Magnetohydrodynamics and Turbulence

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EXAMPLE SHEET II

These problems will be discussed in the 2nd Example Class (28.11.05, 14:30 in MR4).
Problems marked with * are optional: omit them if you are in a hurry. I probably will not have time discuss them unless you ask me to do so.
There will be another Example Class in Easter Term — we shall deal any questions you might have before the exam.

1. Uniform Collapse. A simple model of star formation envisions a sphere of galactic plasma with density $n = 1\ \mathrm{cm}^{-3}$ undergoing a gravitational collapse to a spherical star with density $n = 10^{26}\ \mathrm{cm}^{-3}$. The magnetic field in the galactic plasma is $\sim 3 \times 10^{-6}$ G. Assuming that flux is frozen, estimate the magnetic field in a star.

2.* Flux Concentration. Consider a simple 2D model of incompressible convective motion:

$$\mathbf{u} = U\left(-\sin\frac{\pi x}{L}\cos\frac{\pi z}{L},\ 0,\ \cos\frac{\pi x}{L}\sin\frac{\pi z}{L}\right). \tag{1}$$

1. In the neighbourhood of the stagnation point $(0, 0, 0)$, linearise the flow, assume vertical magnetic field, $\mathbf{B} = (0, 0, B(t, x))$ and derive an equation for $B(t, x)$. Suppose $B(0, x) = B_0 = \mathrm{const}$ It should be clear to you from your equation that magnetic field is being swept towards $x = 0$. What is the time scale of this sweeping? Given the magnetic Reynolds number $\mathrm{Rm} = UL/\eta \gg 1$, show that flux conservation holds on this time scale.
2. Find a steady solution of your equation. Use flux conservation and $B(x) = B(-x)$ to determine the constants of integration (in terms of B_0 and Rm). What is the width of the region around $x = 0$ where the flux is concentrated? What is the magnitude of the field there?
3. Can you think of a heuristic argument based on the induction equation that would tell you that these answers were to be expected?

3. Zeldovich Antidynamo Theorem. Consider the case of an arbitrary 2D velocity field: $\mathbf{u} = (u_x, u_y, 0)$. Assume incompressibility. Show that, in a finite system (specifically, you may work in a periodic box), this velocity field is not a dynamo, i.e., any initial magnetic field will always eventually decay. This is one of the classical antidynamo results: the Zeldovich Theorem.
Hint. Consider separately the equations for B_z and for the (x, y)-plane magnetic field. Show that B_z decays (consider time evolution of the volume integral of B_z^2), then write B_x, B_y in terms of one scalar function (this is possible because $\partial B_x/\partial x + \partial B_y/\partial y = 0$) and show that it decays as well.

4.* X-Point Collapse. Let us set up the following initial magnetic-field configuration:

$$\mathbf{B}_0(\mathbf{r}_0) = B_0\hat{\mathbf{z}} + \hat{\mathbf{z}} \times \nabla_0\psi(x_0, y_0), \tag{2}$$

where $\mathbf{r}_0 = (x_0, y_0, z_0)$, $B_0 = \mathrm{const}$, and

$$\psi(x_0, y_0) = \frac{1}{2}\left(x_0^2 - y_0^2\right). \tag{3}$$

1. Draw the field lines in the (x_0, y_0) plane to see that this is an X-point.

2. Use Lagrangian MHD

$$\rho_0 \frac{\partial^2 \mathbf{r}}{\partial t^2} = -J \left(\nabla_0 \mathbf{r}\right)^{-1} \cdot \nabla_0 \left(\frac{p_0}{J^\gamma} + \frac{|\mathbf{B}_0 \cdot \nabla_0 \mathbf{r}|^2}{8\pi J^2} \right) + \frac{1}{4\pi} \mathbf{B}_0 \cdot \nabla_0 \left(\frac{\mathbf{B}_0}{J} \cdot \nabla_0 \mathbf{r} \right), \tag{4}$$

