Big) + \underbrace{\frac{\vec{B}\cdot\nabla\vec{B}}{4\pi}}_{=\frac{(\vec{B}_0\cdot\nabla_0\vec{x})\cdot\nabla(\vec{B}_0\cdot\nabla_0\vec{x}/J)}{4\pi J}}$$

$$\frac{(\vec{B}_0\cdot\nabla_0\vec{x})\cdot\nabla}{}=\nabla_0, \qquad \frac{(\vec{B}_0\cdot\nabla_0\vec{x})\cdot\nabla(\vec{B}_0\cdot\nabla_0\vec{x}/J)}{4\pi J} = \frac{\vec{B}_0\cdot\nabla_0(\vec{B}_0\cdot\nabla_0\vec{x}/J)}{4\pi J}$$

$$\frac{\rho_0}{J}\frac{\partial^2\vec{\xi}}{\partial t^2} = -\nabla\left(\frac{p_0}{J^\gamma} + \frac{|\vec{B}_0\cdot\nabla_0\vec{x}|^2}{8\pi J^2}\right) + \frac{\vec{B}_0}{J}\cdot\nabla_0\frac{\vec{B}_0}{J}\cdot\nabla_0\vec{x}\,\frac{1}{4\pi}$$

$$\cancel{(\mathbb{1}+\nabla_0\vec{\xi})\cdot\frac{\partial^2\vec{\xi}}{\partial t^2} = -\left(\frac{\rho_0}{J}\right)^{-1}\nabla_0\left(\frac{p_0}{J^\gamma}+\frac{|\vec{B}_0\cdot\nabla_0\vec{x}|^2}{8\pi J^2}\right)+\frac{\vec{B}_0}{\rho_0}\cdot\nabla_0\frac{\vec{B}_0}{J}\cdot\nabla_0\vec{x}\,\frac{1}{4\pi}}$$

Now $\nabla = \nabla \vec{x}_0 \cdot \nabla_0 = (\nabla_0 \vec{x})^{-1} \cdot \nabla_0 = (\mathbb{1} + \nabla_0 \vec{\xi})^{-1} \cdot \nabla_0$

Thus, finally we have

$$\boxed{\rho_0 \frac{\partial^2 \vec{\xi}}{\partial t^2} = - J (\nabla_0 \vec{x})^{-1} \cdot \nabla_0 \left(\frac{p_0}{J^\gamma} + \frac{|\vec{B}_0 \cdot \nabla_0 \vec{x}|^2}{8\pi J^2} \right) + \frac{1}{4\pi} \vec{B}_0 \cdot \nabla_0 \left(\frac{\vec{B}_0}{J} \cdot \nabla_0 \vec{x} \right) \equiv \vec{F}[\vec{\xi}]}$$

force operator

where $\nabla_0 \vec{x} = \mathbb{1} + \nabla_0 \vec{\xi}$

and $J = |\det \nabla_0 \vec{x}| = \frac{1}{6} \epsilon_{ijk} \epsilon_{lmn} \frac{\partial x_i}{\partial x_{0m}} \frac{\partial x_j}{\partial x_{0n}} \frac{\partial x_k}{\partial x_{0l}}$

(Newcomb 1962)

Energy of this system:

$$\mathcal{E} = \int d^3x \left[\frac{1}{2} \rho u^2 + \frac{p}{\gamma - 1} + \frac{B^2}{8\pi} \right] =$$

$$= \int d^3x_0 \, J \left[\frac{1}{2} \frac{\rho_0}{J} \left(\frac{\partial \vec{\xi}}{\partial t} \right)^2 + \frac{1}{\gamma - 1} \frac{p_0}{J^\gamma} + \frac{|\vec{B}_0 \cdot \nabla_0 \vec{x}|^2}{8\pi J^2} \right] =$$

$$= \int d^3x_0 \left[\frac{1}{2} \rho_0 \left(\frac{\partial \vec{\xi}}{\partial t} \right)^2 + \frac{p_0 J^{1-\gamma}}{\gamma - 1} + \frac{|\vec{B}_0 \cdot \nabla_0 \vec{x}|^2}{8\pi J} \right]$$

NB: The equation for $\vec{\xi}$ can be derived from the following action principle:

action $\quad \mathcal{S}[\vec{\xi}] = \int_{t_0}^{t} dt \int d^3x_0 \, \mathcal{L}[\vec{\xi}, \dot{\vec{\xi}}], \quad \frac{\delta \mathcal{S}}{\delta \vec{\xi}} = 0$

Lagr. density $\quad \mathcal{L} = \frac{1}{2} \rho_0 |\dot{\vec{\xi}}|^2 - \frac{p_0 J^{1-\gamma}}{\gamma - 1} - \frac{|\vec{B}_0 \cdot \nabla_0 \vec{x}|^2}{8\pi J}$

You may check your math. stamina and prove this.
(otherwise see refs. on the course page)