where $\mathbf{r}(t, \mathbf{r}_0) = (x, y, z)$ and $J = |\det \nabla_0 \mathbf{r}|$, and seek solutions in the form

$$x = \xi(t) x_0, \quad y = \eta(t) y_0, \quad z = z_0. \tag{5}$$

Show that $\xi(t)$ and $\eta(t)$ satisfy the following equations

$$\frac{d^2\xi}{dt^2} = \eta \left(\frac{1}{\eta^2} - \frac{1}{\xi^2} \right), \tag{6}$$

$$\frac{d^2\eta}{dt^2} = \xi \left(\frac{1}{\xi^2} - \frac{1}{\eta^2} \right). \tag{7}$$

3. Consider the possibility that, as time goes on, $\eta(t) \to 0$ (becomes small) and $\xi(t) = \xi_c + \dots$, where ξ_c is some constant. Find solutions that satisfy this assumption. The answer is

$$\xi(t) \approx \xi_c + \frac{9}{4} \left(\frac{2}{9\xi_c} \right)^{1/3} (t_c - t)^{4/3}, \tag{8}$$

$$\eta(t) \approx \left(\frac{9\xi_c}{2} \right)^{1/3} (t_c - t)^{2/3} \tag{9}$$

as $t \to t_c$, where t_c is some finite time constant. This is called *the Syrovatskii solution* for the X-point collapse.

4. Calculate the magnetic field as a function of time and convince yourself that the initial X-point configuration collapses explosively (in a finite time) to a sheet along the x axis. What do you think happens after t reaches t_c?

5. Now do a similar calculation for incompressible Lagrangian MHD ($J = 1$). Remember that total pressure is now determined by the condition $J = 1$. Show that the solution in this case is

$$\xi(t) = e^{S(t)}, \quad \eta(t) = e^{-S(t)}, \tag{10}$$

where $S(t)$ is an arbitrary function of time. If, e.g., $S(t) = \Lambda t$, show that this means the X-point collapses exponentially. This is called *the Chapman-Kendall solution.*

5. Conservation Laws for Incompressible MHD. Consider equations of incompressible MHD:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{\nabla p}{\rho} + \frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi\rho} + \nu \nabla^2 \mathbf{u} + \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0, \tag{11}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} + \eta \nabla^2 \mathbf{B}, \tag{12}$$

where $\mathbf{f}$ is some forcing function and $\rho = \text{const}$. Derive the evolution equations for the kinetic energy $\int d^3x\, \rho u^2/2$, magnetic energy $\int d^3x\, B^2/8\pi$ and cross-helicity $\int d^3x\, \mathbf{u} \cdot \mathbf{B}$ directly from these equations and make sure your result is consistent with the incompressible limit of the more general equations I derived in class (you should find the incompressible derivation much simpler). Assume that all surface integrals vanish. Can you interpret all terms in your energy equations? Under what assumption are the total energy (kinetic + magnetic) and cross-helicity conserved? When energy is "lost" in this system, where does it go?

6. Kinetic Alfvén Waves. There is an approximation, often used at very small scales, in which one assumes that magnetic field lines are frozen not into the mass flow $\mathbf{u}_i$ (ion velocity — it satisfies the usual Navier-Stokes equation with a Lorentz force) but into the electron flow velocity $\mathbf{u}_e$, which can be expressed in terms the ion velocity $\mathbf{u}_i$ and the current density $\mathbf{j} = en(\mathbf{u}_i - \mathbf{u}_e)$, where e is electron charge and $n = n_i = n_e$ is the ion/electron density. This is called the Electron (or Hall) MHD.

1. Under the above assumption, write a closed system of equations for $\mathbf{B}$ and $\mathbf{u}_i$, assuming incompressibility and neglecting viscosity and Ohmic diffusion.

2. Consider the static equilibrium with a straight uniform magnetic field $\mathbf{B}_0 = B_0\hat{\mathbf{z}} = \text{const}$. Derive the dispersion relation for waves in this system. You will find the following definitions useful: $v_A = B_0/(4\pi n m_i)^{1/2}$ is the Alfvén speed (m_i is the ion mass), $d_i = c(m_i/4\pi e^2 n)^{1/2}$ is called the ion inertial scale or ion skin depth (e is the elementary charge, n number density of ions/electrons).

3. Obtain an explicit formula for the frequency $\omega = \omega(\mathbf{k})$ from your dispersion relation. Under what conditions do you recover the Alfvén waves?

4. Assume $kd_i \gg 1$ (k is the absolute value of the wave vector) and find the corresponding limiting form of the dispersion relation. The waves you have obtained are called the kinetic Alfvén waves (KAW).

7. KAW Turbulence. Interstellar and solar-wind turbulence at scales smaller than d_i (see Problem 6) can be described by an approximation whereby the magnetic field is frozen into the electron flow $\mathbf{u}_e$, while the ions can be considered motionless, $\mathbf{u}_i = 0$.

1. Show that this turbulence obeys the following equation (a trvial task if you have done Problem 6):

$$\frac{\partial \mathbf{B}}{\partial t} = -\frac{c}{4\pi e n}\nabla \times [(\nabla \times \mathbf{B}) \times \mathbf{B}]\,. \tag{13}$$

   Consider a static equilibrium with a straight uniform magnetic field in the z direction, so that $\mathbf{B} = B_0\hat{\mathbf{z}} + \delta\mathbf{B}$. Show that the linear waves in this system have the dispersion relation

$$\omega(\mathbf{k}) = \pm k_\parallel v_A k d_i, \tag{14}$$

   where $v_A = B_0/\sqrt{4\pi n m_i}$, $\mathbf{k} = (k_\perp, 0, k_\parallel)$, $k = |\mathbf{k}|$. These waves are called KInetic Alfvén Waves (KAW). You have, in fact, already obtained this dispersion relation in Problem 6 as a limiting case of the more general dispersion relation taht held for non-zero $\mathbf{u}_i$.

2. Now your task is to work out scalings for the KAW turbulence in a way similar to how this is done for the Alfvén-wave turbulence. As usual, assume that interactions in scale space are local and the energy flux is constant:

$$\epsilon \sim \left(\frac{\delta B_l}{B_0}\right)^2 \frac{v_A^2}{\tau_l} \sim \text{const}, \tag{15}$$

   where τ_l is the cascade time, which you will have to determine. In terms of the typical magnetic-field fluctuation δB_l of scale l, what is the characteristic time associated with the nonlinearity in Eq. (13)? Formulate the assumption of weak interactions. Under this assumption, calculate τ_l and show that

$$\frac{\delta B_l}{B_0} \sim \left(\frac{\epsilon}{v_A^3 d_i} \frac{l_\perp^3}{l_\parallel} \right)^{1/4}, \tag{16}$$

where $l_\perp$ and $l_\parallel$ are characteristic scales perpendicular and parallel to the background field, respectively.

3. If the turbulence were isotropic, what would be scaling of the spectrum of the magnetic field with the wavenumber?

4. Now let us assume that the turbulence is anisotropic, $l_\parallel \gg l_\perp$, and critically balanced, i.e., the interactions are strong and the time for a wave to cascade is comparable to the wave period. Show that in this case, $\delta B_l \propto l_\perp^{2/3}$ and $l_\parallel \propto l_\perp^{1/3}$. This is the picture confirmed by numerical simulations.

8. Reduced MHD. In my lectures on RMHD, I left the derivation of some of the results as an exercise:

1. derivation of the equations for ϕ and ψ;
2. derivation of the equations for Elsasser variables $\zeta^\pm$ and $z_\parallel^\pm$;
3. derivation of the 5 conservation laws.

Work them out.